YOKOHAMA MATHEMATICAL
JOURNAL VOL. 40, 1992

LIMITING BAHAVIOR OF GENERALIZED QUADRATIC FORMS GENERATED BY REGULAR SEQUENCES III

By

KEN-ICHI YOSHIHARA

(Received November 17, 1990; Revised October 6, 1991)

Abstract. In this paper, we consider the limit distributions of sums of $\sum\sum W_n(\xi_i, \xi_j)$ when $\{\xi_i\}$ is strongly mixing and for each n, $W_n(x, y)$ admits the eigenvalue expansion.

1. Main results

Let $\{\xi_i\}$ be a strictly stationary sequence of random variables which are defined on a probability space $(\Omega, \mathcal{F}, P)$ and take values on a measurable space $(X, \mathcal{A})$, We say that $\{\xi_i\}$ satisfies the strongly mixing condition if

$$(1.1)\qquad \alpha(n)=\sup_{A\in\mathcal{M}_{-\infty}^{0},\, B\in\mathcal{M}_{n}^{\infty}} |P(AB)-P(A)P(B)| \longrightarrow 0 \quad \text{as } n\to\infty$$

and that $\{\xi_i\}$ satisfies the absolute regularity condition if

$$(1.2)\qquad \beta(n)=E\{\sup_{B\in\mathcal{M}_{n}^{\infty}} |P(B\,|\,\mathcal{M}_{-\infty}^{0})-P(B)|\} \longrightarrow 0 \quad \text{as } n\to\infty .$$

Here $\mathcal{M}_a^b$ denotes the σ-algebra generated by $\xi_a, \cdots, \xi_b$. It is obvious that if $\{\xi_i\}$ is absolutely regular, then it is strongly mixing since $\beta(n)\leqq\alpha(n)$.

In this paper, we consider only the strongly mixing case. Of course, the results obtained remain valid when $\{\xi_i\}$ is absolutely regular.

Next, let F be the distribution of ξ_1. Let L_2 be the space of all Borel measurable functions which are defined on X and are square-integrable with respect to F. For each $n(\geqq 1)$ let $W_n(x, y)$ be a symmetric kernel, i.e., a symmetric square-integrable function with respect to $F\times F$. Suppose that for all $n(\geqq 1)$ and for all $x\in X$

$$(1.3)\qquad EW_n(\xi_1, x)=0 .$$

For each $n(\geqq 1)$, let A_n be a linear operator mapping from L_2 into L_2 such that

$$(1.4)\qquad A_n : h \longrightarrow EW_n(\xi_1, \cdot)h(\xi_1) .$$

1991 Mathematics Subject Classification: Primary 60F17; Secondary 62E20.

Key words and phrases: strongly mixing, quadratic form, eigenvalue expansion, weak convergence, Wiener integral.

Let $\{h_{n,i}\}$ (with $h_{n,0}(x)=1$) and $\{\lambda_{n,i}\}$ be eigenvectors and eigenvalues of this operator, respectively. Then, it is obvious that for each n and each $i\{h_{n,i}(\xi_j): j\geqq 1\}$ satisfies the same mixing conditions as that of $\{\xi_j\}$.

We assume that for all $n(\geqq 1)$ and for all $j(\geqq 0)$ the following relations hold:

(1.5) $$|\lambda_{n,j}|\geqq|\lambda_{n,j+1}|,$$

(1.6) $$Eh_{n,j}(\xi_1)=0,\quad Eh_{n,j}^2(\xi_1)=n^{-1},$$

(1.7) $$Eh_{n,i}(\xi_1)h_{n,j}(\xi_1)=\begin{cases}1 & \text{if } i=j,\\ 0 & \text{if } i\neq j,\end{cases}$$

(1.8) $$EW_n(\xi_1, x)h_{n,j}(\xi_1)=\lambda_{n,j}h_{n,j}(x).$$

Then, we have

(1.9) $$\sum_{j=1}^{\infty}\lambda_{p,j}^2<\infty$$

and

(1.10) $$W_n(x, y)=\sum_{j=1}^{\infty}\lambda_{n,j}h_{n,j}(x)h_{n,j}(y).$$

Define processes $U_n=\{U_n(t): 0\leqq t\leqq 1\}$ by

(1.11) $$U_n(t)=\sum_{1\leqq i<j\leqq[nt]}W_n(\xi_i, \xi_j)\quad(0\leqq t\leqq 1).$$

The following theorem is a generalization of a result in [2].

Theorem. *Let $\{\xi_i\}$ be a strongly mixing strictly stationary sequence of random variables taking values in $(X, \mathcal{A})$. Suppose there exist two positive numbers ρ and $\delta(0<\delta<1)$ such that*

(1.12) $$K_0=\sup_{n\geqq 1}\sup_{j\geqq 1}E|n^{1/2}h_{n,j}(\xi_1)|^{4+\rho+\delta}<\infty,$$

(1.13) $$\sum_{n=1}^{\infty}n^{1+\rho/2}\alpha^{\delta/(4+\rho+\delta)}(n)<\infty.$$

Further, suppose the following relations hold:

(1.14) $$\lim_{n\to\infty}\lambda_{n,j}=\lambda_j\quad\textit{uniformly in } j,$$

(1.15) $$\sup_n\sum_{j=1}^{\infty}|\lambda_{n,j}|<\infty\quad\textit{and}\quad\sum_{j=1}^{\infty}|\lambda_j|<\infty,$$

(1.16) $$\lim_{n\to\infty}n\sum_{i=1}^{\infty}Eh_{n,j}(\xi_0)h_{n,j}(\xi_i)=a_j\quad\textit{uniformly in } j.$$

Finally, suppose that for each $j(\geqq 1)$

(1.17) $$\{\textstyle\sum_{i=1}^{[nt]}h_{n,j}(\xi_i): 0\leqq t\leqq 1\}$$
$$\xrightarrow{D} B_j=\{B_j(t): 0\leqq t\leqq 1\}\quad\textit{in } D[0, 1]\quad(n\to\infty),$$

where B_j's are Wiener processes such that

(1.18) $$EB_{j_1}(s)B_{j_2}(t)=\lim_{n\to\infty}\sum_{i_1=1}^{[ns]}\sum_{i_2=1}^{[nt]}Eh_{n,j_1}(\xi_{i_1})h_{n,j_2}(\xi_{i_2}).$$

Then, we have

(1.19) $$U_n\xrightarrow{D}U=\left\{\sum_{j=1}^{\infty}\lambda_j\left(\int_0^t B_j(s)dB_j(s)+a_jt\right):\ 0\leqq t\leqq 1\right\}$$

in $D[0, 1]$ $(n\to\infty)$.

As a special case, we consider the case where $W_n(x, y)$ is irrelevant to n, i.e.,

(1.20) $$W_n(x, y)=G(x, y)\quad\text{for all } n(\geqq 1).$$

Define μ_j and $g_j(x)$ by

$$\lambda_{n,j}=\mu_j,\quad h_{n,j}(x)=g_j(x)$$

and put

(1.21) $$\sigma_j^2=1+2\sum_{i=1}^{\infty}Eg_j(\xi_1)g_j(\xi_{j+1}).$$

Assume that

(1.22) $$\inf_{j\geq 1}\sigma_j^2>0.$$

Let $V_n=\{V_n(t):\ 0\leqq t\leqq 1\}$ be the element of $D[0, 1]$ defined by

(1.23) $$V_n(t)=\frac{1}{n}\sum_{1\leq i<j\leq[nt]}G(\xi_i, \xi_j)\quad(t\in[0, 1]).$$

Let $B_j'=\{B_j'(t):\ 0\leqq t\leqq 1\}$ $(j=1, 2, \cdots)$ be Wiener processes such that

(1.24) $$EB_i'(s)B_j'(t)=\lim_{n\to\infty}\frac{1}{n\sigma_i\sigma_j}\sum_{l=1}^{[ns]}\sum_{l'=1}^{[nt]}Eg_i(\xi_l)g_j(\xi_{l'})\quad(s, t\in[0, 1]).$$

Let $V=\{V(t):\ 0\leqq t\leqq 1\}$ be the element of $D[0, 1]$ defined by

(1.25) $$V(t)=\sum_{k=0}^{\infty}\frac{1}{2}\lambda_k\sigma_k^2\{(B_k'(t))^2-t\sigma_k^{-2}\}\quad(t\in[0, 1]).$$

Then, Theorem can be written as follows.

Corollary. *Let $\{\xi_j\}$ be as in Theorem. Suppose* (1.4)-(1.8) *and* (1.21) *hold and there exist two positive numbers ρ and δ $(0<\delta<1)$ such that*

(1.26) $$\sup_{j\geq 1}\|g_j(\xi_1)\|_{4+\rho+\delta}<\infty$$

and (1.12) *holds. Then*

$$V_n\xrightarrow{D}V\quad\text{in } D[0, 1] \text{ as } n\to\infty.$$

2. Auxiliary results

In what follows, c, with or without subscript, denotes an absolute constant, and put $\|\xi\|_r^r = E|\xi|^r (r \geqq 2)$ when the expectation of $|\xi|^r$ is finite. For a given triangular array $\{\xi_{n,j}\}$, let $M_{h,k}^{(n)}$ be the σ-algebra generated by $\xi_{n,j}$ $(h \leqq j \leqq k)$ where $n \geqq 1$ is an integer and $1 \leqq h < k \leqq N_n$. Put

$$\alpha_*(k) = \sup_{n \geq 1} \sum_{1 \leq h \leq N_n - k} \alpha(M_{h,n}^{(n)}, M_{h+k, N_n}^{(n)}) \longrightarrow 0 \quad (k \to \infty).$$

Lemma 2.1. *Let* $\{\eta_{n,i}: 1 \leqq i \leqq N_n, n \geqq 1\}$ *be a triangular array of strictly stationary strongly mixing sequences of zero-mean random variables. Suppose there exist two positive numbers* $r(\geqq 2)$ *and* $\delta (0 < \delta < 1)$ *such that*

$$\sup_n E|\eta_{n,1}|^{r+\delta} < \infty, \tag{2.1}$$

$$\sum_{n=1}^\infty n^{r/2-1} \alpha_*^{\delta/(r+\delta)}(n) < \infty. \tag{2.2}$$

Put

$$K_n = \max\{n\|\eta_{n,1}\|_{r+\delta}^r, n^{r/2}\|\eta_{n,1}\|_{2+\delta}^r\}. \tag{2.3}$$

Then, the following inequalities hold:

$$E|\sum_{i=1}^n \eta_{n,i}|^r \leqq cK_n, \tag{2.4}$$

$$E|\sum_{1 \leq i < j \leq n} \eta_{n,i}\eta_{n,j}|^{r/2} \leqq cK_n. \tag{2.5}$$

Proof. (2.4) is easily proved by modifying the proof of Theorem in [3]. (2.5) is obtained from (2.3) since

$$E|\sum_{1 \leq i < j \leq n} \eta_{n,i}\eta_{n,j}|^{r/2} = 2^{-r/2} E|(\sum_{i=1}^n \eta_{n,i})^2 - \sum_{i=1}^n \eta_{n,i}^2|^{r/2}$$
$$\leqq 2^{-r/2} E|(\sum_{i=1}^n \eta_{n,i})^2|^{r/2} = 2^{-r/2} E|\sum_{i=1}^n \eta_{n,i}|^r. \quad \square$$

By the methods in [6] and [7] we can prove the following lemma.

Lemma 2.2. *Let* $\{\eta_{n,i}: 1 \leqq i \leqq N_n, n \geqq 1\}$ *be a triangular array of strictly stationary strongly mixing sequences of zero-mean random variables. Suppose there exist two positive numbers* ρ *and* δ $(0 < \delta < 1)$ *such that*

$$\sup_{n \geq 1} E|N_n^{1/2}\eta_{n,1}|^{4+\rho+\delta} < \infty, \tag{2.6}$$

$$\sum_{n=1}^\infty n^{1+\rho/2}\{\alpha(n)\}^{\delta/(4+\rho+\delta)} < \infty. \tag{2.7}$$

Suppose

$$a = \lim_{n \to \infty} N_n \sum_{i=1}^{N_n} E\eta_{n,1}\eta_{n,i+1} \tag{2.8}$$

exists. Finally, suppose that

(2.9) $$\{\sum_{i=1}^{[N_n t]}\eta_{n,i}:\ 0\leqq t\leqq 1\}\xrightarrow{D}\{B(t):\ 0\leqq t\leqq 1\}\quad \text{in } D[0,1] \quad \text{as } n\to\infty.$$

Then, we have that as $n\to\infty$

(2.10) $$\{\sum_{1\leq i_1<i_2\leq[N_n t]}\eta_{n,i_1}\eta_{n,i_2}:\ 0\leqq t\leqq 1\}$$
$$\xrightarrow{D}\left\{\int_0^t B(s)dB(s)+at:\ 0\leqq t\leqq 1\right\}\quad \text{in } D[0,1].$$

Lemma 2. 2 can be easily extended to the multidimensional case as follows.

Lemma 2.3. *Let $\{(\eta_{n,j}^{(1)},\cdots,\eta_{n,j}^{(d)}):\ 1\leqq j\leqq N_n,\ n\geqq 1\}$ be a triangular array of strictly stationary strongly mixing sequence of d-dimensional zeromean random vectors. Suppose for each l $(1\leqq l\leqq d)$ $\{\eta_{n,j}^{(l)},\ 1\leqq j\leqq N_n,\ n\geqq 1\}$ satisfies conditions* (2.6)-(2.9). *Suppose further*

(2.11) $$a_l=\lim_{n\to\infty}N_n\sum_{j=1}^{N_n}E\eta_{n,1}^{(l)}\eta_{n,j+1}^{(l)}\quad(1\leqq l\leqq d)\quad(n\to\infty)$$

exist. Then, we have that as $n\to\infty$

(2.12) $$\{\sum_{l=1}^{d}\mu_l\sum_{1\leq i_1<i_2\leq[N_n t]}\eta_{n,i_1}^{(l)}\eta_{n,i_2}^{(l)}:\ 0\leqq t\leqq 1\}$$
$$\xrightarrow{D}\left\{\sum_{j=1}^{d}\int_0^t B_j(s)dB_j(s)+a_jt:\ 0\leqq t\leqq 1\right\}\quad \text{in } D[0,1]$$

where $\mu_1,\cdots,\mu_d$ are arbitrary canstants and $\{B_j(t):\ 0\leqq t\leqq 1\}(j=1,\cdots,d)$ are a collection of Wiener processes such that

(2.13) $$EB_i(s)B_j(t)=\lim_{n\to\infty}\sum_{1\leq l\leq[N_n s]}\sum_{1\leq l'\leq[N_n t]}E\eta_{n,l}^{(i)}\eta_{n,l'}^{(j)}.$$

3. Proof of Theorem

We use the method in [7].

Lemma 3.1. *Suppose conditions of Theorem are satisfied. Then, for any $\varepsilon(>0)$ and any $t(0<t\leqq 1)$*

(3.1) $$\lim_{N\to\infty}P(|\sum_{1\leq i<j\leq[nt]}\sum_{k=N}^{\infty}\lambda_{n,k}h_{n,k}(\xi_i)h_{n,k}(\xi_j)|>\varepsilon)=0$$

holds uniformly in n.

Proof. We note that

$$I_N=E|\sum_{1\leq i<j\leq[nt]}\sum_{k=N}^{\infty}\lambda_{n,k}h_{n,k}(\xi_i)h_{n,k}(\xi_j)|^2$$

$$\leqq \sum_{k=N}^{\infty} \lambda_{n,k}^2 E\,|\sum_{1\leq i<j\leq[nt]} h_{n,k}(\xi_i)h_{n,k}(\xi_j)|^2+2\sum_{k'>k\geq N}|\lambda_{n,k}|\cdot|\lambda_{n,k'}|$$

$$\times|E\{\sum_{1\leq i<j\leq[nt]} h_{n,k}(\xi_i)h_{n,k}(\xi_j)\sum_{1\leq i'<j'\leq[nt]} h_{n,k'}(\xi_{i'})h_{n,k'}(\xi_{j'})\}|\,.$$

Since by Lemma 2.1 (with $r=4$), (1.12) and (2.5)

$$E\,|\sum_{1\leq i<j\leq[nt]} h_{n,k}(\xi_i)h_{n,k}(\xi_j)|^2\leqq c$$

for all t $(0<t\leqq 1)$ and n, we have

$$I_N\leqq c(\textstyle\sum_{k=N}^{\infty}|\lambda_{n,k}|)^2 \tag{3.2}$$

for all n. Now (3.1) follows from (1.14) and (3.2). □

Lemma 3.2. *For any $\varepsilon(>0)$ and for any t $(0<t\leqq 1)$*

$$\lim_{N\to\infty} P\Big(\Big|\textstyle\sum_{k=N}^{\infty}\lambda_k\Big\{\int_0^t B_k(s)dB_k(s)+a_k t\Big\}\Big|>\varepsilon\Big)=0 \tag{3.3}$$

holds.

Proof. Since

$$E\Big|\textstyle\sum_{k=N}^{\infty}\lambda_k\Big(\int_0^t B_k(s)dB_k(s)+a_k t\Big)\Big|$$

$$\leqq \textstyle\sum_{k=N}^{\infty}|\lambda_k|\Big\{E\Big|\int_0^t B_k(s)dB_k(s)\Big|+|a_k|\Big\}\leqq c\sum_{k=N}^{\infty}|\lambda_k|\,,$$

(3.3) follows from (1.14). □

Lemma 3.3. *Suppose conditions of Theorem are satisfied. Let N be fixed arbitrary. Then, for each t $(0<t\leqq 1)$*

$$\sum_{1\leq i<j\leq[nt]}\textstyle\sum_{k=1}^{N}\lambda_{n,k}h_{n,k}(\xi_i)h_{n,k}(\xi_j) \tag{3.4}$$

$$\xrightarrow{D} \textstyle\sum_{k=1}^{N}\lambda_k\Big\{\int_0^t B_k(s)dB_k(s)+a_k t\Big\}\quad(n\to\infty).$$

Proof. (3.4) is easily obtained from (1.13), (1.16) and Lemma 2.3. □

Lemma 3.4. *Suppose conditions of Theorem are satisfied. Then, for each t $(0<t\leqq 1)$*

$$U_n(t)\longrightarrow U(t)\quad(n\to\infty). \tag{3.4}$$

Proof. Let t $(0<t\leqq 1)$ be fixed. Let ε_1 and ε_2 be arbitrary small positive numbers. Then, by Lemmas 3.1 and 3.2 we can choose N so that

$$P(|\sum_{1\leq i<j\leq[nt]}\textstyle\sum_{k=N}^{\infty}\lambda_{n,k}h_{n,k}(\xi_i)h_{n,k}(\xi_j)|>\varepsilon_2)<\varepsilon_1/2 \tag{3.5}$$

holds uniformly in n and

$$P\left(\left|\sum_{k=N}^{\infty}\lambda_k\left\{\int_0^t B_k(s)dB_k(s)+a_k t\right\}\right|>\varepsilon_2\right)<\varepsilon_1/2. \tag{3.6}$$

Therefore, we have

$$\begin{aligned}
&P\Big(\sum_{1\leq i<j\leq[nt]}\sum_{k=1}^{N}\lambda_{n,k}h_{n,k}(\xi_i)h_{n,k}(\xi_j)<u-\varepsilon_2\Big)\\
&\quad -P\left(\sum_{k=1}^{N}\lambda_k\left\{\int_0^t B_k(s)dB_k(s)+a_k t\right\}<u+\varepsilon_2\right)-\varepsilon_1\\
&\leqq P(U_n(t)<u)-P(U(t)<u) \qquad (3.7)\\
&\leqq P\Big(\sum_{1\leq i<j\leq[nt]}\sum_{k=1}^{N}\lambda_{n,k}h_{n,k}(\xi_i)h_{n,k}(\xi_j)<u+\varepsilon_2\Big)\\
&\quad -P\left(\sum_{k=1}^{N}\lambda_k\left\{\int_0^t B_k(s)dB_k(s)+a_k t\right\}<u-\varepsilon_2\right)+\varepsilon_1.
\end{aligned}$$

So, using Lemma 3.3 (with N fixed) we have

$$\begin{aligned}
&\lim_{n\to\infty}|P(U_n(t)<u)-P(U(t)<u)| \qquad (3.8)\\
&\leqq\varepsilon_1+P\left(u-\varepsilon_2\leqq\sum_{k=1}^{N}\lambda_k\left\{\int_0^t B_k(s)dB_k(s)+a_k t\right\}\leqq u+\varepsilon_2\right).
\end{aligned}$$

Now, the desired conclusion follows from (3.8), the continuity of the distribution of $\int_0^t B_k(s)dB_k(s)+a_k t$ and the arbitrariness of ε_1 and ε_2. □

Lemma 3.5. *Suppose conditions of Theorem are satisfied. Then,* $\{U_n\}$ *is tight.*

Proof. Since for any s and t $(0\leqq s<t\leqq 1)$

$$\begin{aligned}
U_n(t)-U_n(s)=\sum_{k=1}^{\infty}\lambda_{n,k}\Big\{&\sum_{i=1}^{[ns]}h_{n,k}(\xi_i)\sum_{j=[ns]+1}^{[nt]}h_{n,k}(\xi_j)\\
&+\sum_{[ns]+1\leq i<j\leq[nt]}h_{n,k}(\xi_i)h_{n,k}(\xi_j)\Big\},
\end{aligned}$$

to prove that $\{U_n\}$ is tight, it suffices to show that for all n sufficiently large

$$E\left|\sum_{k=1}^{\infty}\lambda_{n,k}\left\{\sum_{i=1}^{[ns]}h_{n,k}(\xi_i)\sum_{j=[ns]+1}^{[nt]}h_{n,k}(\xi_j)\right\}\right|^{2+\rho/2}\leqq c(t-s)^{1+\rho/4} \tag{3.9}$$

and

$$E\Big|\sum_{k=1}^{\infty}\lambda_{n,k}\Big\{\sum_{[ns]+1\leq i<j\leq[nt]}h_{n,k}(\xi_i)h_{n,k}(\xi_j)\Big\}\Big|^{2+\rho/2}\leqq c(t-s)^{1+\rho/4} \tag{3.10}$$

(cf. [1]). Let n be fixed and put $h_{n,k}(\xi_i)=\eta_{k,i}$ $(1\leqq i\leqq n,\ k\geqq 1)$. Further, put $\sum_{k=1}^{\infty}|\lambda_{n,k}|=d$, $[ns]=m$ and $[nt]=l$. Then, by the Schwarz inequality, the Jensen inequality and Lemma 2.1 (with $r=4+\rho$) we have

LHS of (3.9)

$$
\begin{aligned}
&\leqq E\{(\textstyle\sum_{k=1}^{\infty}|\lambda_{n,k}||\sum_{i=1}^{m}\eta_{k,i}|^2)(\sum_{k=1}^{\infty}|\lambda_{n,k}||\sum_{j=m+1}^{l}\eta_{k,j}|^2)\}^{1+\rho/4}\\
&\leqq [|E\{\textstyle\sum_{k=1}^{\infty}|\lambda_{n,k}||\sum_{i=1}^{m}\eta_{k,i}|^2\}^{2+\rho/2}\cdot E\{\sum_{k=1}^{\infty}|\lambda_{n,k}||\sum_{j=m+1}^{l}\eta_{k,j}|^2\}^{2+\rho/2}]^{1/2}\\
&= d^{2+\delta/2}[E\{\textstyle\sum_{k=1}^{\infty}d^{-1}|\lambda_{n,k}||\sum_{i=1}^{m}\eta_{k,i}|^2\}^{2+\rho/2}\\
&\quad\cdot E\{\textstyle\sum_{k=1}^{\infty}d^{-1}|\lambda_{n,k}||\sum_{j=m+1}^{l}\eta_{k,j}|^2\}^{2+\rho/2}]^{1/2} \qquad (3.11)\\
&\leqq d^{2+\delta/2}[E\{\textstyle\sum_{k=1}^{\infty}d^{-1}|\lambda_{n,k}||\sum_{i=1}^{m}\eta_{k,i}|^{4+\rho}\}\\
&\quad\cdot E\{\textstyle\sum_{k=1}^{\infty}d^{-1}|\lambda_{n,k}||\sum_{j=m+1}^{l}\eta_{k,j}|^{4+\rho}\}]^{1/2}\\
&\leqq c[\sup_{k\geq 1}E|\textstyle\sum_{i=1}^{m}\eta_{k,i}|^{4+\rho}]^{1/2}[\sup_{k\geq 1}E|\sum_{j=m+1}^{l}\eta_{k,j}|^{4+\rho}]^{1/2}\\
&\leqq cn^{-2-\rho/2}m^{1+\rho/4}(l-m)^{1+\rho/4}\leqq c(t-s)^{1+\rho/4}.
\end{aligned}
$$

Similarly, by the Jensen inequality and Lemma 2.2

$$
\begin{aligned}
\text{LHS of (3.10)} &\leqq d^{2+\rho/2}E\{\textstyle\sum_{k=1}^{\infty}d^{-1}|\lambda_{n,k}||\sum_{m\leq i<j\leq l}\eta_{k,i}\eta_{k,j}|^{2+\rho/2}\\
&\leqq d^{2+\rho/2}\sup_{k\geq 1}E|\textstyle\sum_{m\leq i<j\leq l}\eta_{k,i}\eta_{k,j}|^{2+\rho/2} \qquad (3.12)\\
&\leqq cn^{-2-\rho/2}(l-m)^{2+\rho/2}\leqq c(t-s)^{2+\rho/2},
\end{aligned}
$$

which implies (3.10). Thus, we have the desired conclusion. □

Proof of Theorem. By Lemma 3.4 we can easily show that each finite dimensional distribution of U_n converges weakly to that of U. Hence, from Lemma 3.5 Theorem follows. □

Acknowledgement. I would like to express my thanks to the refree for his careful reading of an earlier version of the manuscript leading to the correction of errors.

Reference

[1] P. Billingsley, *Convergence of probability measures*, Wiley, New York, 1968.

[2] Sh. A. Khashimov, On limit theorems for the distributions of generalized von Mises' functionals, *Theory Prob. Appl.*, **33** (1988), 566-571.

[3] R. Yokoyama, Moment bounds for stationary mixing sequences, *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, **52** (1980), 45-57.

[4] K. Yoshihara, Limiting behavior of generalized quadratic forms generated by absolutely regular sequences, *Proc. Fourth Prague Conf. Asymptotic Statistics*, 539-547, 1988.

[5] K. Yoshihara, Limiting behavior of generalized quadratic forms generated by absolutely regular sequences II, *Yokohama Math. J.*, **37** (1989), 110-123.

[6] K. Yoshihara, Weak convergence to some processes defined by multiple Wiener integrals, *Yokohama Math. J.*, **38** (1990), 57-75.

[7] K. Yoshihara, Limit distributions of degenerate *U*-statistics of degree two for

strongly mixing sequences, (Preprint).
[8] K. Yoshihara, Limiting behavior of *U*-statistics for strongly mixing sequences, *Yokohama Math. J.*, **39** (1991), 107-113.
[9] K. Yoshihara, *Weakly dependent stochastic sequences and their applications*, Vol. 2, *Asymptotic statistics based on weakly dependent deta*, Sanseido, Tokyo, 1992.

Department of Mathematics
Faculty of Engineering
Yokohama National University
Tokiwadai, Hodogaya-ku,
Yokohama 240 Japan