Yokohama Mathematical
Journal Vol. 40, 1992

SUBMERSIONS OF *CR*-SUBMANIFOLDS ON AN ALMOST HERMITIAN MANIFOLD

Sharief Deshmukh, Tahsin Ghazal and Huda Hashem

(Received June 6, 1991; Revised October 28, 1991)

Abstract. Kobayashi [6] has shown that if an almost hermitian manifold $\boldsymbol{B}$ admits a Riemannian submersion $\pi: \boldsymbol{M}\rightarrow\boldsymbol{B}$ of a CR-submanifold $\boldsymbol{M}$ of a Kaehler manifold $\overline{\boldsymbol{M}}$, then $\boldsymbol{B}$ is necessarily a Kaehler manifold. In this paper we consider similar question for the CR-submanifolds of manifolds in different classes of almost hermitian manifolds viz hermitian manifolds, quasi-Kaehler manifolds, and nearly Kaehler manifolds.

1. Introduction

The study of the Riemannian submersion $\pi: M\rightarrow B$ of a Riemannian manifold M onto a Riemannian manifold B is initiated by O'Neill [7]. A submersion π naturally gives rise to two distributions on M called the horizontal and vertical distributions respectively, of which the vertical distribution is always integrable giving rise to the fibers of the submersion which are closed submanifolds of M. Bejancu [1] introduced a special class of submanifolds of an almost hermitian manifold which includes both the class of complex submanifolds as well as the class of totally real submanifolds. These submanifolds of an almost hermitian manifold $\overline{M}$ are known as CR-submanifolds. A CR-submanifold M of an almost hermitian manifold $\overline{M}$ with almost complex structure J requires two orthogonal complementary distributions D and $D^{\perp}$ defined on M such that D is invariant under J and $D^{\perp}$ is totally real (cf. [1], [2]). For a CR-submanifold M of a Kaehler manifold $\overline{M}$, the distribution $D^{\perp}$ is integrable [2]. Kobayashi [6] observed the similarity between the total space of submersion $\pi: M\rightarrow B$ and the CR-submanifold M of a Kaehler manifold $\overline{M}$ in terms of the distributions. Thus he considered the submersion $\pi: M\rightarrow B$ of a CR-submanifold M of a Kaehler manifold $\overline{M}$ onto an almost hermitian manifold B such that the distributions D and $D^{\perp}$ of the CR-submanifold structure of M become respectively the horizontal and vertical distributions required by the submersion π and π restricted to D becomes an isometry which preserves the complex structures, that is, $J'\circ\pi_{*}=\pi_{*}\circ J$ on D, where J and J' are the complex structures of $\overline{M}$ and B respectively. He has shown that under this situation B is necessarily a Kaehler manifold and obtained the relation between the holomorphic sectional curvatures of $\bar{M}$ restricted to D and those of B.

To deal with the similar question for the CR-submanifolds of the manifolds in other classes of almost hermitian manifolds one has the difficulty that the distribution $D^{\perp}$ for these CR-submanifolds are not necessarily integrable to match the requirement of the submersion. To overcome this we have to have the submersions $\pi: M \to B$ of CR-submanifolds M with integrable $D^{\perp}$ onto an almost hermitian manifold B. This additional assumption will be worth provided that does not render the ambient almost hermitian manifold $\bar{M}$ into a Kaehler manifold. Fortunately it works, for, a real hypersurface in S^6 (which is a nearly Kaehler manifold) is a 5-dimensional CR-submanifold with $D^{\perp}$ a 1-dimensional distribution which is integrable and yet S^6 is not a Kaehler manifold (cf. [5]). In the present paper we study the submersions of CR-submanifolds with integrable totally real distribution $D^{\perp}$ of a hermitian manifold, a quasi-Kaehler manifold and a nearly Kaehler manifold onto an almost hermitian manifold and in fact, it is shown that Kobayashi's result can be deduced immediately from our results. In addition we also study the effect of these submersions on the topology of the CR-submanifolds. Lastly for the submersion $\pi: M \to B$ of a CR-submanifold M of S^6 onto an almost hermitian manifold B, it is shown that B is a 4-dimensional Kaehler manifold.

2. Preliminaries

Let $\bar{M}$ be a $2n$-dimensional Riemannian manifold with Rimannian metric g. If there exists an almost complex structure J on $\bar{M}$ which is compatible with the metric, that is, it satisfies

$$g(JX, JY)=g(X, Y), \qquad X, Y \in \chi(\bar{M}),$$

then $\bar{M}$ is called an almost hermitian manifold, where $\chi(\bar{M})$ is the Lie algebra of vector fields on $\bar{M}$. The Nijenhuis torsion N for an almost hermitian manifold $\bar{M}$ is defined by

(2.1) $$N(X, Y)=[JX, JY]-[X, Y]-J[X, JY]-J[JX, Y], \quad X, Y \in \chi(\bar{M}).$$

If $N \equiv 0$ on an almost hermitian manifold, then it is called a hermitian manifold. Let $\bar{\nabla}$ be the Riemannian connection on the almost hermitian manifold $\bar{M}$. If the almost complex structure J on almost hermitian manifold $\bar{M}$ satisfies

(2.2) $$(\bar{\nabla}_X J)(Y)+(\bar{\nabla}_{JX} J)(JY)=0 \qquad X, Y \in \chi(\bar{M}),$$

then $\bar{M}$ is called a quasi-Kaehler manifold, and if it satisfies

(2.3) $$(\bar{\nabla}_X J)(X)=0, \qquad X \in \chi(\bar{M}),$$

then it is called a nearly Kaehler manifold. Further if on an almost hermitian manifold $\bar{M}$ we have

$$(2.4) \qquad (\bar{\nabla}_X J)(Y)=0\,, \qquad X, Y\in \mathfrak{X}(\bar{M})\,,$$

then $\bar{M}$ is called a Kahler manifold. The Kaehler structure is richer than all the above structures, for it can be easily verified that (2.4) implies $N\equiv 0$, (2.2) and (2.3). However, there are examples showing that the converse is not true.

The tensor field J on an almost hermitian manifold $\bar{M}$ is skew-symmetric and as such for any unit vector field X on $\bar{M}$, $\{X, JX\}$ span a plane section and this gives rise to a sectional curvature $H(X)$ called the holomorphic sectional curvature.

Let M be an m-dimensional submanifold of an almost hermitian manifold $\bar{M}$. If there exist two orthogonal complementary distributions D and $D^{\perp}$ on M satisfying $JD=D$ and $JD^{\perp}\subset\nu$, where J is the almost complex structure on $\bar{M}$ and ν is the normal bundle of M, then M is called a CR-submanifold of $\bar{M}$ (cf. [1]). We denote by the same letter g the induced metric on M. The Riemannian connection $\bar{\nabla}$ on $\bar{M}$ gives rise to the induced Riemannian connectiin ∇ on M and a connection $\nabla^{\perp}$ in the normal bundle ν and they are related by

$$(2.5) \qquad \bar{\nabla}_X Y=\nabla_X Y+h(X, Y)\,, \qquad X, Y\in\mathfrak{X}(M)\,,$$

$$(2.6) \qquad \bar{\nabla}_X N=-\bar{A}_N X+\nabla^{\perp}_X N\,, \qquad X\in\mathfrak{X}(M),\ N\in\nu\,,$$

where h is the second fundamental form and $\bar{A}_N$ is the Weingarten map and they are related by $g(h(X, Y), N)=g(\bar{A}_N X, Y)$, $X, Y\in\mathfrak{X}(M)$. The curvature tensor R of the submanifold M is related to the curvature tensor $\bar{R}$ of $\bar{M}$ by the following Gauss formula

$$(2.7) \qquad R(X, Y\,; Z, W)=\bar{R}(X, Y\,; Z, W)+g(h(Y, Z), h(X, W)) \\ -g(h(X, Z), h(Y, W)) \qquad \text{for } X, Y, Z, W\in\mathfrak{X}(M)\,.$$

For the theory of submersions $\pi: M\to B$ of a Riemannian manifold M onto a Riemannian manifold B we follow [7]. We now state the definition for the submersion of a CR-submanifold of an almost hermitian manifold onto an almost hermitian manifold (cf. [6]).

Definition 2.1. Let M be a CR-submanifold of an almost hermitian manifold $\bar{M}$ with distributions D and $D^{\perp}$ and the normal bundle ν. By a submersion $\pi: M\to B$ of M onto an almost hermitian manifold B we mean a Riemannian submersion $\pi: M\to B$ together with the following conditions:

(i) $D^{\perp}$ is the kernel of π_*, that is, $\pi_* D^{\perp}=\{0\}$,

(ii) $\pi_* D^{\perp}_p=T_{\pi(p)}B$ is complex isometry, where $p\in M$ and $T_{\pi(p)}B$ is the tangent space of B at $\pi(p)$,

(iii) J interchanges $D^{\perp}$ and ν, that is, $JD^{\perp}=\nu$.

Naturally the above definition puts restrictions on the dimension of $\bar{M}$, namely we should have $\dim \bar{M}=\dim D+2\dim D^{\perp}$. We recall that a vector field $X\in\mathfrak{X}(M)$ for this submersion $\pi: M\to B$ is said to be basic vector field if $X\in D$ and X is π related to a vector field on B, that is, there exists a vector X_* on B such that $(\pi_* X)_p=X_{*\pi(p)}$ for each $p\in M$ (cf. [7]). We have the following lemma for basic vector fields [7].

Lemma 2.1. *Let X and Y be basic vector fields on M. Then*
(i) $g(X, Y)=g_*(X_*, Y_*)\circ\pi$, *$g$ is the metric on M and g_* is the Riemannian metric on B.*
(ii) *The horizontal part $\mathcal{H}[X, Y]$ of $[X, Y]$ is a basic vector field and corresponds to $[X_*, Y_*]$, that is $\pi_*\mathcal{H}[X, Y]=[X_*, Y_*]\circ\pi$,*
(iii) $[V, X]\in D^{\perp}$, $V\in D^{\perp}$,
(iv) *$\mathcal{H}(\nabla_Y Y)$ is basic vector field corresponding to $\nabla^*_{X_*}Y_*$, where ∇^* is the Riemannian connection on B.*

Set

$$\tilde{\nabla}^*_X Y=\mathcal{H}(\nabla_X Y), \qquad X, Y\in D,$$

then $\tilde{\nabla}^*_X Y$ is the basic vector field and from above lemma we have

$$\pi_*(\tilde{\nabla}^*_X Y)=\nabla^*_{X_*}Y_*\circ\pi. \tag{2.8}$$

Define a tensor field C on M by setting

$$\nabla_X Y=\tilde{\nabla}^*_X Y+C(X, Y), \qquad X, Y\in D, \tag{2.9}$$

that is, $C(X, Y)$ is the vertical component $\mathcal{V}(\nabla_X Y)$ of $\nabla_X Y$. The tensor field C is known to be a skew-symmetric and it satisfies

$$C(X, Y)=\frac{1}{2}\mathcal{V}[X, Y], \qquad X, Y\in D. \tag{2.10}$$

Also, for $X\in D$ and $V\in D^{\perp}$ define an operator A on M by setting $\nabla_X V=\mathcal{V}(\nabla_X V)+A_X V$, that is, $A_X V$ is the horizontal component of $\nabla_X V$. Since by Lemma 2.1 $[V, X]\in D^{\perp}$ for $X\in D$ and $V\in D^{\perp}$ we have

$$\mathcal{H}(\nabla_X V)=\mathcal{H}(\nabla_V X)=A_X V. \tag{2.11}$$

The operators C and A are related by

$$g(A_X V, Y)=-g(V, C(X, Y)), \qquad X, Y\in D,\ V\in D^{\perp}. \tag{2.12}$$

3. Submersions of CR-submanifolds of hermitian manifolds

Let M be a CR-submanifold of a hermitian manifold $\bar{M}$ and $\pi: M \to B$ be its submersion on an almost hermitian manifold B as described in Definition 2.1. If J and J' are the almost complex structures on $\bar{M}$ and B respectively, then from (ii) in Definition 2.1 we have $\pi_* \circ J = J' \circ \pi_*$ on D. This together with (ii) in Lemma 2.1 and the definition of Nijenhuis torsion gives immediately $N'(X_*, Y_*)=0$, $X_*, Y_* \in \mathfrak{X}(B)$, where N' is the Nijenhuis torsion for almost complex structure J' on B. Thus we have

Theorem 3.1. *Let $\pi: M \to B$ be a submersion of a CR-submanifold M of a hermitian manifold $\bar{M}$ onto an almost hermitian manifold B. Then B is a hermitian manifold.*

We recall that on a Riemannian manifold M a distribution S is said to be parallel if $\nabla_X Y \in S$, $X, Y \in S$, where ∇ is the Riemannian connection on M. From the definition of a Riemannian submersion $\pi: M \to B$ of a Riemannian manifold M onto a Riemannian manifold B, it follows that the vertical distribution is always integrable and its integral submanifolds are the fibers [7]. If in addition $D^{\perp}$ is parallel, then we prove

Proposition 3.1. *Let $\pi: M \to B$ be a submersion of a connected, complete and simply connected CR-submanifold M of a hermitian manifold $\bar{M}$ onto an almost hermitian manifold B. If D is integrable and $D^{\perp}$ is parallel, then M is the Riemannian product $M_1 \times M_2$, where M_1 is an invariant submanifold and M_2 is a totally real submanifold of $\bar{M}$.*

Proof. Since D is integrable for $X, Y \in D$, we have $\mathcal{V}[X, Y]=0$. Then from equation (2.10), we have $C(X, Y)=0$, $X, Y \in D$. Thus from the definition of C we have $\nabla_X Y = \tilde{\nabla}^*_X Y \in D$, that is, D is parallel. Since D and $D^{\perp}$ are both parallel by de Rham's Theorem M is the product $M_1 \times M_2$, where M_1 is integral submanifold of D and M_2 that of $D^{\perp}$. From the properties of D and $D^{\perp}$ it follows that M_1 is invariant submanifold of $\bar{M}$ and M_2 is totally real submanifold of $\bar{M}$.

Next we discuss how the submersion $\pi: M \to B$ of a CR-submanifold M with integrable D effects the topology of M. Let M be a CR-submanifold of a hermitian manifold $\bar{M}$ with almost complex structure J. Assume that $\dim D=2p$ and $\dim M=m$. We choose a local orthonormal frame $\{e_1, \cdots, e_p, Je_1, \cdots, Je_p, e_{2p+1}, \cdots, e_m\}$ on M such that $\{e_1, \cdots, e_p, Je_1, \cdots, Je_p\}$ is a local orthonormal frame of D and $\{e_{2p+1}, \cdots, e_m\}$ is that of $D^{\perp}$. Let $\{\omega^1, \cdots, \omega^{2p}, \omega^{2p+1}, \cdots, \omega^m\}$ be the dual frame of 1-forms to the above local orthonormal frame. Define a

$2p$-form Ω on M by

$$\Omega=\omega^1\wedge\cdots\wedge\omega^{2p}, \tag{3.1}$$

then it can be easily shown that this $2p$-form Ω is independent of the choice of the frame $\{e_1, \cdots, e_p, Je_1, \cdots, Je_p\}$ and is globally defined on M.

Definition 3.1. Let S be a q-dimensional distribution on a Riemannian manifold M. If $\sum_{i=1}^{q}\nabla_{e_i}e_i\in S$, then the distribution S is said to be minimal, where ∇ is the Riemannian connection on M and $\{e_1, \cdots, e_q\}$ is a local orthonormal frame of S.

Theorem 3.2. *Let $\bar{M}$ be a hermitian manifold and M be a closed CR-submanifold of $\bar{M}$ with integrable D. Let B be an almost hermitian manifold and $\pi: M\to B$ a submersion. Then the $2p$-form Ω is closed which defines a canonical de Rham cohomology class $[\Omega]\in H^{2p}(M, \boldsymbol{R})$, where $2p=\dim D$. Moreover the cohomology group $H^{2p}(M, \boldsymbol{R})$ is non-trivial if $D^{\perp}$ is minimal.*

Proof. From definition (3.1) of Ω, we have

$$d\Omega=\sum_{i=1}^{2p}(-1)^{i-1}\omega^1\wedge\cdots\wedge d\omega^i\wedge\cdots\wedge\omega^{2p}.$$

From above equation it follows that $d\Omega=0$ if and only if

$$d\Omega(Z, W, E_1, \cdots, E_{2p-1})=0 \quad\text{and}\quad d\Omega(Z, E_1, \cdots, E_{2p})=0, \tag{3.2}$$

for $Z, W\in D^{\perp}$ and $E_1, \cdots, E_{2p}\in D$. Choosing the vectors $E_1, \cdots, E_{2p}\in D$ as a local orthonormal frame $\{e_1, \cdots, e_p, Je_1, \cdots, Je_p\}$ of D to which $\{\omega^1, \cdots, \omega^{2p}\}$ works as dual frame of 1-forms, we get by a straightforward computation that the first equation in (3.2) holds if and only if $D^{\perp}$ is integrable; and the second equation in (3.2) holds if and only if D is minimal. However, from the definition of submersion it follows that $D^{\perp}$ is integrable. The hypothesis of theorem gives that D is integrable and this together with the proof of Proposition 3.1 gives that D is minimal. Hence the form Ω is closed, and it defines a de Rham cohomology class $[\Omega]$ in $H^{2p}(M, \boldsymbol{R})$.

Now suppose $D^{\perp}$ is minimal and we proceed to show that in this case $H^{2p}(M, \boldsymbol{R})\neq 0$. To accomplish this we show that the form Ω is harmonic which would then make the cohomology class $[\Omega]$ non-trivial. Define a $(m-2p)$-form $\Omega^{\perp}$ on M by setting

$$\Omega^{\perp}=\omega^{2p+1}\wedge\cdots\wedge\omega^{m},$$

where $\{\omega^{2p+1}, \cdots, \omega^{m}\}$ is dual frame to the local orthonormal frame $\{e_{2p+1}, \cdots, e_m\}$ of $D^{\perp}$. Then with the similar argument for Ω, it follows that $d\Omega^{\perp}=0$ if D is integrable and $D^{\perp}$ is minimal. Since both conditions are met, we have $d\Omega^{\perp}=0$.

This proves that the $2p$-form Ω is co-closed, that is $\delta\Omega=0$. Since $d\Omega=\delta\Omega=0$ and M is a closed submanifold, we get that Ω is a harmonic $2p$-form; and this completes the proof.

4. Submersion of CR-submanifolds of quasi-Kaehler manifolds

Let M be a CR-submanifold of a quasi-Kaehler manifold $\bar{M}$. First we prove the following lemma.

Lemma 4.1. *Let M be a CR-submanifold of a quasi-Kaehler manifold $\bar{M}$. If the horizontal distribution D is integrable, then*

$$h(X, JY)=h(JX, Y), \qquad X, Y\in D.$$

Proof. For $X, Y\in D$, we have

$$J[JX, JY]=J\bar{\nabla}_{JX}JY-J\bar{\nabla}_{JY}JX=-(\bar{\nabla}_{JX}J)(JY)-\bar{\nabla}_{JX}Y+(\bar{\nabla}_{JY}J)(JX)+\bar{\nabla}_{JY}X.$$

Using equations (2.2) and (2.5) in above equation we obtain

$$\begin{aligned} J[JX, JY]=&\nabla_X JY+h(X, JY)-J\bar{\nabla}_X Y-\nabla_{JX}Y-h(JX, Y)\\ &-\nabla_Y JX-h(Y, JX)+J\bar{\nabla}_Y X+\nabla_{JY}X+h(JY, X), \end{aligned}$$

which gives

$$J[JX, JY]+J[X, Y]=\nabla_X JY-\nabla_Y JX+\nabla_{JY}X-\nabla_{JX}Y+2(h(X, JY)-h(JX, Y))$$

since D is integrable the terms on the left hand side are tangential to M, equating normal components in the above equation we get $h(X, JY)=h(JX, Y)$, $X, Y\in D$, which proves the Lemma.

Theorem 4.1. *Let $\bar{M}$ be a quasi-Kaehler manifold and M be a CR-submanifold of $\bar{M}$. Let B be an almost hermitian manifold and $\pi: M\to B$ a submersion. Then B is a quasi-Kaehler manifold.*

Proof. Let $X, Y\in D$ be the basic vector fields. From equations (2.5) and (2.9), we have

$$\bar{\nabla}_{XY}=\tilde{\nabla}^*_X Y+C(X, Y)+h(X, Y).$$

Using this relation in $(\bar{\nabla}_X J)(Y)+(\bar{\nabla}_{JX}J)(JY)=0$, we obtain

$$\begin{aligned} &\tilde{\nabla}^*_X JY-\tilde{\nabla}^*_{JX}Y-J\tilde{\nabla}^*_X Y-J\tilde{\nabla}^*_{JX}JY+(C(X, JY)-C(JX, Y))+(h(X, JY)\\ &-h(JX, Y))-J(C(X, Y)+C(JX, JY))-J(h(X, Y)+h(JX, JY))=0, \end{aligned}$$

where J is the almost complex structure on $\bar{M}$. Equating horizontal vertical and normal components in above equation we get

$$\tilde{\nabla}^*_X JY - \tilde{\nabla}^*_{JX} Y - J\tilde{\nabla}^*_X Y - J\tilde{\nabla}^*_{JX} JY = 0\,, \tag{4.1}$$

$$\begin{aligned} C(X, JY) - C(JX, Y) &= J(h(X, Y) + h(JX, JY))\,, \\ J(C(X, Y) + C(JX, JY)) &= h(X, JY) - h(JX, Y)\,. \end{aligned} \tag{4.2}$$

Operating π_* on the equation (4.1) to project it down on B and using Lemma 2.1 together with the equation (2.8), we get

$$(\nabla^*_{X_*} J')(Y_*) + (\nabla^{*\prime}_{JX_*} J')(J'Y_*) = 0\,, \qquad X_*, Y_* \in \mathfrak{X}(B)$$

where $\pi_* X = X_*$ and $\pi_* Y = Y_*$ and J' is the almost complex structure on B. This proves that B is a quasi-Kaehler manifold.

Remark. In the latter half of the Kobayashi's result [6] on the submersion $\pi: M \to B$ of a CR-submanifold M of a Kaehler manifold $\bar{M}$ onto an almost hermitian manifold B, he obtained the following relation

$$\bar{H}(X) = H^*(X_*) - 4\|h(X, X)\|^2 \qquad X \in D,\ \|X\| = 1\,, \tag{4.3}$$

where $\bar{H}$ and H^* are holomorphic sectional curvatures of $\bar{M}$ and B respectively, $\pi_* X = X_*$ and D is the horizontal distribution. However we could not decide whether the relation (4.3) holds in general for the submersion $\pi: M \to B$ of a CR-submanifold M of a quasi-Kaehler manifold $\bar{M}$ onto an almost hermitian manifold B, and instead obtain relation (4.5) (see below) for this situation. For this first we observe that using equations (2.8), (2.9), (2.10), (2.11) and (2.12) together with Lemma 2.1, after a straightforward computation we obtain

$$\begin{aligned} R(X, Y; Z, W) &= R^*(X_*, Y_*; Z_*, W_*) - g(C(Y, Z), C(X, W)) \\ &\quad + g(C(X, Z), C(Y, W)) + 2g(C(X, Y), C(Z, W)), \quad X, Y, Z, W \in D, \end{aligned} \tag{4.4}$$

where R and R^* are curvature tensors of M and B respectively and $\pi_* X = X_*$, $\pi_* Y = Y_*$, $\pi_* Z = Z_*$ and $\pi_* W = W_* \in B$. Then using equation (2.7) and the fact that C is skew symmetric in above equation we get

$$\bar{H}(X) = H^*(X_*) + \|h(X, JX)\|^2 - g(h(JX, JX), h(X, X)) - 3\|C(X, JX)\|^2, \tag{4.5}$$

where $\bar{H}(X) = \bar{R}(X, JX; JX, X)$ and $H^*(X_*) = R^*(X_*, J'X_*; J'X_*, X_*)$ are the holomorphic sectional curvatures of $\bar{M}$ and B respectively.

However if we assume that the horizontal distribution D is integrable, we have the following corollary comparing the holomorphic sectional curvatures of $\bar{M}$ and B. Recall that a CR-submanifold M of an almost hermitian manifold $\bar{M}$ is said to be D-totally geodesic if $h(X, Y) = 0$, $X, Y \in D$.

Corollary 4.1. *Let $\bar{M}$ be a quasi-Kaehler manifold and M be a CR-submanifold of $\bar{M}$ with integrable D. Let B be an almost hermitian manifold and*

$\pi: M \to B$ a submersion. Then the holomorphic sectional curvatures $\bar{H}$ and H^* of $\bar{M}$ and B respectively satisfy

$$\bar{H}(X) \geqq H^*(X_*), \qquad X \in D, \qquad \|X\|=1, \qquad \pi_* X = X_*,$$

and the equality holds if and only if M is D-totally geodesic.

Proof. Since D is integrable by Lemma 4.1, we have $h(X, X) = -h(JX, JX)$. Then taking $Y=X$ in first equation of (4.2) we get $C(X, JX)=0$. Thus the equation (4.5) gives,

$$\bar{H}(X) = H^*(X_*) + \|h(X, JX)\|^2 + \|h(X, X)\|^2, \qquad X \in D, \ \|X\|=1.$$

This proves that $\bar{H}(X) \geqq H^*(X_*)$ with the equality holding if and only if $h(X, X) = 0$ and $h(X, JX)=0$, $X \in D$, $\|X\|=1$. From $h(X, X)=0$, $X \in D$, $\|X\|=1$ and linearity of h it follows immediately that $h(X, Y)=0$, $X, Y \in D$ and proves that M is D-totally geodesic.

5. Submersions of CR-submanifolds of nearly Kaehler manifolds

Let $\bar{M}$ be a nearly Kaehler manifold with almost complex structure J. Then on $\bar{M}$ the equation (2.3) can be equivalently written as

$$(\bar{\nabla}_X J)(Y) + (\bar{\nabla}_Y J)(X) = 0, \qquad X, Y \in \chi(\bar{M}). \tag{5.1}$$

Also using (5.1), we can immediately show that on a nearly Kaehler manifold $\bar{M}$, we have

$$(\bar{\nabla}_X J)(JY) = -J(\bar{\nabla}_X J)(Y) \quad \text{and} \quad (\bar{\nabla}_{JX} J)(Y) = -J(\bar{\nabla}_X J)(Y), \quad X, Y \in \chi(\bar{M}). \tag{5.2}$$

Thus using (5.1), we find that

$$(\bar{\nabla}_{JX} J)(JY) + (\bar{\nabla}_X J)(Y) = -(\bar{\nabla}_X J)(Y) + (\bar{\nabla}_X J)(Y) = 0,$$

that is, a nearly Kaehler manifold is a quasi-Kaehler manifold. We state the following lemma which describes how far is the nearly Kaehler manifold from being a Kaehler manifold.

Lemma 5.1. *Let $\bar{M}$ be a nearly Kaehler manifold. If the Nijenhuis torsion for $\bar{M}$ is zero, then M is a Kaehler manifold.*

Proof. The equation (2.1) can be immediately made into

$$N(X, Y) = J(\bar{\nabla}_{JX} J)(JY) - J(\bar{\nabla}_{JY} J)(JX) + (\bar{\nabla}_X J)(JY) - (\bar{\nabla}_Y J)(JX), \quad X, Y \in \chi(\bar{M}).$$

Since $\bar{M}$ is a nearly Kaehler manifold, it is quasi-Kaehler manifold and thus using equations (5.1), (5.2) and (2.2), we get $N(X, Y) = -4J(\bar{\nabla}_X J)(Y)$. This proves that if $N(X, Y)=0$, then $\overline{M}$ is a Kaehler manifold.

It is known that the six-dimensional sphere S^6 considered as sphere in the space of purely imaginary Cayley numbers admits an almost complex structure making S^6 an almost hermitian manifold which is nearly Kaehler but not a Kaehler manifold. Also the only complete and simply connected 6-dimensional nearly Kaehler manifold of constant holomorphic sectional curvature which is not Kaehler is S^6. Next we prove

Theorem 5.1. *Let $\overline{M}$ be a nearly Kaehler manifold and M be a CR-submanifold of $\overline{M}$. Let B be an almost hermitian manifold and $\pi: M \to B$ a submersion. Then B is a nearly Kaehler manifold. If $\overline{H}$ and H^* are the holomorphic sectional curvatures of $\overline{M}$ and B respectively, then*

$$\overline{H}(X)=H^*(X_*)-4\|h(X, X)\|^2, \qquad X \in D, \ \|X\|=1,$$

where $\pi_* X = X_*$ and D is the horizontal distribution.

Proof. Let $X \in D$ be a basic vector field with $\pi_* X = X_* \in \mathfrak{X}(B)$, then from (2.5) and definition of tensor C, we have

$$\overline{\nabla}_X JX = \tilde{\nabla}^*_X JX + C(X, JX) + h(X, JX)$$

and

$$J\overline{\nabla}_X X = J\tilde{\nabla}^*_X X + JC(X, X) + Jh(X, X).$$

Using (2.3) and equating horizontal vertical and normal components we get $\tilde{\nabla}^*_X JX = J\tilde{\nabla}^*_X X$, $C(X, JX)=Jh(X, X)$ and $JC(X, X)=h(X, JX)$. Operating π_* on the first equation we get $(\nabla^*_{X_*} J')(X_*)=0$, $X_* \in \mathfrak{X}(B)$, that is, B is a nearly Kaehler manifold. The rest of the proof follows using $C(X, JX)=Jh(X, X)$ and $h(X, JX)=JC(X, X)=0$ in the equation (4.5).

As a particular case of above theorem we have the following corollary which is essentially the result of Kobayashi [6].

Corollary 5.1. *Let $\overline{M}$ be a Kaehler manifold and M be a CR-submanifold of $\overline{M}$. Let B be an almost hermitian manifold and $\pi: M \to B$ a submersion. Then B is a Kaehler manifold. If $\overline{H}$ and H^* are the holomorphic sectional curvatures of $\overline{M}$ and B respectively, then*

$$\overline{H}(X)=H^*(X_*)-4\|h(X, X)\|^2, \qquad X \in D, \ \|X\|=1,$$

where $\pi_* X = X_*$ and D is the horizontal distribution.

Proof. Since $\overline{M}$ is a nearly Kaehler manifold with vanishing Nijenhuis torsion. From Theorems 3.1 and 5.1 it follows that B is a nearly Kaehler manifold with vanishing Nijenhuis torsion; and the corollary follows from

Lemma 5.1 and Theorem 5.1.

We also have the $2p$-form Ω defined on the CR-submanifold M of a nearly Kaehler manifold $\bar{M}$ as in equation (3.1) for a CR-submanifold of a hermitian manifold. Next we have the following

Theorem 5.2. *Let $\bar{M}$ be a nearly Kaehler manifold and M be a closed CR-submanifold of $\bar{M}$. Let B be an almost hermitian manifold and $\pi: M\to B$ a submersion. Then the $2p$-form Ω is closed which defines a canonical de Rham cohomology class $[\Omega]$ in $H^{2p}(M, \boldsymbol{R})$, where $2p=\dim D$ the horizontal distribution. Moreover the cohomology group $H^{2p}(M, \boldsymbol{R})$ is non-trivial if D is integrable and the vertical distributor $D^{\perp}$ is minimal.*

Proof. The second part of the theorem follows exactly parallel to Theorem 3.2. To prove the first part it is again similar to that of Theorem 3.2, except in this case we have to show that D is minimal (for this follows directly in Theorem 3.2 owing to the integrability of D). Since $\bar{M}$ is also a quasi-Kaehler manifold, replacing Y by JX, $X\in D$ in equation (2.2), we get

$$\nabla_X X+\nabla_{JX}JX=h(X, X)+h(JX, JX)-J[JX, X]=0, \qquad X\in D .$$

Taking inner product with $Z\in D^{\perp}$ in above equation we get $g(\nabla_X X+\nabla_{JX}JX, Z)=0$, that is, $\nabla_X X+\nabla_{JX}JX\in D$. Since we can choose a local orthonormal frame $\{e_1, \cdots, e_p, Je_1, \cdots, Je_p\}$ for D, we get that D is minimal.

Theorem 5.3. *Let M be a CR-submanifold of the nearly Kaehler S^6 and B an almost hermitian manifold. If $\pi: M\to B$ is a submersion, then B is a 4-dimensional Kaehler manifold and the horizontal distribution is not integrable.*

Proof. Since $\dim M<6$, we have either $\dim D=2$ or $\dim D=4$. First we show that $\dim D=2$ cannot occur. For this suppose $\dim D=2$. Then from definition we have $D^{\perp}$ is integrable and $JD=\nu$ and thus $\dim D^{\perp}=2$, that is, M is a 4-dimensional CR-submanifold. For $X\in D$, we have $(\bar{\nabla}_X J)(X)=0$ and $(\bar{\nabla}_X J)(JX)=-J(\bar{\nabla}_X J)(X)=0$. Since a local frame of D is of the form $\{X, JX\}$, it follows that

$$(\bar{\nabla}_X J)(Y)=0 , \qquad X, Y\in D . \tag{5.3}$$

Next consider a local orthonormal frame $\{e_1, e_2\}$ for $D^{\perp}$, then $\{Je_1, Je_2\}$ is local orthonormal frame of normals. Since the operator $\bar{\nabla}_Z J$ is skew-symmetric, using equation (5.1) we get

$$g((\bar{\nabla}_{e_1}J)(e_2), e_1)=0 , \qquad g((\bar{\nabla}_{e_1}J)(e_2), e_2)=0 ,$$

$$g((\bar{\nabla}_{e_1}J)(e_2), Je_1)=0 , \qquad g((\bar{\nabla}_{e_1}J)(e_2), Je_2)=0 .$$

Hence we have $(\nabla_{e_1}J)(e_2)\in D$. Thus we can verify that

$$(\nabla_Z J)(W)\in D\,, \qquad Z, W\in D^\perp\,. \tag{5.4}$$

Now for $Z, W\in D^\perp$, using (2.5) and (2.6), we have

$$J[Z, W]=-2(\nabla_Z J)(W)+\bar{A}_{JZ}W-\bar{A}_{JW}Z+\nabla_Z^\perp JW-\nabla_W^\perp JZ\,. \tag{5.5}$$

Now for a vector field $U\in\mathfrak{X}(M)$, using the equation (5.1) we have

$$2g(\bar{A}_{JZ}W, U)=2g(h(W, U), JZ)=-g(J(\nabla_W U+\nabla_U W), Z)$$

$$=-g(\nabla_W JU, Z)-g(\nabla_U JW, Z)\,.$$

Using $g(JU, Z)=0$, we get

$$2g(\bar{A}_{JZ}W, U)=-g(J\nabla_W Z, U)+g(\bar{A}_{JW}Z, U)\,.$$

Similarly we get

$$2g(\bar{A}_{JW}Z, U)=-g(J\nabla_Z W, U)+g(\bar{A}_{JZ}W, U)\,.$$

Subtracting these two equations we get

$$g(J\mathcal{H}[Z, W], U)=3g(\bar{A}_{JZ}W-\bar{A}_{JW}Z, U)\,,$$

where we have used the fact that $J\mathcal{V}[Z, W]\in\nu$. Since $U\in\mathfrak{X}(M)$ is arbitrary, from above equation we get

$$J\mathcal{H}[Z, W]=3(\bar{A}_{JZ}W-\bar{A}_{JW}Z)\,, \qquad Z, W\in D^\perp\,. \tag{5.6}$$

From the definition it follows that $D^\perp$ is integrable, that is, $\mathcal{H}[Z, W]=0$, $Z, W\in D^\perp$. Thus from equations (5.5) and (5.6), we get for $V\in D$, that $g((\nabla_Z J)(W), V)=0$. Utilizing this in equation (5.2), we obtain $(\nabla_Z J)(W)=0$, $Z, W\in D^\perp$. Next for $X\in D$ and $Z\in D^\perp$, we get

$$g((\nabla_X J)(Z), X)=0, \qquad g((\nabla_X J)(Z), JX)=0,$$

$$g((\nabla_X J)(Z), Z)=0, \qquad g((\nabla_X J)(Z), JZ)=0.$$

Also for $W\in D^\perp$ with $W\perp Z$, we get

$$g((\nabla_X J)(Z), W)=g(X, (\nabla_Z J)(W))=0$$

and

$$g((\nabla_X J)(Z), JW)=g(JX, (\nabla_Z J)(W))=0\,,$$

where we have used $(\nabla_Z J)(W)=0$ proved in previous paragraph. Taking a local orthonormal frame $\{X, JX, Z, W, JZ, JW\}$ of S^6 where $X\in D$ and $Z, W\in D^\perp$, we have proved that

$$(\nabla_X J)(JX)=0, \qquad (\nabla_X J)(Z)=0, \qquad (\nabla_X J)(W)=0,$$

$$(\nabla_{JX}J)(Z)=0, \qquad (\nabla_Z J)(W)=0, \qquad (\nabla_{JX}J)(W)=0.$$

We also have

$$(\nabla_X J)(JZ)=-J(\nabla_X J)(Z)=0, \qquad (\nabla_X J)(JW)=-J(\nabla_Z J)(W)=0$$

and that

$$(\nabla_Z J)(JW)=-J(\nabla_Z J)(W)=0.$$

All this amounts to $(\nabla_X J)(Y)=0$ for all $X, Y \in \mathfrak{X}(S^6)$ which is a contradiction. Hence dim $D=4$, and by Theorem 5.1 B is a 4-dimensional nearly Kaehler manifold. It is easy to show that a 4-dimensional nearly Kaehler manifold is a Kaehler manifold by taking a local frame $\{X_*, J'X_*, Y_*, J'Y_*\}$ on B. Further the horizontal distribution D is not integrable, for otherwise the integral submanifold of D would be a 4-dimensional invariant submanifold in S^6 which does not exist (cf. [5]).

Acknowledgement. The authors wish to express their sincere thanks to the referee for the improvements.

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Department of Mathematics
College of Sciences
King Saud University
P.O. Box 2455, Riyadh 11451
Saudi Arabia