YOKOHAMA MATHEMATICAL
JOURNAL VOL. 40, 1992

A VIABILITY RESULT FOR NONLINEAR TIME DEPENDENT EVOLUTION INCLUSIONS

By

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(Received June 3, 1991; Revised November 25, 1991)

Abstract. In this paper, we establish the existence of viable solutions for a class of nonlinear time dependent evolution equations. Our proof uses Galerkin approximations and a recent finite dimensional result of Deimling [6]. Then we use our viability theorem to prove the existence of admissible trajectories, for a class of nonlinear, time-varying, feedback control systems. An example of a nonlinear parabolic control system is also worked out in detail.

1. Introduction

In a recent paper, Averginos-Papageorgiou [3], established the existence of viable solutions for a class of nonlinear, time invariant evolution inclusions. Their approach followed that of Shuzhong [12], who examined semilinear, time invariant evolution inclusions using Galerkin approximations.

In this paper, using a very recent viability result due to Deimling [6], we extend the above mentioned works to nonlinear, time varying evolution inclusions. Furthermore, our hypotheses are weaker than those in the other nonlinear work [3]. Then we use the viability result to establish the existence of trajectories for a class of time-varying, infinite dimensional feedback control systems, with state constraints. Finally, an example of a distributed parameter control system is worked out in detail.

2. Preliminaries

Let (Ω, Σ) be a measurable space and X a separable Banach space. By $P_{f(c)}(X)$ we will denote the family of nonempty, closed, (convex) subsets of X. A multifunction (set-valued function) F: $\Omega \rightarrow P_f(X)$ is said to be measurable, if for all $x \in X$, $\omega \rightarrow d(x, F(\omega)) = \inf\{\|x-z\| : z \in F(\omega)\}$ is measurable. Other equivalent definitions of measurability of a $P_f(X)$-valued multifunction, can be found

1991 Mathematics Subject Classification: 34G20.
Key words and phrases: evolution triple, compact embedding, viable solution, Nagumo condition, Galerkin approximation, feedback control system.

in the survey paper of Wagner [13]. A multifunction $G: \Omega \to 2^X \setminus \{\emptyset\}$ is said to be graph measurable, if $GrG=\{(\omega, x) \in \Omega \times X: x \in G(\omega)\} \in \Sigma \times B(X)$, with $B(X)$ being the Borel σ-field of X. For $P_f(X)$-valued multifunctions, measurability implies graph measurability and the converse holds, if there is a σ-finite complete measure $\mu(\cdot)$ on Σ.

Let Y, Z be Hausdorff topological spaces. A multifunction $G: Y \to 2^Z \setminus \{\emptyset\}$ is said to be closed if $GrG=\{(y, z) \in Y \times Z: z \in G(y)\}$ is closed in $Y \times Z$. Note that such a multifunction necessarily has closed values. We will say that $G(\cdot)$ is upper semicontinuous ($u.s.c.$), if for all $C \subseteq Z$ nonempty, closed, $G^-(C)=\{y \in Y: G(y) \cap C \neq \emptyset\}$ is closed in Y. If $G(\cdot)$ is an $u.s.c.$ multifunction with closed values and Z is regular, then $G(\cdot)$ is closed. The converse is true if $\overline{G(Y)}$ is compact. For details we refer to [9] (Theorems 7.1.15 and 7.1.16, p. 78).

Our mathematical setting is the following: Let $T=[0, b]$ and H a separable Hilbert space. Let X be a dense subspace of H carrying the structure of a separable Hilbert space which embeds into H continuously. We will also assume that the embedding is compact. Identifying H with its dual (pivot space), we have $X \to H \to X^*$, with all embeddings being continuous, dense (see [15], p. 416). Such a triple of spaces, is known in the literature as "evolution triple" or "Gelfand triple". By $\|\cdot\|$ (resp. $|\cdot|$, $\|\cdot\|_*$), we will denote the norm of X (resp. of H, X^*). By $\langle \cdot, \cdot \rangle$ we will denote the duality brackets for the pair (X, X^*) and by $(\cdot, \cdot)$ the inner product in H. The two are compatible in the sense that $\langle \cdot, \cdot \rangle|_{X \times H}=(\cdot, \cdot)$. Also by $(\cdot, \cdot)_X$ and by $(\cdot, \cdot)_{X^*}$, we will denote the inner products of X and X^* respectively.

Let $J \in \mathcal{L}(X, X^*)$ be the duality operator (canonical isometry) from X into X^*. Then from the Riesz-Fredholm theorem (see [1], p. 236), we know that there exists a sequence $\{e_n\}_{n \geq 0} \subseteq X$ of eigenvectors of J^{-1} with corresponding eigenvalues $\{\lambda_n\}_{n \geq 0} \subseteq \boldsymbol{R}_+$ s.t.

(1) $e_n=\lambda_n J e_n$, $n=0, 1, 2, \cdots$, where $\lambda_n \downarrow 0$,

(2) $(e_n, e_m)=\delta_{nm}$, $(e_n, e_m)_X=\lambda_n^{-1}\delta_{nm}$ and $(e_n, e_m)_{X^*}=\lambda_n \delta_{nm}$, with δ_{nm} being the Kronecker symbol, $n, m=0, 1, 2, \cdots$,

(3) for every $x^* \in X^*$ we have $x^*=\sum_{k=0}^{\infty} \langle x^*, e_k \rangle e_k$ and in addition

$$X=\left\{x^* \in X^*: \sum_{k=0}^{\infty} \lambda_k^{-1} |\langle x^*, e_k \rangle|^2 < \infty\right\} \quad \text{with} \quad \|x^*\|=\sqrt{\sum_{k=0}^{\infty} \lambda_k^{-1} |\langle x^*, e_k \rangle|^2},$$

$$H=\left\{x^* \in X^*: \sum_{k=0}^{\infty} |\langle x^*, e_k \rangle|^2 < \infty\right\} \quad \text{with} \quad |x^*|=\sqrt{\sum_{k=0}^{\infty} |\langle x^*, e_k \rangle|^2}$$

and

$$X^*=\left\{x^* \in X^*: \sum_{k=0}^{\infty} \lambda_k |\langle x^*, e_k \rangle|^2 < \infty\right\} \quad \text{with} \quad \|x^*\|_*=\sqrt{\sum_{k=0}^{\infty} \lambda_k |\langle x^*, e_k \rangle|^2}.$$

Set $X_n=\operatorname{span}\{e_k\}_{k=0}^{n}$. Then $\{X_n\}_{n \geq 1}$ is a Galerkin scheme for each one of the spaces X, H, X^*; i.e. this is a sequence of nonzero, finite dimensional subspaces of X s.t. $d(x, X_n)\to 0$ as $n\to\infty$ for all $x\in X$. Similarly for H and X^*. For every $x^*\in X^*$ set

$$p_n(x^*)=\sum_{k=0}^{n}\langle x^*, e_k\rangle e_k .$$

Then $p_n(x^*)\in X_n$ and $p_n(\cdot)$ is the projection from X^* (also from H, X) onto X_n.

Let $W(T)=\{x\in L^2(X): \dot{x}\in L^2(X^*)\}$ (in this definition the derivative of $x(\cdot)$ is understood in the sense of vectorial distributions). Furnished with the inner product $(x, y)_{W(T)}=\int_0^b(x(t), y(t))_X dt+\int_0^b(\dot{x}(t), \dot{y}(t))_{X^*}dt$, $W(T)$ becomes a Hilbert space (see [14], Theorem 25.4). It is well-known (see for example, [14], Theorem 25.5 or [15], Proposition 23.23), that $W(T)\to C(T, H)$ continuously. So every function in $W(T)$, after possible modification on a Lebesgue null set in T, equals a continuous function from T into H. Furthermore, since we have assumed that $X\to H$ compactly, we have that $W(T)\to L^2(H)$ compactly (see [15], p. 450).

3. Viability theorem

Let $T=[0, b]$ and (X, H, X^*) an evolution triple as in Section 2. The problem under consideration is the following:

$$\left|\begin{array}{l} \dot{x}(t)+A(t, x(t))\in F(t, x(t)) \quad \text{a.e.}, \\ x(0)=x_0\in K\subseteq H, \\ x(t)\in K \quad \text{for all} \quad t\in T . \end{array}\right| \tag{$*$}$$

We will need the following hypotheses on the data of $(*)$.

$H(A)$: $A: T\times X\to X^*$ is an operator s.t.

(1) $t\to A(t, x)$ is measurable,

(2) $x\to A(t, x)$ is monotone, hemicontinuous (i.e. for all $x, y\in X$, we have $\langle A(t, x)-A(t, y), x-y\rangle\geqq 0$ (monotonicity) and for all $x, y, z\in X$, $\lambda\to\langle A(t, x+\lambda y), z\rangle$ is continuous on $[0, 1]$ (hemicontinuity)),

(3) $\langle A(t, x), x\rangle\geqq c_1\|x\|^2$ a.e. with $c_1>0$,

(4) $\|A(t, x)\|_*\leqq c_2(1+\|x\|)$ a.e. with $c_2>0$.

$H(K)$: K is a nonempty, closed subset of H s.t. $K\cap X_n=p_n(K)$ is compact for each $n\geqq 1$.

$H(F)$: F: $T\times H\to P_f(H)$ is a multifunction s.t.

(1) $(t, x)\to F(t, x)$ is graph measurable,

(2) $x\to F(t, x)$ is sequentially closed in $H\times H_w$ (here H_w denotes the Hilbert space H endowed with the weak topology),

(3) $|F(t, x)|=\sup\{|y| : y\in F(t, x)\}\leqq a(t)+c|x|$ a.e. with $a(\cdot)\in L^2_+$, $c>0$.

Remark. Note that hypothesis $H(A)(2)$ above is weaker than hypothesis $H(A)(1)$ in [3], where the time independent operator A was assumed to be weakly sequentially continuous in x. Also hypothesis $H(F)$ above relaxes the corresponding hypothesis $H(F)$ in [3].

Let $x\in K$. By $T'_K(x)$ we will denote the Bouligand tangent cone to K at x defined in the space X^*; i.e. $v\in T'_K(x)$ if and only if $\varliminf_{\lambda\downarrow 0} d_*(x+\lambda v, K)/\lambda=0$, where $d_*(z, K)=\inf\{\|z-y\|_*: y\in K\}$. When K is convex, then $T'_K(x)=cl[\cup_{\lambda>0}(K-x)/\lambda]$ and is simply known as the tangent cone to K at x. For further details, we refer to [2], p. 407. We will also need the following Nagumo type condition:

H_τ: for every $x\in K\cap X$, $[F(t, x)-A(t, x)]\cap T'_K(x)\neq\emptyset$ for all $t\in T$.

Theorem 3.1. *If hypotheses $H(A)$, $H(K)$, $H(F)$ and H_τ hold, then problem $(*)$ admits a solution $x(\cdot)\in W(T)\to C(T, H)$.*

Proof. Consider the Galerkin approximations for problem $(*)$:

$$\left|\begin{array}{l} \dot{x}_n(t)+p_nA(t, x_n(t))\in p_nF(t, x_n(t)) \quad \text{a.e.}, \\ x_n(0)=p_n(x_0)\in K_n=p_n(K), \\ x(t)\in K_n \quad \text{for all} \quad t\in T. \end{array}\right| \qquad (*)_n$$

Since by hypothesis $H(A)(2)$, $A(t, \cdot)$ is monotone and hemicontinuous, it is demicontinuous (see [15], Prop. 26.4 (c)). So if $x_m\xrightarrow{s}x$ in X, then $A(t, x_m)\xrightarrow{w}A(t, w)$ in X^*. Hence for every $n\geqq 1$, we have $p_nA(t, x_m)\to p_nA(t, x)$ in X_n as $m\to\infty$. Also let $C\subseteq X_n$ be nonempty, closed and set $p_nF_t^-(C)=\{z\in X_n: p_nF(t, z)\cap C\neq\emptyset\}$. Let $\{z_m\}_{m\geq 1}\subseteq p_nF_t^-(C)$ and assume that $z_m\to z$ in X_n. By definition $p_nF(t, z_m)\cap C\neq\emptyset$ for every $m\geqq 1$. Choose $v_m\in p_nF(t, z_m)\cap C$, $m\geqq 1$. Then $\{v_m\}_{m\geq 1}$ is bounded in X_n, thus relatively compact and so by passing to a subsequence if necessary, we may assume that $v_m\to v$ as $m\to\infty$. Clearly $v\in C$ and for every $m\geqq 1$, $[z_m, v_m]\in Gr\,p_nF(t, \cdot)$ and $[z_m, v_m]\to[z, v]$ in X_n as $m\to\infty$. Because of hypothesis $H(F)(2)$, $[z, v]\in Gr\,p_nF(t, \cdot)$, while $[z, v]\in X_n\times X_n$. Hence $[z, v]\in Gr\,p_nF(t, \cdot)\Rightarrow v\in p_nF(t, z)\Rightarrow z\in p_nF_t^-(C)\Rightarrow p_nF_t^-(C)$ is closed $\Rightarrow x\to p_nF(t, x)$ is *u.s.c.* on X_n, hence $x\to p_nF(t, x)-p_nA(t, x)$ is *u.s.c.* on X_n. Furthermore, it is clear from hypotheses $H(A)(1)$ and $H(F)(1)$ that $t\to p_nF(t, x)-p_nA(t, x)$ is measurable. From hypothesis H_τ, we know that for all $(t, x)\in T\times K$, we have

$$[F(t, x)-A(t, x)]\cap T'_K(x)\neq\emptyset\,.$$

Furthermore, from Aubin-Ekeland [2], p. 440, we have $p_n T'_K(x) \subseteqq T_{p_n(K)}(p_n(x)) = T_{K_n}(p_n(x))$ and so we get

$$\begin{aligned}\emptyset \neq p_n[(F(t,\ x)-A(t,\ x)) \cap T'_K(x)] \\ \subseteqq [p_n(F(t,\ x)-A(t,\ x))] \cap p_n T'_K(x) \\ \subseteqq [p_n F(t,\ x)-p_n A(t,\ x)] \cap T_{K_n}(p_n(x)).\end{aligned}$$

Hence for every $(t,\ x) \in T \times X_n$, we have $[p_n F(t,\ x)-p_n A(t,\ x)] \cap T_{K_n}(x) \neq \emptyset$. Finally note that $|p_n F(t,\ x)| = \sup\{\|y\|_{X_n} :\ y \in F(t,\ x)\} \leqq |p_n|_{\mathcal{L}} a(t) + |p_n|_{\mathcal{L}} c|x|$ a.e.. Thus, we have verified all hypotheses of Theorem 1, p. 639 in [6]. Applying that result, we get a solution $x_n(\cdot)$ for the Galerkin approximation problem $(*)_n$. Then we have:

$$\langle \dot{x}_n(t),\ x_n(t)\rangle + \langle p_n A(t,\ x_n(t)),\ x_n(t)\rangle = (p_n f_n(t),\ x_n(t)) \quad \text{a.e.},$$

where $f_n \in L^2(H)$, $f_n(t) \in F(t,\ x_n(t))$ a.e.. Recall that $x_n(t) \in K_n$ and that since p_n is the projection of X^* onto X_n, p_n^* projects X onto X_n (see [7], VI.3.3 and VI.9.19 or [5], p. 258). So we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|x_n(t)|^2 + \langle A(t,\ x_n(t)),\ p_n^* x_n(t)\rangle = (p_n f_n(t),\ x_n(t)) = \langle f_n(t),\ p_n^* x_n(t)\rangle \quad \text{a.e.} \\ &\Longrightarrow \frac{d}{dt}|x_n(t)|^2 + 2\langle A(t,\ x_n(t)),\ x_n(t)\rangle = (f_n(t),\ x_n(t)) \quad \text{a.e.} \\ &\Longrightarrow \frac{d}{dt}|x_n(t)|^2 + 2c_1\|x_n(t)\|^2 \leqq 2|f_n(t)| \cdot |x_n(t)| \quad \text{a.e.} \qquad (1)\end{aligned}$$

Applying Cauchy's inequality with $\varepsilon > 0$ on the right-hand side, we get

$$2|f_n(t)| \cdot |x_n(t)| \leqq 2\beta|f_n(t)| \cdot \|x_n(t)\| \leqq \varepsilon\beta^2|f_n(t)|^2 + \varepsilon^{-1}\|x_n(t)\|^2 \quad \text{a.e.}$$

where $\beta > 0$ is such that $|\cdot| \leqq \beta\|\cdot\|$. It exists since we have assumed that X embeds into H continuously. Choose $\varepsilon > 0$ so that $\varepsilon^{-1} = 2c_1 \Rightarrow (2c_1)^{-1} = \varepsilon$. Then we have

$$\frac{d}{dt}|x_n(t)|^2 \leqq c_3|f_n(t)|^2 \quad \text{with} \quad c_3 = \beta^2/(2c_1),$$

$$\frac{d}{dt}|x_n(t)|^2 \leqq c_3(a(t)+c|x_n(t)|)^2 \leqq 2c_3 a(t)^2 + 2c_3 c^2|x_n(t)|^2 \quad \text{a.e.}$$

$$\Longrightarrow |x_n(t)|^2 \leqq |x_0|^2 + 2c_3\|a\|_2^2 + 2c_3 c^2 \int_0^t |x_n(s)|^2\ ds.$$

Invoking Gronwall's inequality, we deduce that there exists $M_1 > 0$ s.t. for all $t \in T$ and all $n \geqq 1$, we have

$$|x_n(t)| \leqq M_1.$$

Using this bound in inequality (1) above, we get that

$$\frac{d}{dt}|x_n(t)|^2+2c_1\|x_n(t)\|^2\leqq 2|f_n(t)|\cdot M_1 \quad \text{a.e.}$$

$$\Longrightarrow 2c_1\int_0^b\|x_n(t)\|^2dt\leqq|x_0|^2+2M_1\int_0^b(a(t)+cM_1)dt.$$

Thus, there exists $M_2>0$ s.t. for all $n\geqq 1$, we have:

$$\|x_n\|_{L^2(X)}\leqq M_2. \tag{2}$$

Finally, let $h\in L^2(X)=L^2(X^*)^*$. For every $n\geqq 1$, we have:

$$\int_0^b\langle\dot{x}_n(s),\ h(s)\rangle ds=\int_0^b\langle -p_nA(s,\ x_n(s)),\ h(s)\rangle ds+\int_0^b(p_nf_n(s),\ h(s))ds$$

$$\leqq\int_0^b M_3\|A(s,\ x_n(s))\|_*\|h(s)\|ds+\int_0^b M_3|f_n(s)|\beta\|h(s)\|ds$$

where $M_3>0$ is such that $\|p_n\|_{\mathcal{L}}\leqq M_3$ for all $n\geqq 1$ (see [5]). So we have:

$$\int_0^b\langle\dot{x}_n(s),\ h(s)\rangle ds\leqq\int_0^b[c_2(1+\|x_n(s)\|)+(a(s)+cM_1)]\|h(s)\|ds.$$

Since $a(\cdot)\in L_+^2$, $\|x_n(\cdot)\|\in L_+^2$ and $\|h(\cdot)\|\in L_+^2$, applying the Cauchy-Schwartz inequality and recalling that for all $n\geqq 1$ $\|x_n\|_{L^2(X)}\leqq M_2$, we get that there exists $M_4>0$ s.t.

$$((\dot{x}_n,\ h))_0=\int_0^b\langle\dot{x}_n(t),\ h(t)\rangle dt\leqq M_4\|h\|_{L^2(X)},$$

where $((\cdot,\ \cdot))_0$ denotes the duality brackets for pair $(L^2(X),\ L^2(X^*))$. Since $h\in L^2(X)$ was arbitrary, we deduce that for all $n\geqq 1$, we have

$$\|\dot{x}_n\|_{L^2(X^*)}\leqq M_4. \tag{3}$$

From (2) and (3) above, we deduce that $\{x_n\}_{n\geq 1}$ is bounded in $W(T)$. Recall (see Section 2) that $W(T)\to L^2(H)$ compactly. So by passing to a subsequence if necessary, we may assume that $x_n\xrightarrow{s}\hat{x}$ in $L^2(H)$ and since $W(T)\to C(T,\ H)$ continuously (see Section 2), we may assume that $x_n(b)\xrightarrow{s}\hat{x}(b)$ in H. Let $\hat{A}$: $L^2(X)\to L^2(X^*)$ be the Nemitsky operator corresponding to the map $A(t,\ x)$; i.e. $\hat{A}(x)(t)=A(t,\ x(t))$ for all $x\in L^2(X)$. Because of hypothesis $H(A)(4)$, $\{\hat{A}(x_n)\}_{n\geq 1}$ is bounded in the Hilbert space $L^2(X^*)$. So by passing to a subsequence if necessary, we may assume that $\hat{A}(x_n)\xrightarrow{w}v$ in $L^2(X^*)$. Let $\hat{p}_n$: $L^2(X^*)\to L^2(X_n)$ be the lifting of the projection operator p_n: $X^*\to X_n$; i.e. $(\hat{p}_nx)(t)=p_nx(t)$. Recalling that by $((\cdot,\ \cdot))_0$ we denote the duality brackets for the pair $(L^2(X),\ L^2(X^*))$, we have for every $n\geqq 1$:

$$((\dot{x}_n,\ x_n-\hat{x}))_0+((\hat{p}_n\hat{A}(x_n),\ x_n-\hat{x}))_0=((\hat{p}_nf_n,\ x_n-\hat{x}))_0. \tag{4}$$

From the integration by parts formula for functions in $W(T)$ (see [15], Prop. 23.23), we have:

$$((\dot{x}_n-\dot{\hat{x}},\ x_n-\hat{x}))_0=(1/2)|x_n(b)-\hat{x}(b)|^2$$

$$\Longrightarrow ((\dot{x}_n,\ x_n-\hat{x}))_0=(1/2)|x_n(b)-\hat{x}(b)|^2+((\dot{\hat{x}},\ x_n-\hat{x}))_0 \longrightarrow 0 \text{ as } n\to\infty .$$

Also because $\|x_n\|_{C(T,H)}\leqq M_1$ for all $n\geqq 1$ and using hypothesis $H(F)(3)$, we have that $\{f_n\}_{n\geq 1}$ is relatively weakly compact in $L^2(H)$. So we may assume that $f_n\to f$ in $L^2(H)$. Hence we have:

$$\begin{aligned}((\hat{p}_n f_n,\ x_n-\hat{x}))_0&=((f_n,\ x_n-\hat{p}_n^*\hat{x}))_0=(f_n,\ x_n-\hat{p}_n^*\hat{x})_{L^2(H)}\\ &=(f_n,\ x_n-\hat{x})_{L^2(H)}+(f_n,\ \hat{x}-\hat{p}_n^*\hat{x})_{L^2(H)}\\ &=(f_n,\ x_n-\hat{x})_{L^2(H)}+\int_0^b(f_n(t),\ \hat{x}(t)-p_n^*\hat{x}(t))dt .\end{aligned}$$

But $(f_n,\ x_n-\hat{x})_{L^2(H)}\to 0$ and $|\hat{x}(t)\to p_n^*\hat{x}(t)|\to 0$ (see [5], p. 258) as $n\to\infty$. Therefore we have

$$((\hat{p}_n f_n,\ x_n-\hat{x}))_0 \longrightarrow 0 \quad \text{as} \quad n\to\infty .$$

So from (4) above, we have that

$$\lim((\hat{p}_n\hat{A}(x_n),\ x_n-x))_0=0 .$$

Then we have:

$$\begin{aligned}((\hat{A}(x_n),\ x_n-\hat{x}))_0&=((\hat{A}(x_n),\ x_n-\hat{x}+\hat{p}_n^*\hat{x}-\hat{p}_n^*\hat{x}))_0\\ &=((\hat{A}(x_n),\ \hat{p}_n^*\hat{x}-\hat{x}))_0+((\hat{A}(x_n),\ x_n-\hat{p}_n^*\hat{x}))_0\\ &=((\hat{A}(x_n),\ \hat{p}_n^*\hat{x}-\hat{x}))_0+((\hat{p}_n\hat{A}(x_n),\ x_n-\hat{x}))_0 .\end{aligned}$$

But recall that $\lim((\hat{p}_n\hat{A}(x_n),\ x_n-\hat{x}))_0=0$, $\{\hat{A}(x_n)\}_{n\geq 1}$ is bounded and $\|p_n^*\hat{x}(t)-\hat{x}(t)\|\to 0$ as $n\to\infty\Rightarrow\|\hat{p}_n^*\hat{x}-\hat{x}\|_{L^2(X)}\to 0$. Thus we get

$$\lim((\hat{A}(x_n),\ x_n-\hat{x}))_0=0 .$$

Because of hypothesis $H(A)$, $\hat{A}(\cdot)$ is hemicontinuous, monotone, hence it has property (M) (see [15], 583-584). So $\hat{A}(x_n)\xrightarrow{w}\hat{A}(\hat{x})$ in $L^2(X^*)$. Then for every $h\in L^2(X)$, we have $((\hat{p}_n\hat{A}(x_n),\ h))_0=((\hat{A}(x_n),\ \hat{p}_n^*h))_0$. Since $\hat{A}(x_n)\xrightarrow{w}\hat{A}(\hat{x})$ in $L^2(X^*)$ and $\hat{p}_n^*h\xrightarrow{s}h$ in $L^2(X)$, we get that $((\hat{p}_n\hat{A}(x_n),\ h))_0\to((\hat{A}(\hat{x}),\ h))_0$. Because $h\in L^2(X)$ was arbitrary, we deduce that $\hat{p}_n\hat{A}(x_n)\xrightarrow{w}\hat{A}(\hat{x})$ in $L^2(X^*)$. Therefore

$$\dot{x}_n+\hat{p}_n\hat{A}(x_n)\xrightarrow{w} x+\hat{A}(x) \quad \text{in} \quad L^2(X^*)$$

and $f_n\xrightarrow{w}f$ in $L^2(H)$, hence in $L^2(X^*)$ too.

So in the limit as $n \to \infty$, we get that

$$x(t)+A(t, x(t))=f(t) \quad \text{a.e..}$$

Also since $x_n \xrightarrow{s} x$ in $L^2(H)$, by passing to a subsequence if necessary, we may assume that $x_n(t) \xrightarrow{s} x(t)$ a.e.. Then applying Theorem 3.1 of [11], we get

$$f(t) \subseteq \overline{conv}\, w\text{-}\overline{\lim}\{f_n(t)\}_{n\geq 1} \subseteq \overline{conv}\, w\text{-}\overline{\lim} F(t, x_n(t))$$

$$=\overline{conv}\{y \in H: y_{n_k} \to y,\ y_{n_k} \in F(t, x_{n_k}(t)),\ n_1 < n_2 < \cdots < n_k < \cdots\} \quad \text{a.e..}$$

On the other hand, using hypothesis $H(F)(2)$, and since $F(t, x)$ has closed, convex values, we can easily check that $\overline{conv}\, w\text{-}\overline{\lim} F(t, x_n(t)) \subseteq F(t, \hat{x}(t))$ a.e. $\Rightarrow$ $f(t) \in F(t, \hat{x}(t))$ a.e. $\Rightarrow \hat{x}(\cdot) \in W(T) \to C(T, H)$ is the desired viable trajectory of (*).

Q.E.D.

4. Control systems

In this section, we use Theorem 3.1 to establish the existence of trajectories for a class of infinite dimensional, nonlinear feedback control systems, with state constraints.

Let $T=[0, b]$ and (X, H, X^*) an evolution triple as in Section 3. The system under consideration is the following:

$$\left|\begin{array}{c} \dot{x}(t)+A(t, x(t))=f(t, x(t))u(t) \quad \text{a.e.}, \\ x(0)=x_0 \in K \subseteq H, \\ x(t) \in K \quad \text{for all } t \in T,\ u(t) \in U(t, x(t)) \quad \text{a.e.}, \\ u(\cdot)=\text{measurable}. \end{array}\right| \qquad (**)$$

We will need the following hypothesis on the control vector field $f(t, x)u$. Here Y is a separable Banach space, modelling the control space. Also by $\mathcal{L}(Y, H)$ we will denote the Banach space of bounded linear operators from Y into H.

H(f): $f: T \times K \to \mathcal{L}(Y, H)$ is a map s.t.

(1) $t \to f(t, x)u$ is measurable for all $(x, u) \in K \times Y$,

(2) $x \to f(t, x)^*$ is continuous from K into $\mathcal{L}(H, Y^*)$ with the strong operator topology,

(3) $\|f(t, x)\|_{\mathcal{L}} \leq a(t)+c|x|$ a.e. with $a(\cdot) \in L_+^2$, $c>0$.

Also we will assume the following about the control constraint multifunction. Denote by $P_{wkc}(Y)$ the family of nonempty, weakly compact and convex subsets of Y.

$\boldsymbol{H(U)}$: $U: T\times H\to P_{wkc}(Y)$ is a multifunction s.t.

(1) $(t, x)\to U(t, x)$ is graph measurable,

(2) $x\to U(t, x)$ is $u.s.c.$ from H into Y_w, where Y_w denotes the Banach space Y equipped with the weak topology,

(3) $U(t, x)\subseteq W$ a.e. with $W\in P_{wkc}(Y)$

$\boldsymbol{H'_\tau}$: for every $x\in K$, there exists $u\in U(t, x)$ s.t. $f(t, x)u-A(t, x)\in T'_K(x)$ a.e..

Theorem 4.1. *If hypotheses $H(A)$, $H(K)$, $H(f)$, $H(U)$ and H hold, then control system* (**) *admits a viable trajectory $x(\cdot)\in W(T)\to C(T, H)$.*

Proof. Let F: $T\times H\to P_{fc}(H)$ be defined by

$$F(t, x)=f(t, x)U(t, x)=\cup\{f(t, x)u : u\in U(t, x)\}$$

(the orientor field of the control system (**)).

Note that $F(\cdot, \cdot)$ has in fact weakly-compact and convex values in H. First we claim that $(t, x)\to F(t, x)$ is graph measurable. Note that $GrF=proj_{T\times H\times H}\{(t, x, y, u)\in T\times H\times H\times W: y=f(t, x)u, u\in U(t, x)\}$. Because of hypothesis $H(f)$, for every $v\in H$, the map $(t, x, u)\to(v, f(t, x)u)$ is measurable in t and continuous in $(x, u)\in H\times W_w$, where W_w denotes the relative weak topology on $W\subseteq Y$. Since W_w is a compact, metrizable space (see [7], Theorem 3, p. 434), we have that $(t, x, u)\to(u, f(t, x)u)$ is measurable. Since $v\in H$ was arbitrary, Prop. 2.1 of [4], tells us that $(t, x, u)\to f(t, x)u$ is measurable on $T\times H\times W_w$. Also because of hypothesis $H(U)$, $GrU\in B(T)\times B(H)\times B(W)$. But because Y is separable, $B(W)=B(W_w)$ (see for example [8], Cor, 2.4, p. 461). So $GrU\in B(T)\times B(H)\times B(W_w)$. Thus we finally have:

$$\{(t, x, y, u)\in T\times H\times H\times W: y=f(t, x)u, u\in U(t, x)\}$$
$$\in B(T)\times B(H)\times B(H)\times B(W_w).$$

Since W_w is compact, metrizable, we can apply Lemma 2.2, p. 435 of [10], to get that

$$proj_{T\times H\times H}\{(t, x, y, u)\in T\times H\times H\times W: y\in f(t, x)u, u\in U(t, x)\}$$
$$\in B(T)\times B(H)\times B(H)$$
$$\Longrightarrow GrF\in B(T)\times B(H)\times B(H)$$
$$\Longrightarrow F(\cdot, \cdot) \text{ is indeed graph measurable.}$$

Next we will show that $F(t, \cdot)$ is sequentially closed om $H\times H_w$. To this end let $[x_n, y_n]\in GrF(t, \cdot)$, $[x_n, y_n]\to[x, y]$ in $H\times H_w$. Then from the definition of the multifunction $F(\cdot, \cdot)$, we know that there exist $u_n\in U(t, x_n)n\geqq 1$ s.t. $y_n=f(t, x_n)u_n$. Because of hypothesis $H(U)$ and from Theorem 7.4.2 of

[9], we have that $\cup_{n\geq 1}U(t, x_n)$ is relatively w-compact on Y. So by passing to a subsequence if necessary, we may assume that $u_n \xrightarrow{w} u$. Then from hypothesis $H(f)(2)$, we get $f(t, x_n)u_n \xrightarrow{w} f(t, x)u$ in $H \Rightarrow y=f(t, x)u$. Also because of hypothesis $H(U)(2)$, we have $u \in U(t, x)$. Therefore $[x, y] \in GrF(t, \cdot) \Rightarrow GrF(t, \cdot)$ is sequentially closed in $H \times H_w$.

Finally note that $|F(t, x)|=\sup\{|y| : y \in F(t, x)\} \leq a(t)|W|+c|W||x|$ a.e., where $|W|=\sup\{\|u\|_Y : u \in W\}$.

Then consider the following multivalued viability problem:

$$\left|\begin{array}{c} \dot{x}(t)+A(t, x(t)) \in F(t, x(t)) \quad \text{a.e.,} \\ x(0)=x_0 \in K \subseteq H, \\ x(t) \in K,\ t \in T. \end{array}\right| \qquad (**)'$$

We have just checked that $(**)'$ satisfies the hypotheses of Theorem 3.1. So applying that result, we get a viable trajectory $x(\cdot) \in W(T) \to C(T, H)$. Then let

$$R(t)=\{u \in U(t, x(t)) : \dot{x}(t)+A(t, x(t))=f(t, x(t))u\}.$$

From the definition of $F(t, x(t))$, we know that $R(t) \neq \emptyset$ a.e., and by redefining it on a Lebesgue null subset of T., we can have $R(t) \neq \emptyset$ for all $t \in T$. Also

$$GrR=\{(t, u) \in T \times Y : \dot{x}(t)+A(t, x(t))=f(t, x(t))u\} \cap GrU(\cdot, x(\cdot)).$$

Note that the function $(t, u) \to \dot{x}(t)+A(t, x(t))-f(t, x(t))u$ is measurable in t, continuous in u, hence jointly measurable. Thus $\{(t, u) \in T \times Y : \dot{x}(t)+A(t, x(t)) =f(t, x(t))u\} \in B(T) \times B(Y)$. Also from hypothesis $H(U)$, $GrU(\cdot, x(\cdot)) \in B(T) \times B(Y)$. Therefore $GrR \in B(T) \times B(Y)$. Apply Aumann's selection theorem (see [13], Theorem 5.10), to get $u : T \to Y$ measurable s.t. $u(t) \in R(t)$ for all $t \in T$. Then

$$\left|\begin{array}{c} \dot{x}(t)+A(t, x(t))=f(t, x(t))u(t) \quad \text{a.e.,} \\ x(0)=x_0 \in K \subseteq H, \\ x(t) \in K,\ t \in T,\ u(t) \in U(t, x(t)) \quad \text{a.e.,} \\ u(\cdot)=\text{measurable}. \end{array}\right|$$

Thus $x(\cdot) \in W(T) \to C(T, H)$ is the desired viable trajectory of $(**)$.

Q.E.D.

5. An example

In this section, we present an example of a parabolic distributed parameter control system, illustrating the applicability of our work.

So let $T=[0, b]$ and Z a bounded domain in $\boldsymbol{R}^N$, with boundary $\partial Z=\Gamma$. Let $D_i=\partial/\partial z_i$, $i=1, \cdots, N$ and $D^\alpha=D_1^{\alpha_1}\cdots D_N^{\alpha_N}$, where $\alpha=(\alpha_1, \cdots, \alpha_N)$ are N-tuples of nonnegative integers (multiindex) and $|\alpha|=\sum_{k=1}^N \alpha_k$ is the length of the multi-index. Also let $N_m=(N+m)!/N!\,m!$ and let $\eta(x)=\{D^\alpha x:\ |\alpha|\leqq m\}$. The problem under consideration is the following:

$$\left|\begin{array}{l} \dfrac{\partial x(t, z)}{\partial t}+\sum\limits_{|\alpha|\leq m}(-1)^{|\alpha|}D^\alpha A_\alpha(t, z, \eta(x(t, z)))=B(t, z, x(t, z))u(t, z) \ on\ T\times Z, \\ D^\beta x|_{T\times\Gamma}=0,\ |\beta|\leqq m-1,\ x(0, z)=x_0(z) \ on\ Z,\ \|x(t, \cdot)\|_{L^2(Z)}\leqq r,\ t\in T, \\ \qquad \gamma\leqq u(t, z)\leqq r_0(t, z\|x(t, \cdot)\|_{L^2(Z)}) \text{ a.e., } \gamma>0. \end{array}\right| \tag{***}$$

We will need the following hypotheses on the data of (***):

$\boldsymbol{H(A)_1}$: $A_\alpha:\ T\times Z\times \boldsymbol{R}^{N_m}\to\boldsymbol{R}$ are functions s.t.

(1) $(t, z)\to A_\alpha(t, z, \eta)$ is measurable,

(2) $\eta\to A_\alpha(t, z, \eta)$ is continuous,

(3) $\sum_{|\alpha|\leq m}(A_\alpha(t, z, \eta)-A_\alpha(t, z, \eta'))\ (\eta_\alpha-\eta'_\alpha)\geqq 0$ a.e.,

(4) $|A_\alpha(t, z, \eta(|\leqq\varphi_1(t, z)+c_1(t)\|\eta\|$ a.e., with $\varphi_1\in L^\infty(T, L^2(Z))$, $c_1\in L_+^\infty$,

(5) $\sum_{|\alpha|\leq m} A_\alpha(t, z, \eta)\eta_\alpha\geqq c_2\|\eta\|^2$ a.e., $c_2>0$.

$\boldsymbol{H(B)}$: $B:\ T\times Z\to\boldsymbol{R}$ is a function s.t.

(1) $(t, z)\to B(t, z, x)$ is measurable,

(2) $x\to B(t, z, x)$ is continuous,

(3) $|B(t, z, x)|\leqq a(t, z)$ a.e. with $a\in L^\infty(T\times Z)$.

$\boldsymbol{H(r_0)}$: $r_0:\ T\times Z\times\boldsymbol{R}_+\to\boldsymbol{R}_+$ is a function s.t.

(1) $(t, z, v)\to r_0(t, z, v)$ is measurable,

(2) $v\to r_0(t, z, v)$ is $u.s.c.$,

(3) $r_0(t, z, v)\leqq\theta(t, z)$ a.e., with $\theta\in L^\infty(T\times Z)$.

Here $X=H_0^m(Z)$, $H=L^2(Z)$ and $X^*=H_0^m(Z)^*=H^{-m}(Z)$. Then from the Sobolev embedding theorem, we know that (X, H, X^*) is an evolution triple, with all embeddings being compact. Also let $K=rB_H=\{v\in L^2(Z):\ |v|=\|v\|_{L^2(Z)}\leqq r\}$. From Prop. 8, p. 170 of [2] (see also [12], p. 961), we have for $x\in H_0^m(Z)$:

$$T'_K(x)=\begin{cases} H^{-m}(Z) & \text{if } \|x\|_{L^2(Z)}<r \\ \{v\in H^{-m}(Z):\ \langle x, v\rangle_{H_0^m(Z)H^{-m}(Z)}\leqq 0\} & \text{if } \|x\|_{L^2(Z)}=r. \end{cases}$$

Consider the time varying Dirichlet form $a:\ T\times H_0^m(Z)\times H_0^m(Z)\to\boldsymbol{R}$ defined by $a(t, x, y)=\sum_{|\alpha|\leq m}\int_Z A_\alpha(t, z, \eta(x(z)))D^\alpha y(z)dz$, for $x, y\in H_0^m(Z)$.

Using the Cauchy-Schwartz inequality, we get that

$$|a(t, x, y)|\leqq\eta_1\|x\|_{H_0^m(Z)}\|y\|_{H_0^m(Z)}$$

for some $\eta_1>0$. So there exists a generally nonlinear operator $\hat{A}: T\times X\to X^*$ defined by

$$\langle \hat{A}(t, x), y\rangle = a(t, x, y).$$

Then from Fubini's theorem, we have that $t\to\langle\hat{A}(t, x), y\rangle$ is measurable $\Rightarrow t\to\hat{A}(t, x)$ is weakly measurable. But $H^{-m}(Z)$ is a separable Hilbert space. Thus from the Pettis measurability theorem, we deduce that $t\to\hat{A}(t, x)$ is measurable. Also using hypothesis $H(A)_1$, we can easily check that $x\to\hat{A}(t, x)$ is continuous, while from $H(A)_1(5)$ we get that $\langle\hat{A}(t, x), x\rangle\geqq c_2\|x\|^2_{H_0^m(Z)}$.

Next let $Y=L^2(Z)$ (the control space) and $U(t, x)=\{u\in L^2(Z): \gamma\leqq u(z)\leqq r_0(t, z, \|x\|_{L^2(Z)})\}$. Note that $GrU=\{(t, x, u)\in T\times L^2(Z)\times L^2(Z): \gamma|C|^{-1}\leqq\int_C u(z)dz \leqq\int_C r_0(t, z, \|x\|_{L^2(Z)})dz$ for all $C\in B(Z)=$the Borel σ-field of Z, with $|C|>0\}$ (here $|C|$ denotes the Lebesgue measure of C). But recall that $B(Z)$ is countably generated. So $GrU\in B(T)\times B(L^2(Z))\times B(L^2(Z))$. Also let $V\subseteqq Y=L^2(Z)$ be nonempty, w-closed and let $U_t^-(V)=\{x\in L^2(Z)=H: U(t, x)\cap V\neq\emptyset\}$. We claim that this last set is closed. Let $x_n\in U_t^-(V)$, $x_n\xrightarrow{s}x$ in $H=L^2(Z)$. Take $u_n\in U(t, x_n)\cap V$, $n\geqq 1$. Because of hypothesis $H(r_0)(3)$ and by passing to a subsequence, we may assume that $u_n\to u$ in $Y=L^2(Z)$. Then for all $C\in B(Z)$ and by using hypothesis $H(r_0)$, we have

$$\gamma|C|\leqq\int_C u(z)dz\leqq\overline{\lim}\int_C r_0(t, z\|x_n\|_{L^2(Z)})dz\leqq\int_C r_0(t, z, \|x\|_{L^2(Z)})dz$$

$\Rightarrow u\in U(t, x)$ and $u\in V$, since the latter is w-closed.

Hence $x\in U_t^-(V)\Rightarrow U_t^-(V)$ is closed $\Rightarrow U(t, \cdot)$ is $u.s.c.$ from $H=L^2(Z)$ into $Y_w=L^2(Z)_w$. Furthermore $|U(t, x)|=\sup\{\|u\|_{L^2(Z)}: u\in U(t, x)\}\leqq\hat{\theta}$, $\hat{\theta}=\|\theta(\cdot, \cdot)\|_\infty$.

Next define F: $T\times L^2(Z)\to P_{fc}(L^2(Z))$ by

$$F(t, x)=\{y\in L^2(Z): y(z)=B(t, z, x(z))u(z) \text{ a.e. with } u\in U(t, x)\}.$$

Let $\hat{B}(t, x)(z)=B(t, z, x(z))$. Then $F(t, x)=\hat{B}(t, x)U(t, x)$ and as in the proof of Theorem 4.1, we can check that $F(\cdot, \cdot)$ is graph measurable, while $F(t, \cdot)$ is sequentially closed in $L^2(Z)\times L^2(Z)_w$. Also $|F(t, x)|\leqq\|a\|_\infty\|\theta\|_\infty$ a.e..

Then we can rewrite (***) in the following equivalent evolution inclusion form. Assume that $x_0\in L^2(Z)$, $\|x_0\|_{L^2(Z)}\leqq r$.

$$\left|\begin{array}{c} \dot{x}(t)+\hat{A}(t, x(t))\in F(t, x(t)) \quad \text{a.e.}, \\ x(0)=\hat{x}_0\in K\subseteqq H, \\ x(t)\in K,\ t\in T,\ u(t)\in U(t, x(t)) \quad \text{a.e.}, \\ u(\cdot)=\text{measurable}. \end{array}\right| \qquad (***)'$$

We will make the following hypothesis:

$\boldsymbol{H}_\tau''$: for every $x(\cdot)\in H_0^m(Z)$ with $\|x\|_{L^2(Z)}=r$, we can find $u\in U(t, x)$ s.t. $\int_Z B(t, z, x(z))u(z)x(z)dz\leqq 0$ (sign condition).

Then given $\|x\|_{L^2(Z)}=r$, for this particular u, we will have

$$\langle \hat{B}(t, x)u, x\rangle \leqq 0$$

with $\hat{B}(t, x)(z)=B(t, z, x(z))$, and so

$$\begin{aligned}\langle \hat{B}(t, x)u-\hat{A}(t, x), x\rangle &\leqq -\hat{c}_2\|x\|^2_{H_0^m(Z)}\leqq 0\\ &\Longrightarrow \hat{B}(t, x)u-\hat{A}(t, x)\in T'_K(x)\\ &\Longrightarrow [F(t, x)-\hat{A}(t, x)]\cap T'_K(x)\neq\emptyset\end{aligned}$$

for all $(t, x)\in T\times H_0^m(Z)$, $\|x\|_{L^2(Z)}=b$. So we have satisfied hypotheses $H(F)$ and H_τ. Thus all hypotheses of Theorem 4.1 have been verified, and we have:

Theorem 5.1. *If hypotheses $H(A)_1$, $H(B)$, $H(r_0)$ and H_τ'' hold, then problem* (***) *admits a soluon $x(\cdot, \cdot)\in L^2(T, H_0^m(Z))\cap C(T, L^2(Z))$ with $\partial x/\partial t\in L^2(T, H^{-m}(Z))$.*

Acknowledgement The author would like to thank the referee for his very careful reading of the paper and for his many corrections and remarks that improved the content of this work.

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