YOKOHAMA MATHEMATICAL
JOURNAL VOL. 40, 1993

SPACES WHOSE ALL NONEMPTY CLOPEN SUBSPACES ARE HOMEOMORPHIC

By

TOSHIJI TERADA

(Received November 26, 1991)

Summary. A zero-dimensional topological space is called h-homogeneous if all nonempty clopen subspaces are homeomorphic. The Cantor discontinuum, the space of rational numbers and the space of irrational numbers are h-homogeneous. We show the following:

(1) If a non-pseudocompact zero-dimensional space Y has a dense set of isolated points, then the product space Y^{κ} is h-homogeneous for any infinite cardinal κ.

(2) If X is a strongly zero-dimensional h-homogeneous space of countable type, then $\beta X-X$ is h-homogeneous.

1. Introduction

All topological spaces considered here are Tychonoff. A space X is called homogeneous if for any two points $x, y\in X$ there exists a homeomorphism f from X onto itself which maps x to y. A zero-dimensional (in the ind sense) space X is said to be h-homogeneous [8] if all nonempty clopen (closed and open) subspaces are homeomorphic. It is well known [3] that every first countable h-homogeneous space is homogeneous. At a glance, the class of h-homogeneous spaces seems special. But some important zero-dimensional spaces have this property. For example, the Cantor discontinuum $\boldsymbol{C}$, the space $\boldsymbol{Q}$ of rational numbers, the space $\boldsymbol{P}$ of irrational numbers, the Sorgenfrey line and the remainder $\beta\omega-\omega$ of the Stone-Čech compactification of ω are all h-homogeneous. Further, Motorov ([6], [7]) showed the following interesting result: If X is a first countable zero-dimensional space which has a dense set of isolated points, then the ω-power X^{ω} is h-homogeneous and hence homogeneous. Recently, Balcar and Dow [2] showed that, for an infinite compact extremally disconnected space X, the dynamical system (X, Hom) is minimal if and only if

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1991 Mathematics Subject Classification: Primary 54C05; Secondary 54D40.
Key words and phrases: h-homogeneous, pseudocompact, zero-dimensional, Stone-Čech compactification.

X is h-homogeneous, where Hom is the group of all autohomeomorphisms on X under the operation of composition $\circ$.

A Boolean algebra B is called homogeneous if for any non-zero element a of B, the relative algebra $B\upharpoonright a$ $(=\{x\in B: x\leqq a\})$ is isomorphic to B. It is well known that a zero-dimensional compact space X is h-homogeneous if and only if the Boolean algebra Clop(X) of all clopen subsets of X is homogeneous (see [5]). Hence it follows that the study of h-homogeneous compact spaces is equivalent to the study of homogeneous Boolean algebras. For a non-compact space X, the h-homogeneity of X implies also the homogeneity of the Boolean algebra Clop(X) of all clopen subsets of X, but the converse implication is not generally true. Further, since the space $\boldsymbol{Q}$ of rational numbers and the space $\boldsymbol{P}$ of irrational numbers are h-homogeneous as noted above, the study of non-compact h-homogeneous spaces seems to be also interesting.

In this paper, using some properties of the Stone-Čech compactifications of spaces, we will obtain some results concerning non-compact h-homogeneous spaces.

2. Some basic facts and results

For a topological property $\mathcal{P}$, a space X is called nowhere locally $\mathcal{P}$ if no point of X has a neighborhood which has the property $\mathcal{P}$. At first, let us note the following trivial property of h-homogeneous spaces.

Lemma 2.1. *Let X be an h-homogeneous space.*

(1) *If X is not compact, then X is nowhere locally compact.*

(2) *If X is not pseudocompact, then X is nowhere locally pseudocompact.*

The next lemma is also easy and well known.

Lemma 2.2. *Let p be a point in a space X and let q be a point in a space Y. If there exist two sequences $\{U_n: n\in\omega\}$, $\{V_n: n\in\omega\}$ of disjoint clopen subsets of X, Y respectively with the following properties;*

(a) *U_n and V_n are homeomorphic for each $n\in\omega$,*

(b) $X=\cup\{U_n: n\in\omega\}\cup\{p\}$, $Y=\cup\{V_n: n\in\omega\}\cup\{q\}$,

(c) *$\{\cup\{U_n: k\leqq n\}\cup\{p\}: k\in\omega\}$ is a neighborhood base of p in X,*
$\{\cup\{V_n: k\leqq n\}\cup\{q\}: k\in\omega\}$ is a neighborhood base of q in Y,

then there is a homeomorphism f from X onto Y such that $f(p)=q$.

In case only the condition (a) above is satisfied, it is obvious that $\cup\{U_n: n\in\omega\}$ and $\cup\{V_n: n\in\omega\}$ are homeomorphic. A space X is called strongly zero-dimensional if its Stone-Čech compactification is zero-dimensional.

Proposition 2.3. *Let X be a non-pseudocompact, strongly zero-dimensional, h-homogeneous space. Then all nonempty cozero-sets of X are homeomorphic.*

Proof. Every nonempty cozero-set can be expressed as the disjoint union of a sequence of nonempty clopen subsets of X. Since X is h-homogeneous, every nonempty cozero-set is homeomorphic to the topological discrete union of infinitely countable copies of X.

Motorov [6] showed essentially that, if a first countable zero-dimensional space X has a π-base consisting of clopen subsets which are homeomorphic to X, then X is h-homogeneous. The following theorem says that the first countability can be omitted in case X is not pseudocompact.

Theorem 2.4. *Let X be a non-pseudocompact zero-dimensional space. If X has a π-base consisting of clopen subsets which are homeomorphic to X, then X is h-homogeneous.*

Proof. Let A be an arbitrary nonempty clopen subset of X. Since A contains a homeomorphic copy of X as a clopen subset, A is not pseudocompact. Hence A can be expressed as the disjoint union of a sequence of nonempty clopen subsets of X. Let

$$A=A_1\oplus A_2\oplus A_3\oplus \cdots .$$

Then from the assumption it follows that A_1 contains a clopen subset B_1 which is homeomorphic to X. Let $C_1=A_1-B_1$ and let $B_2=X-C_1$. Then $A_1=B_1\oplus C_1$ and $X=C_1\oplus B_2$. Next, since A_2 contains a copy of X as a clopen subset, A_2 contains a copy B_2' of B_2 as a clopen subset. Then we can express that $A_2=B_2'\oplus C_2$ and $X=C_2\oplus B_3$. Continuing this procedure, we obtain a sequence $\{B_n, C_n : n\in\omega\}$ of pairs of clopen subsets of X which satisfy the following conditions:

$$A_i\approx B_i\oplus C_i; \qquad X=C_i\oplus B_{i+1} \qquad (i=1, 2, 3, \cdots).$$

Then

$$\begin{aligned} A&\approx(B_1\oplus C_1)\oplus(B_2\oplus C_2)\oplus(B_3\oplus C_3)\oplus \cdots \\ &\approx B_1\oplus(C_1\oplus B_2)\oplus(C_2\oplus B_3)\oplus \cdots \\ &\approx X\oplus X\oplus X\oplus \cdots . \end{aligned}$$

The last space does not depend on A. Hence all nonempty clopen subspaces of X are homeomorphic.

3. Product spaces and h-homogeneity

Motorov [7] showed some interesting results about the h-homogeneity of products of first countable compact spaces. Here we will show the similar results for non-pseudocompact spaces.

Theorem 3.1. *Let Y be a non-pseudocompact zero-dimensional space. Let κ be an arbitrary cardinal and let $X=Y^{\kappa}$. If Y has a π-base $\mathcal{B}$ consisting of clopen subsets U which satisfy $U\times X\approx X$, then X is h-homogeneous.*

Proof. Let $\mathcal{C}$ be the family of canonical open subsets V of X which are defined as follows:

$$V=\prod\{V_{\alpha}:\alpha\in\kappa\}.$$

where $V_{\alpha}=X$ or $V_{\alpha}\in\mathcal{B}$ for each α. Then it is obvious that $\mathcal{C}$ is a π-base of X consisting of clopen subsets which are homeomorphic to X. Hence it follows that X is h-homogeneous from Theorem 2.4.

Corollary 3.2. *Let Y be a non-pseudocompact zero-dimensional space. If the set of all isolated points of Y is dense in Y, then Y^{κ} is h-homogeneous for any infinite cardinal κ.*

It is trivial that any product of homogeneous spaces is homogeneous. But the author does not know whether every product of h-homogeneous spaces is h-homogeneous. However we have the following.

Theorem 3.3. *Let $\{Y_{\lambda}:\lambda\in\Lambda\}$ be a family of h-homogeneous spaces and let $X=\prod\{Y_{\lambda}:\lambda\in\Lambda\}$. If X is compact or non-pseudocompact, then X is h-homogeneous.*

Proof. A canonical open subset U of X will be called a canonical clopen subset, when every factor of U is clopen. If X is non-pseudocompact, then X is h-homogeneous by Theorem 2.4, since every canonical clopen subset of X is homeomorphic to X. In case X is compact, every nonempty clopen subset of X can be expressed as a finite disjoint union of canonical clopen subsets. Since we can suppose that the cardinality of at least one space Y_{λ} is infinite, every discrete union of finite copies of X is homeomorphic to X. It follows that X is h-homogeneous.

Motorov (see [1]) suggested that for every first countable zero-dimensional compact space X, the ω-power X^{ω} is homogeneous. And it is not known whether there is a first countable zero-dimensional space whose ω-power is not homogeneous [4]. Similary the following question is natural.

Question. Is there a first countable zero-dimensional space whose ω-power is not h-homogeneous?

We have the following, when the first countability is omitted.

Theorem 3.4. *Let X be a space such that $\chi(x, X) \geqq \omega_1$ for any point $x \in X$. Then $(X \oplus \{p\})^\omega$ is neither h-homogeneous nor homogeneous.*

Proof. Let $Y = (X \oplus \{p\})^\omega$. Then Y contains a point which has a countable neighborhood base. However Y contains a clopen subset which is homeomorphic to $X \times Y$. And no point of $X \times Y$ has a countable neighborhood base.

Let D be the discrete space consisting of two points. Then D^{ω_1} is a zero-dimensional compact space such that $\chi(x, D^{\omega_1}) \geqq \omega_1$ for any $x \in D^{\omega_1}$. Hence there is a (compact) zero-dimensional space whose ω-power is not h-homogeneous.

Remark. Let $\mathcal{P}$ be a class of zero-dimensional spaces. If $\mathcal{P}$ is clopen hereditary and closed under finite discrete unions, then the following assertions are equivalent:

(1) For every space X in $\mathcal{P}$, the ω-power X^ω is h-homogeneous.

(2) For any spaces A, B in $\mathcal{P}$, the ω-power $(A \oplus B)^\omega$ is homeomorphic to $A \times (A \oplus B)^\omega$.

4. Stone-Čech compactifications and h-homogeneity

As noted in Introduction, a compact zero-dimensional space X is h-homogeneous if and only if the Boolean algebra of all clopen subsets is homogeneous in the sense of Boolean algebra. Let X be a zero-dimensional space which is not compact. Then, since the h-homogeneity of X implies obviously the h-homogeneity of its Stone-Čech compactification βX and the Boolean algebra of all clopen subsets of X is isomorphic to the Boolean algebra of all clopen subsets of βX, the h-homogeneity of X implies the homogeneity of the Boolean algebra of all clopen subsets of X. However the converse of this assertion is not generally true. We will use some basic properties concerning the Stone-Čech compactifications (see [9]) in this section.

Theorem 4.1. *If X is a strongly zero-dimensional h-homogeneous space, then βX is h-homogeneous, Let X be a first countable zero-dimensional space. If βX is h-homogeneous, then X is h-homogeneous.*

Proof. Since the first assertion is obvious, we will show the second assertion. Let A be a nonempty clopen subset of X. Then $\mathrm{cl}_{\beta X} A$ is clopen in βX and homeomorphic to βA. Hence, from the h-homogeneity of βX, it follows that βA is homeomorphic to βX. Since any point of $\beta X - X$ does not have a countable neighborhood base in βX and any point of $\beta A - A$ does not have a countable neighborhood base in βA, the spaces A and X must be homeomorphic.

Next, we will study the h-homogeneity of remainders of Stone-Čech compactifications. It is well known that the remainder $\beta\omega - \omega$ of the Stone-Čech compactification of ω is h-homogeneous. A space X is called a space of countable type if every compact subset is included in a compact subset with a countable neighborhood base. For a closed subset F of a space X, the quotient space of X obtained by collapsing F to one point is expressed by X/F.

Theorem 4.2. *Let X be a strongly zero-dimensional space of countable type. Suppose that, for every non-compact regular closed subset A with the compact boundary $\mathrm{Bd}_X A$, there is a compact subset F of X containing $\mathrm{Bd}_X A$ such that $(A \cup F)/F$ and X/F are homeomorphic. Then $\beta X - X$ is h-homogeneous.*

Proof. Let U be an arbitrary nonempty clopen subset of $\beta X - X$. We can assume that the complement $V = \beta X - X - U$ is not empty. Then $\mathrm{cl}_{\beta X} U \cap \mathrm{cl}_{\beta X} V$ is a compact subset of X. There is a compact subset C with a countable neighborhood base in X such that $\mathrm{cl}_{\beta X} U \cap \mathrm{cl}_{\beta X} V \subset C$. Then there is a real-valued continuous function f on $\mathrm{cl}_{\beta X}(\beta X - X) \cup C$ such that $f(U) \subset (0, \infty)$ and $f(V) \subset (-\infty, 0)$. Let

$$A = \mathrm{cl}_X(\beta f^{-1}((0, \infty)) \cap X),$$

where βf is the real-valued continuous extension of f to βX. Then A is a regular closed subset of X whose boundary is compact. By the assumption, $\mathrm{Bd}_X A$ is contained in a compact subset F of X such that $(A \cup F)/F$ and X/F are homeomorphic. Since $A \cup F$ is C^*-embedded in X and $U = \mathrm{cl}_{\beta X}(A \cup F) - (A \cup F)$, the clopen set U is homeomorphic to the remainder of the Stone-Čech compactification of $(A \cup F)/F$. Since $\beta X - X$ is homeomorphic to the remainder of the Stone-Čech compactification of X/F, the clopen set U is homeomorphic to $\beta X - X$.

Corollary 4.3. *If X is a strongly zero-dimensional h-homogeneous space of countable type, then $\beta X - X$ is h-homogeneous.*

Proof. Let A be a non-compact regular closed subset of X with the compact boundary $\mathrm{Bd}_X A$. Then $\mathrm{Bd}_X A$ is included in a compact subset F with a countable neighborhood base. By Lemma 2.2, it is obvious that $(A \cup F)/F$ and X/F are homeomorphic.

References

[1] A.V. Arhangel'skii, Topological homogeneity, topological groups and their continuous images, *Russian Math. Surveys*, **42** (1987), 83-131.

[2] B. Balcar and A. Dow, Dynamical systems on compact extremally disconnected spaces, *Topology and its Appl.*, **41** (1991), 41-56.

[3] E.K. van Douwen, A compact space with a measure that knows which sets are homeomorphic, *Advances in Math.*, **52** (1984), 1-33.

[4] G. Gruenhage and Zhou Hao-xuan, Homogeneity of X^{ω}, preprint.

[5] S. Koppelberg, *Handbook of Boolean algebras*, vol. 1, North-Holland, 1989.

[6] D.B. Motorov, Homogeneity and π-networks, *Vestnik Moskov. Univ. Mat.*, **44** (1989), 31-34.

[7] ———, Zero-dimensional and linearly ordered bicompacta: properties of homogeneity type, *Russian Math. Surveys*, **44** (1989), 190-191.

[8] A.V. Ostrovskii, Continuous images of the Cantor product $C\times Q$ of a perfect set C and the rational numbers Q, *Seminar on General Topology, Moskov. Gos. Univ.*, Moscow, 1981, 78-85.

[9] R.C. Walker, *The Stone-Čech compactification*, Springer-Verlag, 1974.

Department of Mathematics
Faculty of Engineering
Yokohama National University
156, Tokiwadai, Hodogaya
Yokohama, 240
Japan