

BOUNDS FOR THE ABSOLUTE REGULARITY COEFFICIENT OF A STATIONARY RENEWAL PROCESS

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Abstract. We give an upper bound for the absolute regularity coefficient of a stationary renewal process in terms of the total variation of the difference between the corresponding Palm and the usual renewal measure.

1. Introduction and main result

There is a huge and scattered literature on limit theorems for partial sums (integrals or other functionals) of discrete and continuous time random processes satisfying certain mixing conditions, see, for example, the review paper by Bradley [1]. The assumed mixing conditions ensure asymptotic independence between such parts of the random process whose index sets are separated far enough. To make this idea precise for a certain dependence structure we need measures of dependence expressed by so-called mixing coefficients. Let (Ω, \mathcal{F}, P) be a fixed probability space. An important and in many situations quite natural measure of dependence between two arbitrary sub- σ -fields \mathcal{A} and $\mathcal{B} \subset \mathcal{F}$ is the *absolute regularity* (or *β -mixing* or *weak Bernoulli*) *coefficient*

$$\beta(\mathcal{A}, \mathcal{B}) := E \sup_{B \in \mathcal{B}} |P(B/\mathcal{A}) - P(B)|$$

which was first studied by Volkonskii and Rozanov [7] who attributed this measure of dependence to Kolmogorov. As has been shown in [7] $\beta(\mathcal{A}, \mathcal{B})$ can be described in a different way. Let $P_{\mathcal{A} \otimes \mathcal{B}}$ be the restriction to the product- σ -field $\mathcal{A} \otimes \mathcal{B}$ of the measure on $\Omega \times \Omega$ induced by P and the diagonal mapping $\omega \mapsto (\omega, \omega)$ and $P_{\mathcal{A}}, P_{\mathcal{B}}$ denote respectively the restrictions to \mathcal{A} and \mathcal{B} of P . Then we have

$$(1) \quad \begin{aligned} \beta(\mathcal{A}, \mathcal{B}) &= \sup_{C \in \mathcal{A} \otimes \mathcal{B}} |P_{\mathcal{A} \otimes \mathcal{B}}(C) - (P_{\mathcal{A}} \times P_{\mathcal{B}})(C)| \\ &= \frac{1}{2} \sup \sum_{i=1}^I \sum_{j=1}^J |P(A_i \cap B_j) - P(A_i)P(B_j)|, \end{aligned}$$

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where the latter sup is taken over all pairs of finite partitions $\{A_1, \dots, A_I\}$ and $\{B_1, \dots, B_J\}$ of Ω such that $A_i \in \mathcal{A}$, $i=1, \dots, I$ and $B_j \in \mathcal{B}$, $j=1, \dots, J$. Now, let $X = \{X(t), t \geq 0\}$ be a real-valued random process on (Ω, \mathcal{F}, P) and for $0 \leq a \leq b \leq \infty$ the notation \mathcal{F}_a^b will mean the σ -field generated by the family $\{X(t), a \leq t \leq b\}$. The random process X is then said to be *absolutely regular* (or β -mixing or *weak Bernoulli*) if

$$(2) \quad \beta(r) := \sup_{t \geq 0} \beta(\mathcal{F}_0^t, \mathcal{F}_{t+r}^\infty) \longrightarrow 0$$

as $r \rightarrow \infty$. Indeed much research has been done in the last years to study the structure of stationary absolutely regular processes and to find conditions (imposed on the moments and the rate of decay of $\beta(r)$) to hold the CLT, see [8], [2], [1] and references therein. However there is still a conspicuous lack of models which satisfy (2) because bounds of $\beta(r)$ are rather difficult to obtain. The purpose of this note is to establish an apparently sharp estimate of $\beta(r)$ for renewal processes which enables to derive rates of decay based on the well-known convergence rates in Blackwell's renewal theorem, see e.g. [3], [5] and [6]. The proving technique used here is quite elementary and therefore it might be of own interest. Let X_1, X_2, \dots be independent and identically distributed random variables on (Ω, \mathcal{F}, P) with distribution function F satisfying

$$(3) \quad F(0+) = 1 - F(\infty) = 0, \quad m := \int_{R_+} x dF(x) < \infty,$$

where $R_+ = (0, \infty)$. Further, let \hat{X}_1 be a positive random variable on (Ω, \mathcal{F}, P) independent of X_1, X_2, \dots with distribution function \hat{F} . Set

$$S_0 = \hat{S}_0 := 0, \quad S_n := \sum_{i=1}^n X_i, \quad \hat{S}_n := \hat{X}_1 + \sum_{i=2}^n X_i$$

for $n \geq 1$, and consider the counting measure N respectively \hat{N} given by

$$N(a, b] := \#\{n : S_n \in (a, b]\} \quad \text{and} \quad \hat{N}(a, b] := \#\{n : \hat{S}_n \in (a, b]\}$$

for $0 \leq a < b < \infty$. The connection between the counting measure N and the partial sums S_n , $n \geq 0$ can be expressed by the following identity:

$$(4) \quad \{N(0, t_1] = n_1, N(t_1, t_2] = n_2, \dots, N(t_{k-1}, t_k] = n_k\} \\ = \{S_{n_1} \leq t_1 < S_{n_1+1}, S_{n_1+n_2} \leq t_2 < S_{n_1+n_2+1}, \dots, S_{n_1+\dots+n_k} \leq t_k < S_{n_1+\dots+n_k+1}\}$$

for $n_1, \dots, n_k \geq 0$, $0 < t_1 < \dots < t_k < \infty$, $k \geq 1$. An analogous relation holds between \hat{N} and the partial sums \hat{S}_n , $n \geq 0$. Finally, we define the Palm renewal measure H by

$$H(B) := EN(B) = \sum_{n=1}^{\infty} P(S_n \in B),$$

and the usual renewal measure \hat{H} by

$$\hat{H}(B) := E\hat{N}(B) = \sum_{n=1}^{\infty} P(\hat{S}_n \in B)$$

for any Borel set $B \subset R_+$. It is well-known from renewal theory that in case the renewal process \hat{N} is stationary, i.e. $\hat{F}(x) = \frac{1}{m} \int_0^x (1-F(t))dt$, we have

$$(5) \quad \hat{H}(B) = L(B)/m, \quad B \subset R_+,$$

where L denotes the Lebesgue measure on R_+ . For brevity we introduce the signed σ -finite measure ΔH on R_+ by

$$(6) \quad (\Delta H)(B) := H(B) - L(B)/m, \quad B \subset R_+.$$

We are now in a position to formulate the main result of this note.

Theorem. *Let $X(t) = \hat{N}(0, t]$, $t \in R_+$, be a stationary renewal process on (Ω, \mathcal{F}, P) as defined above such that (3) is satisfied. Then we have*

$$(7) \quad \beta(\mathcal{F}_a^c, \mathcal{F}_b^c) \leq 2 \sup_{0 \leq y \leq a} \int_{b-y}^{c-y} |(\Delta H)(dx)|$$

for every $a, b, c \in R_+$ with $a < b < c$.

2. Proof of Theorem

A monoton class argument shows that for the absolutely regularity coefficient on the left side of (7) it is enough to consider the expression

$$\begin{aligned} \Delta_{t_0, t_1, \dots, t_l}^{s_0, s_1, \dots, s_k} := \\ \frac{1}{2} \sum_{\substack{n_1, \dots, n_k \geq 0 \\ m_1, \dots, m_l \geq 0}} P(\hat{N}(0, s_1] = n_1, \dots, \hat{N}(s_{k-1}, s_k] = n_k, \hat{N}(t_0, t_1] = m_1, \dots, \hat{N}(t_{l-1}, t_l] = m_l) \\ - P(\hat{N}(0, s_1] = n_1, \dots, \hat{N}(s_{k-1}, s_k] = n_k) P(\hat{N}(t_0, t_1] = m_1, \dots, \hat{N}(t_{l-1}, t_l] = m_l)) \end{aligned}$$

and to find an upper bound of it which does not depend on the partitions $0 = s_0 < s_1 < \dots < s_k = a$, $b = t_0 < t_1 < \dots < t_l = c$ for $k, l \geq 1$. With the abbreviation $\max(n_1, \dots, n_k) =: n_1 \vee \dots \vee n_k$ we may write

$$\{\hat{N}(0, s_k] = 0\} = \Omega \setminus \bigcup_{n_1 \vee \dots \vee n_k \geq 1} \{\hat{N}(0, s_1] = n_1, \dots, \hat{N}(s_{k-1}, s_k] = n_k\}$$

and so together with (4) and

$$\{S_{n_1 + \dots + n_k} \leq a < S_{n_1 + \dots + n_{k+1}}\} = \{S_{n_1 + \dots + n_k} \leq a\} \setminus \{S_{n_1 + \dots + n_{k+1}} \leq a\}$$

the number $\Delta_{t_0, t_1, \dots, t_l}^{s_0, s_1, \dots, s_k}$ is less than or equal to

$$\begin{aligned}
& \sum_{\substack{n_1 \vee \dots \vee n_k \geq 1 \\ m_1, \dots, m_l \geq 0}} \left| \sum_{n \geq 0} \left\{ P(\hat{N}(0, s_1] = n_1, \dots, \hat{N}(s_{k-2}, s_{k-1}] = n_{k-1}, \hat{S}_{n_1 + \dots + n_k} \leq a, \right. \right. \\
& \quad \left. \left. \hat{S}_{n_1 + \dots + n_k + n} \leq t_0 < \hat{S}_{n_1 + \dots + n_k + n + 1}, \dots, \hat{S}_{n_1 + \dots + n_k + n + m_1 + \dots + m_l} \leq t_l \right. \right. \\
& \quad \left. \left. < \hat{S}_{n_1 + \dots + n_k + n + m_1 + \dots + m_{l+1}} \right) - P(\hat{N}(0, s_1] = n_1, \dots, \hat{N}(s_{k-2}, s_{k-1}] = n_{k-1}, \hat{S}_{n_1 + \dots + n_k} \leq a, \right. \\
& \quad \left. \left. \hat{S}_{n_1 + \dots + n_k + n} \leq t_0 < \hat{S}_{n_1 + \dots + n_k + n + 1}) P(\hat{N}(t_0, t_1] = m_1, \dots, \hat{N}(t_{l-1}, t_l] = m_l) \right\} \right| \\
& + \sum_{\substack{n_1 \vee \dots \vee n_k \geq 1 \\ m_1, \dots, m_l \geq 0}} \left| \sum_{n \geq 0} \left\{ \dots \text{with event } \{\hat{S}_{n_1 + \dots + n_{k+1}} \leq a\} \text{ instead of } \{\hat{S}_{n_1 + \dots + n_k} \leq a\} \right\} \right| \\
& = T_1 + T_2.
\end{aligned}$$

We shall first treat the sum T_1 in some detail. From the total probability rule using the independence between $(\hat{S}_1, \dots, \hat{S}_{n_1 + \dots + n_k})$ and $(\hat{S}_{n_1 + \dots + n_k + p} - \hat{S}_{n_1 + \dots + n_k}, p \geq 1)$ and the stationarity of the renewal process $X(t)$ we obtain that T_1 is equal to

$$\begin{aligned}
& \sum_{\substack{n_1 \vee \dots \vee n_k \geq 1 \\ m_1, \dots, m_l \geq 0}} \left| \sum_{n \geq 0} \int_0^a P(\hat{N}(0, s_1] = n_1, \dots, \hat{N}(s_{k-2}, s_{k-1}] = n_{k-1} / \hat{S}_{n_1 + \dots + n_k} = t) \right. \\
& \quad \times \left\{ P(S_n \leq t_0 - t < S_{n+1}, \dots, S_{n+m_1 + \dots + m_l} \leq t_l - t < S_{n+m_1 + \dots + m_{l+1}}) \right. \\
& \quad \left. - P(S_n \leq t_0 - t < S_{n+1}) P(\hat{N}(t_0 - t, t_1 - t] = m_1, \dots, \hat{N}(t_{l-1} - t, t_l - t] = m_l) \right\} \\
& \quad \times P(\hat{S}_{n_1 + \dots + n_k} \in dt) \Big| \\
& = \sum_{\substack{n_1 \vee \dots \vee n_k \geq 1 \\ m_1, \dots, m_l \geq 0}} \left| \int_0^a P(\hat{N}(0, s_1] = n_1, \dots, \hat{N}(s_{k-2}, s_{k-1}] = n_{k-1} / \hat{S}_{n_1 + \dots + n_k} = t) \right. \\
& \quad \times \delta_{0, m_1, \dots, m_l}(t_0 - t, t_1 - t, \dots, t_l - t) P(\hat{S}_{n_1 + \dots + n_k} \in dt) \Big|,
\end{aligned}$$

where $\delta_{0, m_1, \dots, m_l}(t_0 - t, t_1 - t, \dots, t_l - t)$ denotes the sum

$$\begin{aligned}
& \sum_{n \geq 0} \left\{ P\left(\bigcap_{q=0}^l \{S_{n+m_0+m_1+\dots+m_q} \leq t_q - t < S_{n+m_0+m_1+\dots+m_{q+1}}\}\right) \right. \\
& \quad \left. - P\left(\bigcap_{q=0}^l \{\hat{S}_{n+m_0+m_1+\dots+m_q} \leq t_q - t < \hat{S}_{n+m_0+m_1+\dots+m_{q+1}}\}\right) \right\}
\end{aligned}$$

with $m_0 = 0$. In the same way T_2 can be shown to be equal to

$$\begin{aligned}
& \sum_{\substack{n_1 \vee \dots \vee n_k \geq 1 \\ m_1, \dots, m_l \geq 0}} \left| \int_0^a P(\hat{N}(0, s_1] = n_1, \dots, \hat{N}(s_{k-2}, s_{k-1}] = n_{k-1} / \hat{S}_{n_1 + \dots + n_{k+1}} = t) \right. \\
& \quad \times \delta_{0, m_1, \dots, m_l}(t_0 - t, t_1 - t, \dots, t_l - t) P(\hat{S}_{n_1 + \dots + n_{k+1}} \in dt) \Big|.
\end{aligned}$$

In view of (4) and by definition of H and \hat{H} we find that

$$\begin{aligned}
 (8) \quad & \delta_{0,0,\dots,0}(t_0-t, t_1-t, \dots, t_l-t) \\
 &= \sum_{n \geq 0} \left\{ P(S_n \leq t_0-t, t_l-t < S_{n+1}) - P(\hat{S}_n \leq t_0-t, t_l-t < \hat{S}_{n+1}) \right\} \\
 &= \sum_{n \geq 1} \left\{ P(S_n \leq t_0-t) - P(\hat{S}_n \leq t_0-t) \right\} - \sum_{n \geq 1} \left\{ P(S_n \leq t_l-t) - P(\hat{S}_n \leq t_l-t) \right\} \\
 &\quad + \sum_{n \geq 1} \left\{ P(t_0-t < S_n, S_{n+1} \leq t_l-t) - P(t_0-t < \hat{S}_n, \hat{S}_{n+1} \leq t_l-t) \right\} \\
 &= \int_0^{t_0-t} (\Delta H)(ds) - \int_0^{t_l-t} (\Delta H)(ds) + \int_{t_0-t}^{t_l-t} P(S_1 \leq t_l-t-s) (\Delta H)(ds) \\
 &= - \int_{t_0-t}^{t_l-t} P(N(0, t_l-t-s] = 0) (\Delta H)(ds).
 \end{aligned}$$

Next we consider l -tuples (m_1, \dots, m_l) for which $m_1 \vee \dots \vee m_l \geq 1$. Let, for example, $m_p \geq 1$ for some $p \in \{1, \dots, l\}$ and $m_q = 0$ for $q < p$. In this case $\delta_{0, m_1, \dots, m_l}(t_0-t, t_1-t, \dots, t_l-t)$ can be rewritten as

$$\begin{aligned}
 & \sum_{n \geq 0} \left\{ P(S_n \leq t_0-t, t_{p-1}-t < S_{n+1}, S_{n+m_p} \leq t_p-t < S_{n+m_{p+1}}, \dots, \right. \\
 & \quad \left. S_{n+m_p+\dots+m_l} \leq t_l-t < S_{n+m_p+\dots+m_{l+1}}) \right. \\
 & \quad \left. - P(\hat{S}_n \leq t_0-t, t_{p-1}-t < \hat{S}_{n+1}, \hat{S}_{n+m_p} \leq t_p-t < \hat{S}_{n+m_{p+1}}, \dots, \right. \\
 & \quad \left. \hat{S}_{n+m_p+\dots+m_l} \leq t_l-t < \hat{S}_{n+m_p+\dots+m_{l+1}}) \right\} \\
 &= \sum_{n \geq 0} \left\{ P(t_{p-1}-t < S_{n+1}, S_{n+m_p} \leq t_p-t < S_{n+m_{p+1}}, \dots, \right. \\
 & \quad \left. S_{n+m_p+\dots+m_l} \leq t_l-t \leq S_{n+m_p+\dots+m_{l+1}}) \right. \\
 & \quad \left. - P(t_{p-1}-t < \hat{S}_{n+1}, \hat{S}_{n+m_p} \leq t_p-t < \hat{S}_{n+m_{p+1}}, \dots, \right. \\
 & \quad \left. \hat{S}_{n+m_p+\dots+m_l} \leq t_l-t < \hat{S}_{n+m_p+\dots+m_{l+1}}) \right\} \\
 & - \sum_{n \geq 1} \left\{ P(t_0-t < S_n \leq t_{p-1}-t, t_{p-1}-t < S_{n+1}, S_{n+m_p} \leq t_p-t < S_{n+m_{p+1}}, \dots, \right. \\
 & \quad \left. S_{n+m_p+\dots+m_l} \leq t_l-t < S_{n+m_p+\dots+m_{l+1}}) \right. \\
 & \quad \left. - P(t_0-t < \hat{S}_n \leq t_{p-1}-t, t_{p-1}-t < \hat{S}_{n+1}, \hat{S}_{n+m_p} \leq t_p-t \leq \hat{S}_{n+m_{p+1}}, \dots, \right. \\
 & \quad \left. \hat{S}_{n+m_p+\dots+m_l} \leq t_l-t < \hat{S}_{n+m_p+\dots+m_{l+1}}) \right\} \\
 & - \sum_{n \geq 1} \left\{ P(t_{p-1}-t < S_n, S_{n+m_p} \leq t_p-t < S_{n+m_{p+1}}, \dots, \right. \\
 & \quad \left. S_{n+m_p+\dots+m_l} \leq t_l-t < S_{n+m_p+\dots+m_{l+1}}) \right. \\
 & \quad \left. - P(t_{p-1}-t < \hat{S}_n, \hat{S}_{n+m_p} \leq t_p-t < \hat{S}_{n+m_{p+1}}, \dots, \right. \\
 & \quad \left. \hat{S}_{n+m_p+\dots+m_l} \leq t_l-t < \hat{S}_{n+m_p+\dots+m_{l+1}}) \right\}.
 \end{aligned}$$

Again taking into account (4) and the definition of the renewal measures H and \hat{H} we may proceed the latter equality as follows:

$$\begin{aligned}
&= \int_{t_{p-1}-t}^{t_{p-t}} P(S_{m_{p-1}} \leq t_p - t - s < S_{m_p}, \dots, \\
&\quad S_{m_p + \dots + m_{l-1}} \leq t_l - t - s < S_{m_p + \dots + m_l}) (\Delta H)(ds) \\
&- \int_{t_0-t}^{t_{p-1}-t} P(t_{p-1} - t - s < S_1, S_{m_p} \leq t_p - t - s < S_{m_{p+1}}, \dots, \\
&\quad S_{m_p + \dots + m_l} \leq t_l - t - s < S_{m_p + \dots + m_{l+1}}) (\Delta H)(ds) \\
&- \int_{t_{p-1}-t}^{t_{p-t}} P(S_{m_p} \leq t_p - t - s < S_{m_{p+1}}, \dots, \\
&\quad S_{m_p + \dots + m_l} \leq t_l - t - s < S_{m_p + \dots + m_{l+1}}) (\Delta H)(ds) \\
&= \int_{t_{p-1}-t}^{t_{p-t}} P(N(0, t_p - t - s] = m_p - 1, \dots, N(t_{l-1} - t - s, t_l - t - s] = m_l) (\Delta H)(ds) \\
&- \int_{t_0-t}^{t_{p-1}-t} P(N(0, t_{p-1} - t - s] = 0, N(t_{p-1} - t - s, t_p - t - s] = m_p, \dots, \\
&\quad N(t_{l-1} - t - s, t_l - t - s] = m_l) (\Delta H)(ds) \\
&- \int_{t_{p-1}-t}^{t_{p-t}} P(N(0, t_p - t - s] = m_p, \dots, N(t_{l-1} - t - s, t_l - t - s] = m_l) (\Delta H)(ds).
\end{aligned}$$

In the next step we estimate the sum

$$\sum_{m_1 \vee \dots \vee m_l \geq 1} |\delta_{0, m_1, \dots, m_l}(t_0 - t, t_1 - t, \dots, t_l - t)|$$

which is less than or equal to

$$\sum_{p=1}^l \sum_{m_p \geq 1, m_{p+1}, \dots, m_l \geq 0} |\delta_{0, \dots, 0, m_p, \dots, m_l}(t_0 - t, \dots, t_p - t, t_{p+1} - t, \dots, t_l - t)|.$$

From the foregoing integral representations of $\delta_{0, \dots, 0, m_p, \dots, m_l}(t_0 - t, \dots, t_p - t, t_{p+1} - t, \dots, t_l - t)$ the latter sum is easily seen to be less than or equal to the following expression:

$$\begin{aligned}
&\sum_{p=2}^l \int_{t_0-t}^{t_{p-1}-t} P(N(0, t_{p-1} - t - s] = 0, N(t_{p-1} - t - s, t_p - t - s] \geq 1) |(\Delta H)(ds)| \\
&+ \int_{t_0-t}^{t_l-t} |(\Delta H)(ds)| + \sum_{p=1}^l \int_{t_{p-1}-t}^{t_{p-t}} P(N(0, t_p - t - s] \geq 1) |(\Delta H)(ds)| \\
&= 2 \int_{b-t}^{c-t} |(\Delta H)(ds)| - \int_{b-t}^{c-t} P(N(0, c - t - s] = 0) |(\Delta H)(ds)|.
\end{aligned}$$

Combining this with (8) we arrive at

$$(9) \quad \sum_{m_1, \dots, m_l \geq 0} |\delta_{0, m_1, \dots, m_l}(t_0 - t, t_1 - t, \dots, t_l - t)| \leq 2 \int_{b-t}^{c-t} |(\Delta H)(ds)|.$$

Hence

$$\begin{aligned}
T_1 \leq 2 \sum_{n_1 \vee \dots \vee n_k \geq 1} \int_0^a P(\hat{N}(0, s_1] = n_1, \dots, \hat{N}(s_{k-2}, s_{k-1}] = n_{k-1} / \hat{S}_{n_1 + \dots + n_k} = t) \\
\times \int_{b-t}^{c-t} |(\Delta H)(ds)| P(\hat{S}_{n_1 + \dots + n_k} \in dt)
\end{aligned}$$

$$\begin{aligned} &\leq 2 \sup_{0 \leq t \leq a} \int_{b-t}^{c-t} |(\Delta H)(ds)| \sum_{n_1 \vee \dots \vee n_k \geq 1} P(\hat{N}(0, s_1] = n_1, \dots, \\ &\quad \hat{N}(s_{k-2}, s_{k-1}] = n_{k-1}, \hat{S}_{n_1 + \dots + n_k} \leq a) \\ &\leq 2 \sup_{0 \leq t \leq a} \int_{b-t}^{c-t} |(\Delta H)(ds)| \sum_{n_k \geq 0} P(\hat{N}(s_{k-1}, a] \geq n_k) \end{aligned}$$

and in the same way we find that

$$T_2 \leq 2 \sup_{0 \leq t \leq a} \int_{b-t}^{c-t} |(\Delta H)(ds)| \sum_{n_k \geq 1} P(\hat{N}(s_{k-1}, a] \geq n_k).$$

Finally, the definition of \hat{H} and (5) yield

$$(10) \quad \Delta_{t_0: t_1: \dots: t_l}^{f_0: f_1: \dots: f_l} \leq 2 \sup_{0 \leq t \leq a} \int_{b-t}^{c-t} |(\Delta H)(ds)| (1 + 2(a - s_{k-1})).$$

Since, for any $s \in (s_{k-1}, a)$ (which can be chosen arbitrarily close to a),

$$\Delta_{t_0: \dots: t_{l-1}: t_l}^{f_0: \dots: f_{l-1}: f_l} \leq \Delta_{t_0: \dots: t_{l-1}: t_l}^{f_0: \dots: f_{l-1}: f_l}$$

the estimate (10) can in fact be strengthened to

$$\Delta_{t_0: t_1: \dots: t_l}^{f_0: f_1: \dots: f_l} \leq 2 \sup_{0 \leq t \leq a} \int_{b-t}^{c-t} |(\Delta H)(ds)|$$

for arbitrary partitions $0 = s_0 \leq s_1 \leq \dots \leq s_k = a$ and $b = t_0 \leq t_1 \leq \dots \leq t_l = c$ with $k, l \geq 1$. But this provides exactly the desired inequality (7) and so our Theorem is completely proved.

3. Rates of decay of the absolute regularity coefficient

In this section we discuss some consequences of (7) which follow immediately from the error estimates in Blackwell's renewal theorem as they were obtained by Stone and Wainger [6] (see also [5] or [3]). In what follows an important role will play the notion *a distribution function F on R_+ is spread-out* which means that some convolution power of F has a non-trivial absolutely continuous component with respect to the Lebesgue measure L . Assuming that the conditions of Theorem are satisfied we obtain from (2) and (7) that

$$\beta(r) \leq 2 \int_r^\infty |(\Delta H)(ds)|$$

and consequently

$$(11) \quad \lim_{r \rightarrow \infty} \phi(r) \beta(r) = 0$$

provided that the function $\phi | R_+ \rightarrow R_+$ is non-decreasing and

$$(12) \quad \int_{R_+} \phi(s) |(\Delta H)(ds)| < \infty.$$

Corollary. Let $X(t) = \hat{N}(0, t]$, $t \in R_+$ be a stationary renewal process as in the above Theorem directed by a distribution function F which is spread-out and satisfies (3). Further, let $p \geq 0$, $0 < a \leq 1$ and $c > 0$. Then

$$(13) \quad \int_{R_+} x^{p+2} dF(x) < \infty \text{ implies } \lim_{r \rightarrow \infty} r^p \beta(r) = 0$$

and

$$(14) \quad \int_{R_+} e^{cx^a} dF(x) < \infty \text{ implies } \lim_{r \rightarrow \infty} e^{br^a} \beta(r) = 0$$

for some b , $0 < b \leq c$.

Proof. Both implication (13) and (14) follow from Theorems 3.16 and 3.18 in [3] which say that the assumption ' F is spread-out' and the conditions (13) resp. (14) are sufficient to hold

$$\int_{R_+} s^p |(\Delta H)(ds)| < \infty \text{ resp. } \int_{R_+} e^{bs^a} |(\Delta H)(ds)| < \infty$$

for some b , $0 < b \leq c$. Thus, combining this with (10) and (11) proves the Corollary.

In conclusion it should be noted that already Matthes and Nawrotzki [4] stated the absolute regularity of a stationary renewal process, i. e. $\lim_{r \rightarrow \infty} \beta(r) = 0$, whenever F is spread-out and satisfies (3). However, the technique employed there is not suitable for obtaining quantitative bounds for the rate of $\beta(r)$.

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