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A NOTE ON THE DOEBLIN-LÉVY-KOLMOGOROV-ROGOZIN INEQUALITY FOR CONCENTRATION FUNCTIONS WITH STATIONARY RANDOM VARIABLES

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Abstract. Yoshihara [10] derived bounds for the concentration functions for partial sums of strictly stationary sequences of random variables with the absolutely regular property present. In this work, bounds for uniform and strong mixing sequences of random variables are obtained. They are algebraically simple, and are of the known asymptotic form. Properties of the characteristic function are utilized here.

1. Introduction

The aim of this study is to assess the asymptotic performance order for the concentration function of the partial sums of strictly stationary random variables satisfying some of the usual mixing properties. In recent years, concentration functions have been proposed in a variety of approximation problems (Petrov, 1975, Arak and Zaitsev, 1983). Particular attention has been paid to partial sums of independent variables, as can be seen in Doeblin and Lévy [4], Kolmogorov [8], Rogozin [9] and a decade later in Kesten [7] and Esséen [5]. This is due to the close relationship between distances in the space of probability distributions and concentration functions. The main tool used to determine the concentration function's asymptotic form is properties of convolution functions, and for infinitely divisible distributions, the use of the Lévy spectral measure. Now, if independence is replaced by strictly stationary sequences of random variables, then the results of convolution functions are invalid. Even so, Yoshihara [10] has been able to provide a lower and upper bound for the concentration function for strictly stationary sequences (s.s.) of random variables satisfying the absolutely regular property. The attention of this work is focused

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on obtaining inequalities for the concentration function, but with a uniform and strong mixing property present. The proposed method is based on characteristic functions (c.f.).

2. Notations and main results

Consider a strict sense stationary sequence $X_1, X_2, \cdots$ of real valued random variables (r.v.) on some probability space $(\Omega, \mathfrak{F}, P)$, with $EX_n=0$ and $EX_n^2<\infty$. Let $\hat{F}_X(t)$, $t\in\boldsymbol{R}$, denote the characteristic function (c.f.) of a r.v. X and let $Q(X;\lambda)=\sup_x P(x\leqq X\leqq x+\lambda)$ denote its concentration function for every $\lambda\geqq 0$. For $-\infty\leqq j<t\leqq\infty$, let $\mathfrak{F}_j^t$ denote the σ-field generated by $X_j, X_{j+1}, \cdots, X_t$. Two kinds of measures of mixing are considered here. Allow $\varphi: \boldsymbol{N}\to[0, 1]$ to be a non-increasing sequence, and call the sequence $\{X_j; j\in\boldsymbol{N}\}$ uniform mixing (u.m.) if, for all $n, m\in\boldsymbol{N}$,

(2.1) $$\varphi(n)=\sup\nolimits_{A\in\mathfrak{F}_{n+m}^{\infty}}\|P(A\,|\,\mathfrak{F}_{-\infty}^{m})-P(A)\|_{\infty}\downarrow 0 \quad \text{as } n\uparrow\infty,$$

where $\|\cdot\|_p$ denotes the norm for $\mathcal{L}^p(\Omega, \mathfrak{F}, P)$. And let $\alpha: \boldsymbol{N}\to[0, 1]$ be a non-increasing sequence. The sequence $\{X_j; j\in\boldsymbol{N}\}$ is called strong mixing (s.m.) with mixing coefficient $\alpha(j)$, if

(2.2) $$\alpha(n)=1/2\sup\nolimits_{A\in\mathfrak{F}_{n,m}^{*}}\|P(A\,|\,\mathfrak{F}_{-\infty}^{m})-P(A)\|_1\downarrow 0 \quad \text{as } n\uparrow\infty.$$

In the statements and expressions below, some notations are required. The partial sums of $\{X_j; j\in\boldsymbol{N}\}$ are denoted by $S_n:=X_1+\cdots+X_n$, $n\leqq 1$. For the situations given in the conditions, theorems and in the proofs, the Bernstein's blocking decomposition is required. Define $\xi_m:=\sum_{j\,1m}X_j$, $\eta_m:=\sum_{j\,2m}X_j$, $m=0, 1, \cdots, k-1$ and $\eta_k:=\sum_{j\,3}X_j$, where $\sum_{j\,1m}$ denotes summation over j from $m(p+q)+1$ to $m(p+q)+p$, $\sum_{j\,2m}$ denotes summation over j from $m(p+q)+p+1$ to $(m+1)(p+q)$ and $\sum_{j\,3}$ denotes summation over j from $k(p+q)+1$ to n. Then, $S_n=S_n'+S_n''$, where $S_n'=\sum_{j=0}^{k-1}\xi_j$ and $S_n''=\sum_{j=0}^{k}\eta_j$. The standardized form is presented by $Z_n=Z_n'+Z_n''$, where $Z_n=S_n/s_n$, $Z_n'=S_n'/s_n$, $Z_n''=S_n''/s_n$ and $s_n^2=\mathrm{Var}(S_n)$.

Prior to stating the sufficient conditions of the results of this study, the following notations and comments are in order.

i. The measure $L(x, n)$ depends on ξ_0/s_n; it is non-decreasing in the intervals $(-\infty, 0)$ and $(0, \infty)$, and satisfies the conditions $L(-\infty, n)=0$, $L(\infty, n)=0$ and $\oint_{|x|\leq\delta}x^2dL(x, n)<\infty$, for any finite $\delta>0$. The existence of the measure $L(x, n)$ is secured from the fact that the c.f. of ξ_0 is infinitely divisible and from Theorem 5, p. 32 in Petrov (1975).

ii. The symbol $\oint$ signifies that the point zero is excluded from the domain of integration.

iii. $k=[n/(p+q)]$ and $[x]=\max\{1, \max\{j\in \boldsymbol{N}: j\leq x\}\}$.

iv. The symbol $\ll$ was initiated by Vinogradov, which is used instead of the usual 0-symbol whenever it is found convenient.

We shall consider the following set of conditions:

A. The sequence $\{X_j; j\in \boldsymbol{N}\}$ is strictly stationary with $EX_n=0$ and $EX_n^2<\infty$. For the partial sums S_n and the r.v.'s ξ_j's, we suppose that:

a) $s_n^2/n\to\sigma^2$ as $n\uparrow\infty$, for some $\sigma>0$, where $\sigma^2=EX_1^2+2\sum_{j=1}^{\infty}EX_1X_{j+1}$,

b) $\inf_{n\in N} s_n^2/n>0$,

c) $\sup\{E(S_{m+n}-S_n)^2/n;\ m, n\in \boldsymbol{N}\}<\infty$,

d) $\sup_{n\geq 1} Q(\xi_0/s_n;\lambda)=N<1$, for N some constant.

B. The sequence defined in A satisfies the following conditions:

a) ξ_0 has an infinitely divisible characteristic function,

b) for any $r^2>0$, $\inf_{n\geq 1}\{r^2+\int_{|x|\leq\lambda}x^2dL(x, n)\}=M>0$, for M some constant.

C. The sequence defined in A is a uniform mixing. With respect to integer p, q and k, and the uniform mixing coefficient, we impose the following:

a) p, q, k and p/q tend to infinite as $n\uparrow\infty$,

b) $n^{-1/4}\ll\max\{k^{-1/2}, \sqrt{q/p}, \sqrt{p/n}, k\varphi(q), \{\sum_{j\geq p}\varphi^{1/2}(j)\}^{1/2}\}\ll n^{-1/4}$.

D. The sequence defined in A is a strong mixing, with

a) for any $\beta\in(2,\infty]$ and $\gamma=2/\beta$, $\sup_{i\leq n}\|X_i\|_\beta^2<\infty$,

b) p, q, k and $p/q\uparrow\infty$ as $n\uparrow\infty$,

c) $n^{-1/4}\ll\max\{k^{-1/2}, \sqrt{q/p}, \sqrt{p/n}, k\alpha(q), \{\sum_{j\geq p}\alpha^{1-\gamma}(j)\}^{1/2}\ll n^{-1/4}$.

The aim of this paper is to develop Yoshihara-type results with uniform and strong mixing properties present. Our findings can be summarized as follows:

Theorem 1. *Suppose that the sequence $\{X_j; j\in \boldsymbol{N}\}$ satisfies the set of conditions in* A *and* C. *Then, for any $\lambda>0$*

$$Q(S_n/s_n;\lambda)\ll n^{-1/4}. \tag{3.1}$$

Further, if the set of conditions in B *is satisfied, then for any $\lambda>0$*

$$n^{-1/4}\ll Q(S_n/s_n;\lambda). \tag{3.2}$$

Theorem 2. *Suppose that $\{X_j; j\in \boldsymbol{N}\}$ satisfies the set of conditions in* A *and* D, *then* (3.1) *holds. Further, if the set of conditions in* B *is satisfied, then* (3.2) *holds.*

Variations of these theorems are also possible. For example, one may relax some of the conditions given above. In particular, strict stationary assumptions may be substituted by weak stationary assumptions. Of course, this may be done at the expense of algebraic simplicity. Or one may allow stable distributions with some specified exponent, as is seen in [10].

3. Auxiliary results

In establishing Theorems 1-2, we are aided by using a series of Lemmas. Specifically, the following two Lemmas are consequences of some well known results. The first was extracted from [2] (p. 17) for the uniform mixing and the second is from [3] for the strong mixing.

Lemma 1. *Let $\{X_j ; j \in N\}$ be a sequence of s.s.r.v.'s satisfying the uniform mixing property. Then, for all $k, j \in \{1, 2, \cdots, n\}$ with $k \neq j$ the following inequality holds:*

$$|EX_k X_j| \leqq 2\varphi^{1/r}(|k-j|)\|X_j\|_r \|X_k\|_s$$

for $r, s>1$ and $r^{-1}+s^{-1}=1$.

Lemma 2. *Let $\{X_j ; j \in N\}$ be a sequence of s.s.r.v's satisfying the strong mixing property. Let $\beta \in (2, \infty]$ and $\gamma=2/\beta$. Then, for all $k, j \in \{1, 2, \cdots, n\}$ with $k \neq j$, the following inequality holds:*

$$|EX_k X_j| \leqq 12\alpha^{1-\gamma}(|k-j|)\|X_k\|_\beta \|X_j\|_\beta .$$

The Lemma presented below can also be found in [1] (p. 41).

Lemma 3. *Suppose X and Y are real-valued independent variables. Then, for any $\lambda \geqq 0$,*

$$Q(X \pm Y ; \lambda) \leqq \min\{Q(X ; \lambda), Q(Y ; \lambda)\} .$$

Lemmas 4 and 5 relate concentration functions and characteristic functions through inequalities. Both of them are restatements of Lemmas 3 and 4 in Petrov (1975, pp. 38, 41).

Lemma 4. *For any real-valued random variable, with c.f. $\hat{F}(t)$, $t \in \boldsymbol{R}$ and concentration function $Q(X; \lambda)$, $\lambda \geqq 0$, we have that*

$$Q(X; \lambda) \leqq \left(\frac{96}{95}\right)^2 \max(\lambda, 1/a) \int_{|t| \leq a} |\hat{F}(t)|\, dt$$

for every $a>0$.

Lemma 5. *For any real-valued random X, with c.f. $\hat{F}(t)$, $t\in\boldsymbol{R}$ and concentration function $Q(X;\lambda)$, $\lambda\geqq 0$, we have that*

$$Q(X;\lambda)\geqq\frac{95\lambda}{256\pi(1+2a\lambda)}\int_{|t|\leq a}|\hat{F}(t)|\,dt$$

for every non-negative a.

The following Lemma can also be found in [1].

Lemma 6. *Suppose that $G(\cdot)$ is a probability measure related to a symmetric random variable X satisfying*

$$\int_{-\infty}^{\infty}\frac{x^2}{1+x^2}G(dx)<\infty,$$

then, for any $r^2\geqq 0$ and $\lambda\geqq 0$, it yields

$$\int_{|t|\leq\lambda^{-1}}\exp\left(-r^2t^2/2-\left(\int_{-\infty}^{\infty}(1-\cos tx)G(dx)\right)\right)dt\leqq c(r^2+D(X;\lambda))^{-1/2},$$

where $D(X;\lambda)=\int_{|x|\leq\lambda}x^2G(dx)+\lambda^2\int_{|x|>\lambda}G(dx)$, and $c>0$.

4. Proofs

This segment of our work investigates how some arguments used for independence are also engaged for the dependence case. Specifically, we show some key expressions employed for s.s. sequences of u.m.r.v.'s to pursue the analysis using some classical results. The same ideas can also be carried out for s.m. sequences of r.v.'s. It can be seen that the use of truncated arguments is not required here ([10]). The symbol "c" denotes a generic positive constant, not necessarily the same at each appearance, while c_j, $j=1, 2, \cdots$ denotes particular versions of c. We shall begin with the proof of Theorem 1.

Proof of Theorem 1. From the definition of Z_n it is not hard to see that

$$||E\exp(itZ_n)|-|E\exp(itZ'_n)||\leqq E|\exp(itZ''_n)-1|=2E|\sin(tZ''_n/2)|. \tag{4.1}$$

Calling upon Lemma 4 for $a=\lambda^{-1}$ and (4.1), one obtains that

$$Q(Z_n;\lambda)\leqq c_1\lambda\int_{|t|\leq\lambda^{-1}}|E\exp(itZ'_n)|\,dt+2c_1\lambda\int_{|t|\leq\lambda^{-1}}E|\sin(tZ''_n/2)|\,dt. \tag{4.2}$$

The verification of the second assertion being of order $n^{-1/4}$ will be accomplished by noting that since $|\sin x|\leqq|x|$ and since $\|\cdot\|_r\leqq\|\cdot\|_s$ for $r\leqq s$, it yields

$$2c_1\lambda\int_{|t|\leq\lambda^{-1}}E|\sin(tZ''_n/2)|\,dt\leqq 2c_1\|Z''_n\|_2/\lambda. \tag{4.3}$$

In conjunction with A(a), A(b), A(c) and Lemma 1, for $r=2$, it is quite apparent that if C holds,

$$EZ_n''^2 \leqq s_n^{-2}\sum_{j=0}^{k} E\eta_j^2 + 2s_n^{-2}\sum_{1\leq t\leq n}\sum_{t+p\leq j\leq n} |EX_tX_j| \tag{4.4}$$

$$\leqq c_2 s_n^{-2}(kq+p+q) + 4ns_n^{-2}\sum_{j>p}\varphi^{1/2}(j)\sup_{j\leq n}\|X_j\|_2^2$$

$$\ll n^{-1/2}.$$

The completion of showing that (4.3) is bounded above by an expression of order $n^{-1/4}$ is thus achieved. The proof that the first assertion is of the same order as (4.3) is slightly more complex. We proceed as follows: Via the stationary property and the arguments on p. 318 of [6], where instead of the strong mixing coefficient $\alpha(\cdot)$ we substitute the uniform mixing coefficient $\varphi(\cdot)$, it can be seen that

$$\left|\left|E\prod_{j=0}^{k-1}\exp(it\xi_j/s_n)\right| - |E\exp(it\xi_0/s_n)|^k\right| \leqq 16k\varphi(q). \tag{4.5}$$

By inserting (4.5) into the first assertion of (4.2), it follows that

$$c_1\lambda\int_{|t|\leq\lambda^{-1}}|E\exp(itZ_n')|\,dt \leqq c_1\lambda\int_{|t|\leq\lambda^{-1}}|E\exp(it\xi_0/s_n)|^k dt + 16c_1k\varphi(q). \tag{4.6}$$

Referring to condition C, the proof of the first part of Theorem 1 will be accomplished by showing that only the first term in the right hand side of (4.6) is of order $n^{-1/4}$. Define $\xi_0^* = \xi_0 - \xi_0'$, where ξ_0' is an independent and equiprobable random variable to ξ_0. Since $x \leqq e^{x-1}$ and from Lemma 6, it follows that

$$c_1\lambda\int_{|t|\leq\lambda^{-1}}|E\exp(it\xi_0/s_n)|^k dt \leqq c_1\lambda\int_{|t|\leq\lambda^{-1}}\exp(-(k/2)E(1-\cos(t\xi_0^*/s_n)))dt \tag{4.7}$$

$$\leqq c_2\lambda(kD(\xi_0^*/s_n\,;\lambda))^{-1/2}.$$

Now, because $D(\xi_0^*/s_n\,;\lambda)$ is a non-decreasing function with respect to λ, we have that

$$D(\xi_0^*/s_n\,;\lambda) \geqq D(\xi_0^*/s_n\,;\lambda/2) \geqq (\lambda^2/4)P(|\xi_0^*| > \lambda s_n/2).$$

From Lemma 3, one also observes that

$$P(|\xi_0^*| \leqq \lambda s_n/2) \leqq Q(\xi_0^*/s_n\,;\lambda) \leqq Q(\xi_0/s_n\,;\lambda).$$

Hence the proof of the first part of the Theorem is completed from A(d) and C(b). In obtaining the lower bound, we are aided by certain ideas used in Petrov (1975, p. 42). Employing (4.1), (4.5), Lemma 5 and condition B, then, as in Petrov, it follows that for $r^2 > 0$

$$Q(Z_n\,;\lambda) \geqq c_3\lambda\int_{|t|\leq\lambda^{-1}}\exp\left(-kr^2t^2 - 2k\int_{-\infty}^{\infty}(1-\cos tx)\,dL(x,n)\right)dt \tag{4.8}$$

$$-16k\varphi(q) - 2c_3\lambda\int_{|t|\leq\lambda^{-1}}E|\sin(tZ_n''/2)|\,dt.$$

Proceeding in exactly the same fashion as Petrov (1979, p. 42), it can be obtained that for $\lambda \geqq 0$, and fixed,

$$Q(Z_n;\lambda) \geqq c_4 M^{-1/2} n^{-1/4} - 16k\varphi(q) - 2c_3\lambda \int_{|t|\leq \lambda^{-1}} E|\sin(tZ_n''/2)|\,dt. \tag{4.9}$$

This completes the proof of Theorem 1.

Next we continue with the proof of Theorem 2.

Proof of Theorem 2. To prove Theorem 2, it only remains to show equivalent inequalities to (4.4) and (4.5) for strong mixing sequences of r.v.'s. The rest of the proof is exactly the same as Theorem 1, except instead of using condition C, we operate with condition D. Noting that if condition D is satisfied, it immediately follows from A(a), A(b), A(c) and Lemma 2 that, as in (4.4),

$$EZ_n''^2 \leqq c_5 s_n^{-2}(kq+p+q) + 2ns_n^{-2} \sum_{j>p} \alpha^{1-\gamma}(j) \sup_{j\leq n} \|X_j\|_\beta^2 \ll n^{-1/2}. \tag{4.10}$$

The last statement yields from D(c). Applying the same arguments as in [6] p. 318, it is clear that

$$\left| \left| E \prod_{j=0}^{k-1} \exp(it\xi_j/s_n) \right| - \left| \prod_{j=0}^{k-1} E \exp(it\xi_j/s_n) \right| \right| \leqq 16k\alpha(q). \tag{4.11}$$

This completes the proof of Theorem 2.

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