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ABOUT POLYNOMIALS ORTHOGONAL ON TWO SYMMETRIC INTERVALS

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Abstract. The investigations devoted to the theory of orthogonal polynomials discuss generally the case when the corresponding weight function is positive on the whole interval of orthogonality. First Ahieser [1] and Brzecka [2] introduced systems of orthogonal polynomials with weight functions which vanish on subsets of the interval of orthogonality with positive measure. The interest towards such systems is challenged by the possibility to get generalizations of the classical orthogonal polynomials of Jacobi, Laguerre and Hermite by means of appropriate choice of the weight function [3]. This paper considers a general class of polynomials orthogonal on two final symmetric intervals. We introduce also the corresponding functions of second kind to these polynomials, as suitable solutions of a linear second-order recurrence equation. The asymptotic properties of the polynomials and their functions of second kind allow us to give a description of the region and mode of convergence of the series on them. The representation problem and some boundary properties of such series are also discussed.

1. Introduction

Let us denote by G the class of functions $f(\theta)\geqq 0$, defined on the interval $[-\pi, \pi]$, for which the integrals

$$\int_{-\pi}^{\pi} f(\theta)d\theta>0 \qquad \text{and} \qquad \int_{-\pi}^{\pi}|\ln f(\theta)|\,d\theta$$

exist. According to [4], the corresponding function

$$D(z)=\exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln f(\theta)\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}d\theta\right\}$$

is holomorphic for $|z|<1$ and $D(0)>0$. If $f(\theta)\in G$ is the weight function on the unit circle $z=e^{i\theta}$ of the system of orthonormal polynomials $\{\varphi_n(z)\}_{n=0}^{\infty}$, it is known that

$$\varphi_n(z)=z^n\{\bar{D}(z^{-1})\}^{-1}[1+\eta(n)O(1)]$$

uniformly on n and $|z|\geqq R>1$, where $\lim_{n\to\infty}\eta(n)=0$.

In the case, when the weight function $f(\theta)$ satisfies the inequality

$$|f(\theta+\delta)-f(\theta)|<L|\ln\delta|^{-1-\lambda}, \tag{1.1}$$

then for $|z|=1$, the following asymptotic formula is valid

$$\varphi_n(z)=z^n\{\bar{D}(z^{-1})\}^{-1}+\varepsilon_n(z)$$

where $L>0$, $\lambda>0$, $|\varepsilon_n(z)|<C(\ln n)^{-\lambda}$ and $C=C(L, \lambda)>0$.

Let the polynomials $p_n^*(x)=x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n$ are orthogonal on the interval $[-1, 1]$ with respect to a weight function $w(x)$ such that $w(\cos\theta)|\sin\theta|=f(\theta)\in G$. By direct relation between the systems $\{\varphi_n(z)\}_{n=0}^{\infty}$ and $\{p_n^*(x)\}_{n=0}^{\infty}$. Szego [4] proved the validity of the following asymptotic formula

$$p_n^*(x)=(2\pi)^{-1/2}\zeta^n[\bar{D}(\zeta^{-1})]^{-1}[1+\eta(n)O(1)], \tag{1.2}$$

where $x\in C\setminus[-1, 1]$, $x=1/2(\zeta+\zeta^{-1})$, $|\zeta|>1$ and $\lim_{n\to\infty}\eta(n)=0$. In the case when $w(\cos\theta)|\sin\theta|=f(\theta)$ satisfies the inequality (1.1), it can be proved that

$$(1-x^2)^{1/4}[w(x)]^{1/2}p_n^*(x)=(2/\pi)^{1/2}\cos[n\theta+\gamma(\theta)]+O[(\ln n)^{-\lambda}]$$

uniformly on $x\in[-1, 1]$, where $|D(e^{i\theta})|^{-1}D(e^{i\theta})=e^{i\gamma(\theta)}$.

The functions

$$q_n^*(\xi)=\int_{-1}^{1}\frac{w(x)p_n^*(x)}{\xi-x}dx, \qquad \xi\in C\setminus[-1, 1]$$

are the functions of second kind corresponding to the polynomials $p_n^*(x)$. An asymptotic formula for these functions was found by Baret [5], who proved that if $\xi=1/2(\xi+\xi^{-1})$ then

$$1/2(\zeta-\zeta^{-1})q_n^*(\xi)=(2\pi)^{1/2}\zeta^{-n}D(\zeta^{-1})[1+\eta(n)O(1)] \tag{1.3}$$

uniformly on $|\zeta|\geqq r>0$.

2. Asymptotic formulas

Let us consider the polynomials $p_n^*(x)=x^n+b_1x^{n-1}+\cdots+b_{n-1}x+b_n$, orthogonal on the intervals $\sigma_\alpha=[-1, -\alpha]\cup[\alpha, 1]$ $(0\leqq\alpha<1)$ with respect to a non-negative summable weight function

$$\omega(x)=\begin{cases}\omega_1(x), & x\in[-1, -\alpha]\\ \omega_2(x), & x\in[\alpha, 1]\\ 0, & x\notin\sigma_\alpha.\end{cases}$$

Theorem 1. *If*

$$\omega_1(x)=\frac{x+s}{x-s}\omega_2(-x) \tag{2.1}$$

where $-\alpha \leqq s \leqq \alpha$ *and* $w(t)=(x+s)^{-1}\omega_2(x)$, $2x^2=(1-\alpha^2)t+\alpha^2+1$, *then*

$$P_{2n}^*(x)=[(1-\alpha^2)/2]^n p_n^*(t) .$$

Proof. If $Q(t)$ is a polynom of degree less than n then

$$\int_{-1}^{1} w(t)p_n^*(t)Q(t)dt=0$$

and hence

$$\int_{\alpha}^{1} \frac{2x}{x+s}\omega_2(x)p_n^*\Big(\frac{2x^2-\alpha^2-1}{1-\alpha^2}\Big)Q(x^2)dx=0 . \tag{2.2}$$

From (2.1) it follows immediately that for an arbitrary even polynom $Q^+(x)$ of degree less than $2n$,

$$\int_{\sigma_\alpha} \omega(x)p_n^*\Big(\frac{2x^2-\alpha^2-1}{1-\alpha^2}\Big)Q^+(x)dx=0 . \tag{2.3}$$

Further we can assure that (2.3) also holds for an arbitrary odd polynom $Q^-(x)$ of degree less than $2n$. Indeed, according to (2.2),

$$\int_{\sigma_\alpha} \omega(x)p_n^*\Big(\frac{2x^2-\alpha^2-1}{1-\alpha^2}\Big)Q^-(x)dx$$

$$=\int_{-1}^{-\alpha} x\omega_1(x)p_n^*\Big(\frac{2x^2-\alpha^2-1}{1-\alpha^2}\Big)Q^+(x)dx+\int_{\alpha}^{1} x\omega_2(x)p_n^*\Big(\frac{2x^2-\alpha^2-1}{1-\alpha^2}\Big)Q^+(x)dx$$

$$=s\int_{\alpha}^{1} \frac{2x}{x+s}\omega_2(x)p_n^*\Big(\frac{2x^2-\alpha^2-1}{1-a^2}\Big)Q^+(x)dx=0$$

and the theorem is proved. □

Theorem 2. *Under the conditions of Theorem* 1, *if* $p(x)$ *is an even function and* $\omega_2(x)=(x+s)p(x)$, *then*

$$P_{2n+1}^*(x)=\frac{P_{2n+2}^*(x)-m_{2n}P_{2n}^*(x)}{x+s} \tag{2.4}$$

where

$$m_{2n}=[P_{2n}^*(-s)]^{-1}P_{2n+2}^*(-s) .$$

Proof. Let $Q^+(x)$ and $Q^-(x)$ are the even and the odd part respectively of an arbitrary polynom $Q(x)$ of a degree less than $2n+1$. Since $\omega(x)=|x+s|p(x)$, the truth of this statement follows from the equalities

$$\int_{\sigma_\alpha} \omega(x)\frac{P_{2n+2}^*(x)-m_{2n}P_{2n}^*(x)}{x+s}Q^*(x)dx=0 ,$$

$$\int_{\sigma_\alpha} \omega(x)\frac{P_{2n+2}^*(x)-m_{2n}P_{2n}^*(x)}{x+s}Q^-(x)dx$$

$$=\int_{\alpha}^{1}\frac{2x}{x+s}\omega_2(x)[P_{2n+2}^*(x)-m_{2n}P_{2n}^*(x)]Q^+(x)dx=0\,. \quad \square$$

Our further considerations are connected with the orthogonal on σ_α polynomials $P_n(x)=\gamma_n x^n+\gamma_{n-1}x^{n-1}+\cdots+\gamma_1 x+\gamma_0$ ($\gamma_n>0$) for which

$$\int_{\sigma_\alpha}\omega(x)[P_n(x)]^2dx=1\,,$$

as well as with the orthogonal polynomials $p_n(t)=c_n t^n+c_{n-1}t^{n-1}+\cdots+c_1 t+c_0$ ($c_n>0$) for which

$$\int_{-1}^{1}w(x)[p_n(t)]^2dt=1\,.$$

Obviously $p_n^*(t)=A_n p_n(t)$, where $A_n^2=\int_{-1}^{1}w(t)p_n^*(t)t^n dt$ and $P_{2n}^*(x)=B_{2n}P_{2n}(x)$ where $B_{2n}^2=\int_{\sigma_\alpha}|x+s|p(x)P_{2n}^*(x)x^{2n}dx$. From Theorem 1 and Theorem 2 we conclude that

$$B_{2n}^2=[(1-\alpha^2)/2]^{2n+1}\int_{-1}^{1}w(t)p_n^*(t)t^n dt\,,$$

i.e., $B_{2n}=[(1-\alpha^2)/2]^{n+1/2}A_n$. This allows us to establish that

$$P_{2n}(x)=[(1-\alpha^2)/2]^{-1/2}p_n\left(\frac{2x^2-\alpha^2-1}{1-\alpha^2}\right).$$

From this equality and the asymptotic formula (1.2) we get that for $z\in C\setminus\sigma_\alpha$,

(2.5) $$P_{2n}(z)=[(1-\alpha^2)/2]^{-1/2}(2\pi)^{-1/2}\zeta^n[D(\zeta^{-1})]^{-1}[1+\eta(n)O(\zeta)]$$

uniformly on $|\zeta|\geq r>1$, where $\zeta=\dfrac{[(z^2-\alpha^2)^{1/2}+(z^2-1)^{1/2}]^2}{1-\alpha^2}$.

By means of (2.4) and the equalities

$$P_{2n+1}^*(x)=P_{2n+1}(x)B_{2n+1},\quad B_{2n+1}^2=\int_{\sigma_\alpha}|x+s|p(x)P_{2n+1}^*(x)x^{2n+1}dx,$$

it is easy to get that,

$$\begin{aligned}B_{2n+1}^2&=\int_{\sigma_\alpha}|x+s|p(x)P_{2n+1}^*(x)x^{2n}(x+s)dx\\&=-\frac{P_{2n+2}^*(-s)}{P_{2n}^*(-s)}[(1-\alpha^2)/2]^{2n+1}\int_{-1}^{1}w(t)p_n^*(t)t^n dt\\&=-\frac{P_{2n+2}^*(-s)}{P_{2n}^*(-s)}B_{2n}^2\,.\end{aligned}$$

Then the following representation holds,

(2.6) $$P_{2n+1}(x)=\left[-\frac{B_{2n+2}P_{2n}(-s)}{B_{2n}P_{2n+2}(-s)}\right]^{1/2}$$
$$\times\left[P_{2n+2}(x)-P_{2n}(x)\frac{P_{2n+2}(-s)}{P_{2n}(-s)}\right]\Big/(x+s).$$

Using the asymptotic formula [4]

$$\int_{-1}^{1} w(t)[p_n^*(t)]^2 dt \sim 2^{1-2n}\pi\lambda(t), \qquad n\to\infty$$

where

$$\lambda(t)=\exp\left[\frac{1}{\pi}\int_{-1}^{1}\frac{\ln w(t)}{(1-t^2)^{1/2}}dt\right],$$

we conclude that $A_n \sim 2^{1/2-n}\pi^{1/2}[\lambda(t)]^{1/2}$. Hence if we denote

$$\kappa_s=\frac{[(s^2-\alpha^2)^{1/2}+(s^2-1)^{1/2}]^2}{1-\alpha^2}=\frac{P_{2n+2}(-s)}{P_{2n}(-s)},$$

it is evident that

$$-\frac{B_{2n+2}P_{2n}(-s)}{B_{2n}P_{2n+2}(-s)}\sim\left[-\frac{1-\alpha^2}{4\kappa_s}\right]^{1/2}.$$

From (2.5) and (2.6) it follows that for $z\in \boldsymbol{C}\setminus\sigma_\alpha$,

(2.7) $$P_{2n+1}(z)=K(\alpha)A(\zeta)\zeta^{n+1/2}[D(\zeta^{-1})]^{-1}[1+\eta(n)O(1)]$$

uniformly on $|\zeta|\geqq r>1$, where $K(\alpha)=(2\pi)^{-1/2}[(1-\alpha^2)/2]^{-1/2}\cdot[(\alpha^2-1)/4\kappa_s]^{1/2}$.

Let us note that a similar approach was applied to the functions of second kind

$$Q_n(z)=\int_{\sigma_\alpha}\frac{\omega(x)P_n(x)}{z-x}dx, \qquad z\in\boldsymbol{C}\setminus\sigma_\alpha .$$

Using some relations between these functions and the functions

$$q_n(\xi)=\int_{-1}^{1}\frac{w(t)p_n(t)}{\xi-t}dt, \qquad \xi\in\boldsymbol{C}\setminus[-1, 1],$$

the following asymptotic formulas have been proved [6];

(2.8) $$Q_{2n}(z)=[(1-\alpha^2)/4\pi]^{-1/2}(z+s)[1/2(\zeta-\zeta^{-1})]^{-1}$$
$$\times\zeta^{-n}D(\zeta^{-1}([1+\eta(n)O(1)], \qquad z\in\boldsymbol{C}\setminus\sigma_\alpha$$

and

(2.9) $$Q_{2n+1}(z)=4\pi K(\alpha)B(\zeta)(\zeta-\zeta^{-1})^{-1}\zeta^{-n-1/2}$$
$$\times D(\zeta^{-1})[1+\eta(n)O(1)], \qquad z\in\boldsymbol{C}\setminus\sigma_\alpha$$

uniformly on $|\zeta|\geqq r>1$, where $B(\zeta)=\zeta^{1/2}(\zeta^{-1}-\kappa_s)$.

3. Series convergence and Christoffel-Darboux formula

In this section we study the region and the mode of convergence of the series

$$\sum_{n=0}^{\infty} a_n P_n(z) \tag{3.1}$$

and

$$\sum_{n=0}^{\infty} b_n Q_n(z)\,. \tag{3.2}$$

Let us denote $\xi_\alpha(z)=\dfrac{(z^2-\alpha^2)^{1/2}+(z^2-1)^{1/2}}{(1-\alpha^2)^{1/2}}=\rho e^{i\theta}$ $(\rho>1)$. Since

$$z^2=\frac{1+\alpha^2}{2}+\frac{1-\alpha^2}{4}(\rho^2e^{2i\theta}+\rho^{-2}e^{-2i\theta}),$$

then, if $z=u+iv$, we get immediately that

$$u^2-v^2-\frac{1+\alpha^2}{2}=\frac{1-\alpha^2}{4}(\rho^2+\rho^{-2})\cos 2\theta\,,$$

$$2uv=\frac{1-\alpha^2}{4}(\rho^2+\rho^{-2})\sin 2\theta\,.$$

Let us further denote with Γ_ρ the curves

$$\frac{\left(u^2-v^2-\frac{1+\alpha^2}{2}\right)^2}{\left[\frac{1-\alpha^2}{4}(\rho^2+\rho^{-2})\right]^2}+\frac{4u^2v^2}{\left[\frac{1-\alpha^2}{4}(\rho^2+\rho^{-2})\right]^2}=1$$

and $G_\rho=\operatorname{int}\Gamma_\rho$.

Theorem 3. *If* $\limsup\limits_{n\to\infty}|a_n|^{1/n}=1/\rho<1$, *then the series* (3.1) *is absolutely and uniformly convergent in every region* G_{ρ_0} *where* $\rho_0<\rho$.

Proof. The asymptotic formulas (2.4) and (2.7) enable us to conclude that $P_n(z)\sim\xi_\alpha^n(z)$, $n\to\infty$. According to the condition of the theorem, the series (3.1) converges for $|\xi_\alpha(z)|<\rho$, i.e., for $z\in G_\rho$. Let $z_0\in G_\rho$ and $\rho_0=|\xi_\alpha(z_0)|<\rho$. The series (3.1) converges for $z=z_0$ and hence $\lim a_nP_n(z_0)=0$. Then there exists a constant $A>0$ such that $|a_nP_n(z_0)|<A$. Let z be such that $r=|\xi_\alpha(z)|<\rho_0$. It is clear that

$$|a_nP_n(z)|=|a_nP_n(z_0)|\cdot\frac{|P_n(z)|}{|P_n(z_0)|}<A\frac{|P_n(z)|}{|P_n(z_0)|}$$

and therefore $|a_nP_n(z)|<L(r/\rho_0)^n$. The last inequality leads us to the absolutely convergence of (3.1) in the region G_{ρ_0}.

Let z be such that $|\xi_\alpha(z)|=\rho_0$. It is evident that the series $\sum_{n=0}^{\infty}|a_n|\rho_0^n$ converges. For every z for which $|\xi_\alpha|\leqq\rho_0$, we have $|a_n\xi_\alpha^n(z)|\leqq|a_n|\rho_0^n$ and this explains the uniformly convergence of $\sum_{n=0}^{\infty}a_n\xi_\alpha^n(z)$ in the region G_{ρ_0}. □

The region and the mode of convergence of the series (3.2) can be discussed in a similar manner. From the formulas (2.8) and (2.9) it follows that $Q_n(z)\sim\xi_\alpha^{-n}(z)$, $n\to\infty$. If we assume that limsup $|b_n|^{1/n}=1/\rho<1$, it can be proved as in Theorem 3 that the series (3.2) converges in $C\setminus G_\rho$ and disconverges in G_ρ.

Our further considerations in this paper oblige us to note that, according to the general theory, every system of orthogonal polynomials as well as the corresponding functions of second kind are solutions of a linear second-order recurrence equation. In accordance with the notations in [7], this means that if $n\geqq1$

$$k_nP_{n+1}(z)+(z-\alpha_n)P_n(z)+k_{n-1}P_{n-1}(z)=0\,, \tag{3.3}$$

$$k_nQ_{n+1}(w)+(w-\alpha_n)Q_n(w)+k_{n-1}Q_{n-1}(w)=0\,. \tag{3.4}$$

If we define

$$\Delta_n(z,\,w)=k_n[P_n(z)Q_{n+1}(w)-P_{n+1}(z)Q_n(w)],\qquad n=0,\,1,\,2,\,\cdots,$$

then the relations (3.3) and (3.4) give further that for $n=1,\,2,\,\cdots$

$$\Delta_n(z,\,w)+(w-z)P_n(z)Q_n(w)-\Delta_{n-1}(z,\,w)=0\,. \tag{3.5}$$

Then if $\nu\geqq0$ is an integer, from (3.5) it follows that

$$\Delta_\nu(z,\,w)+(w-z)\sum_{n=0}^{\nu}P_n(z)Q_n(w)-\Delta(z,\,w)=0\,, \tag{3.6}$$

where $\Delta(z,\,w)=\Delta_0(z,\,w)+(w-z)P_0(z)Q_0(w)$. As in [7], it is easy to show that in fact $\Delta(z,\,w)=1$. Therefore, the equality (3.6) leads us to the following representation of the Cauchy kernel namely ($w\neq z$, $z\in C$, $w\in C\setminus\sigma_\alpha$)

$$\frac{1}{w-z}=\sum_{n=0}^{\nu}P_n(z)Q_n(w)+\frac{\Delta_\nu(z,\,w)}{w-z}\,, \tag{3.7}$$

where

$$\Delta_\nu(z,\,w)=k_\nu[P_\nu(z)Q_{\nu+1}(w)-P_{\nu+1}(z)Q_\nu(w)]\,. \tag{3.8}$$

We call the relation (3.7) formula of Christoffel-Darboux type for the systems $\{P_n(z)\}_{n=0}^{\infty}$ and $\{Q_n(z)\}_{n=0}^{\infty}$. In the next section we emphasize on the important role of this formula in studying the problem of representing analytic functions by series of the kind (3.1) and (3.2).

4. Series expansions

Lemma 1. *Let $\rho>1$ and φ be a complex function absolutely integrable on the contour Γ_ρ. Then the function*

$$f(z)=\frac{1}{2\pi i}\int_{\Gamma_\rho}\frac{\varphi(w)}{w-z}dw, \qquad z\in \boldsymbol{C}\setminus\Gamma_\rho \tag{4.1}$$

can be represented in G_ρ by a series (3.1) *with coefficients*

$$a_n=\frac{1}{2\pi i}\int_{\Gamma_\rho}\varphi(w)Q_n(w)dw, \qquad n=0, 1, 2, \cdots. \tag{4.2}$$

Proof. If we multiply (3.7) by $(2\pi i)^{-1}\varphi(w)$ and integrate along Γ_ρ it follows that for every $z\in\boldsymbol{C}\setminus\Gamma_\rho$,

$$f(z)=\sum_{n=0}^{\nu}a_nP_n(z)+R_\nu(z)$$

where $R_\nu(z)=\frac{1}{2\pi i}\int_{\Gamma_\rho}\frac{\varphi(w)\Delta_\nu(z, w)}{w-z}dw$ and $\{a_n\}_{n=0}^{\infty}$ are given by (4.2). Further, if $z\in G_\rho$, the asymptotic formulas (2.5), (2.7), (2.8) and (2.9) yield that

$$|R_\nu(z)|=0\left\{\left(\frac{|\xi_\alpha(z)|}{\rho}\right)^\nu\int_{\Gamma_\rho}|\varphi(w)|\,ds\right\}.$$

But $|\xi_\alpha(z)|<\rho$ when $z\in G_\rho$ and hence $\lim_{\nu\to\infty}R_\nu(z)=0$. Then we get a representation of the function $f(z)$ in the region G_ρ by a series (3.1) with coefficients (4.2). □

In the same way we can prove

Lemma 2. *Let $\rho>1$ and ψ be a complex function absolutely integrable on Γ_ρ. Then the function*

$$g(z)=\frac{1}{2\pi i}\int_{\Gamma_\rho}\frac{\psi(w)}{w-z}dw, \qquad z\in\boldsymbol{C}\setminus\Gamma_\rho$$

can be represented in G_ρ^* by a series (3.2) *with coefficients*

$$b_n=-\frac{1}{2\pi i}\int_{\Gamma_\rho}\psi(w)P_n(w)dw, \qquad n=0, 1, 2, \cdots.$$

An usual application of Cauchy integral formula and the above lemmas lead to the following statements

Theorem 4. *Let $1<\rho_0<+\infty$ and f be a complex function holomorphic in G_{ρ_0} and continuous on G_{ρ_0}. Then f can be represented in G_{ρ_0} by a series* (3.1) *with coefficients*

$$a_n=\frac{1}{2\pi i}\int_{\Gamma_\rho} f(w)Q_n(w)dw, \qquad n=0,\ 1,\ 2,\ \cdots.$$

Theorem 5. *Let* $1<\rho_0<+\infty$, f *be a complex function holomorphic in* $G^*_{\rho_0}=C\setminus G_{\rho_0}$, *continuous on* $G^*_{\rho_0}$ *and* $f(\infty)=0$. *Then* f *can be represented in* $G^*_{\rho_0}$ *by a series* (3.2) *with coefficients*

$$b_n=-\frac{1}{2\pi i}\int_{\Gamma_{\rho_0}} f(w)P_n(w)dw, \qquad n=0,\ 1,\ 2,\ \cdots.$$

On the basis of the Christoffel-Darboux formula (3.7) and the asymptotic properties discussed in Section 2, it is possible to study the behaviour of the series (3.1) at a boundary point of its region of convergence. As in the classical theory of Fourier series, suitable asymptotic formula for the partial sums will play an essential role.

Theorem 6. *Let* $1<\rho<+\infty$ *and* φ *be a complex function absolutely integrable on the contour* Γ_ρ. *If for a point* $w_0\in\Gamma_\rho$

$$\int_{\Gamma_\rho}\left|\frac{\varphi(w)-\varphi(w_0)}{w-w_0}\right|ds<+\infty,$$

then the series (3.1) *representing the function* (4.1) *in the region* G_ρ *is convergent at the point* w_0 *with a sum*

$$\frac{1}{2\pi i}\int_{\Gamma_\rho}\frac{\varphi(w)}{w-w_0}dw+\frac{1}{2}\varphi(w_0). \tag{4.3}$$

Remark. The integral in (4.3) is considered as a main value in the sense of Cauchy.

Proof. If we denote the partial sum

$$S_\nu(f,\ z)=\sum_{n=0}^{\nu} a_nP_n(z),$$

where the coefficients are given by (4.2), from Lemma 1 and Christoffel-Darboux formula (3.7) it follows that

$$S_\nu(f,\ w_0)=\frac{1}{2\pi i}\int_{\Gamma_\rho}\frac{1-\Delta_\nu(w_0,\ w)}{w-w_0}\varphi(w)dw.$$

Further if $z\in C\setminus\sigma_\alpha$ using the asymptotic formulas (2.5), (2.7), (2.8) and (2.9) as well as Stirling's formula, we can write that

$$\Delta_\nu(z,\ w)=\xi^\nu_\alpha(z)\xi^{-\nu}_\alpha(w)\,[R(z,\ w)+\delta_\nu(z,\ w)], \tag{4.4}$$

where $R(z,\ w)$ and $\{\delta_\nu(z,\ w)\}_{\nu=0}^{\infty}$ are holomorphic in the region $(C\setminus\sigma_\alpha)\times(C\setminus\sigma_\alpha)$ and moreover $\lim_{\nu\to+\infty}\delta_\nu(z,\ w)=0$ uniformly on every compact subset of this region.

The relation (4.4) gives further that $R(z, z)+\delta_\nu(z, z)=1$, therefore $R(z, z)=1$ and $\delta_\nu(z, z)=0$ $(\nu=1, 2, 3, \cdots)$. From (4.4) we get the representation

$$(4.5)\qquad S_\nu(f, w_0)=\frac{1}{2\pi i}\int_{\Gamma_\rho}\frac{1-\xi_\alpha^\nu(w_0)\xi_\alpha^{-\nu}(w)R(w_0, w)}{w-w_0}\varphi(w)dw+r_\nu(w_0),$$

where

$$r_\nu(w_0)=-\frac{1}{2\pi i}\int_{\Gamma_\rho}\frac{\xi_\alpha^\nu(w_0)\xi_\alpha^{-\nu}(w)\delta_\nu(w_0, w)}{w-w_0}\varphi(w)dw\,.$$

Since $\delta_\nu(w_0, w_0)=0$, the sequence $\{(w-w_0)^{-1}\delta_\nu(w_0, w)\}_{\nu=1}^\infty$ ($w\in \boldsymbol{C}\setminus\sigma_\alpha$ is fixed) tends uniformly to zero on every compact subset of $\boldsymbol{C}\setminus\sigma_\alpha$ and hence the sequence $\{r_\nu(w_0)\}_{\nu=1}^\infty$ tends to zero when $\nu\to+\infty$. In view of the relations

$$(4.6)\qquad \frac{1}{2\pi i}\int_{\Gamma_\rho}\frac{\varphi(w)}{w-z_0}dw=\frac{1}{2\pi i}\int_{\Gamma_\rho}\frac{\varphi(w)-\varphi(w_0)}{w-z_0}dw+\varphi(w_0),\quad z\in G_\rho,$$

$$(4.7)\qquad \frac{1}{2\pi i}\int_{\Gamma_\rho}\frac{\varphi(w)-\varphi(w_0)}{w-w_0}dw=\frac{1}{2\pi i}\int_{\Gamma_\rho}\frac{\varphi(w)}{w-w_0}dw-\frac{1}{2}\varphi(w_0),\quad w_0\in\Gamma_\rho,$$

it is sufficient to prove the statement in the case when $\varphi(w_0)=0$. By this assumption (4.5) yields that

$$(4.8)\qquad S_\nu(f\,;\, w_0)=\frac{1}{2\pi i}\int_{\Gamma_\rho}\frac{\varphi(w)}{w-w_0}dw+J_\nu(w_0)+o(1),$$

where

$$J_\nu(w_0)=-\frac{1}{2\pi i}\int_{\Gamma_\rho}(w-w_0)^{-1}\varphi(w)R(w_0, w)\xi_\alpha^\nu(w_0)\xi_\alpha^{-\nu}(w)dw\,.$$

If we substitute $[\xi_\alpha(w)]^{-1}\xi_\alpha(w_0)=\exp i\theta$ $(0\leqq\theta\leqq 2\pi)$ in the above integral, we can write that

$$J_\nu(w_0)=\int_0^{2\pi}\Phi(\theta)\exp(i\nu\theta)d\theta\,,$$

where $\Phi(\theta)$ is an absolutely integrable complex-value function. Then the Riemann-Lebesgue lemma allows us to conclude that if $\nu\to\infty$, $J_\nu(w_0)$ tends to zero and in view of (4.8) the theorem is proved. □

Corollary. *Let* $1<\rho<+\infty$ *and* f *be a complex function continuous on* $\bar{G}_\rho$ *and holomorphic in* G_ρ. *If at a point* $w_0\in\Gamma_\rho$ *this function satisfies the Holder condition, i.e.*

$$|f(w)-f(w_0)|\leqq M|w-w_0|^\mu,\qquad 0<\mu\leqq 1,\qquad w\in\Gamma_\rho,$$

then the series (3.1) *representing* f *in* G_ρ *is convergent at the point* w_0 *with a sum* $f(w_0)$.

5. Cesaro summability

We study the (C, 1)-summability of the series (3.1) at a boundary point of its region of convergence in this final section of the paper. Our main result concerning this problem is the following statement.

Theorem 7. *Let* $0\leqq\alpha<1$, $[(1-\alpha)^{-1}(1+\alpha)]^{1/2}<\rho<+\infty$ *and* φ *be a complex function absolutely integrable on* Γ_ρ. *If* φ *is continuous in a neighbourhood of a point* $w_0\in\Gamma_\rho$ *and the integral in* (4.3) *exists as a main value, then the series* (3.1) *representing the function* (4.1) *in the region* G_ρ *is* (C, 1)*-summable at a point* w_0 *with a sum given by* (4.3).

Proof. In view of the relations (4.6) and (4.7) we may suppose that $\varphi(w_0)=0$. If we denote

$$\sigma_\nu(f;\, w_0)=(\nu+1)^{-1}\sum_{n=0}^{\nu} S_\nu(f;\, w_0),$$

from (4.5) and (4.8) we get that

$$\sigma_\nu(f;\, w_0)=\frac{1}{2\pi i}\int_{\Gamma_\rho}\frac{1-(\nu+1)^{-1}\sum_{n=0}^{\nu}\xi_\alpha^n(w_0)\xi_\alpha^{-n}(w)}{w-w_0}\varphi(w)dw+o(1). \tag{5.1}$$

Let $\gamma_\rho^\nu(w_0)=\{|\xi-w_0|=\nu^{-1}\}\cap\Gamma_\rho$ $(\nu=1, 2, 3, \cdots)$ and $\Gamma_\rho^\nu(w_0)=\Gamma_\rho-\gamma_\rho^\nu(w_0)$. If we define

$$R_\nu(w_0)=\sigma_\nu(f;\, w_0)-\frac{1}{2\pi i}\int_{\Gamma_\rho^\nu(w_0)}\frac{\varphi(w)}{w-w_0}dw,$$

then, in order to prove Theorem 7, it is sufficient to show that $\lim_{\nu\to+\infty} R_\nu(w_0)=0$. In view of (5.1) we have that

$$R_\nu(w_0)=\frac{1}{2\pi i}\int_{\Gamma_\rho^\nu(w_0)}L_\nu(w_0,\, w)\varphi(w)dw+\frac{1}{2\pi i}\int_{\gamma_\rho^\nu(w_0)}K_\nu(w_0,\, w)\varphi(w)dw+o(1),$$

where

$$L_\nu(w_0,\, w)=K_\nu(w_0,\, w)-(w-w_0)^{-1}$$

and

$$K_\nu(w_0,\, w)=\frac{\nu+1-\sum_{n=0}^{\nu}\xi_\alpha^n(w_0)\xi_\alpha^{-n}(w)}{(\nu+1)(w-w_0)}.$$

If we denote $u=\xi_\alpha(w_0)\xi_\alpha^{-1}(w)$, from the representation

$$K_\nu(w_0,\, w)=\frac{1}{\nu+1}\cdot\frac{1-\xi_\alpha(w_0)\xi_\alpha^{-1}(w)}{w-w_0}\cdot\frac{\nu+1-\sum_{n=0}^{\nu}\xi_\alpha^n(w_0)\xi_\alpha^{-n}(w)}{1-\xi_\alpha(w_0)\xi_\alpha^{-1}(w)},$$

we get immediately that

$$\frac{\nu+1-\sum_{n=0}^{\nu}u^n}{1-u}=\nu+(\nu-1)u+(\nu-2)u^2+\cdots+2u^{\nu-2}+u^{\nu-1}.$$

Hence for $w \in \gamma_\rho^\nu(w_0)$

$$K_\nu(w_0, w) = o\left[\frac{1}{\nu+1}\frac{\nu(\nu+1)}{2}\right] = o(\nu). \tag{5.2}$$

If $l_\nu(w_0)$ is the lenght of the arc $\gamma_\rho^\nu(w_0)$, then in view of (5.2) and the assumption that $\varphi(w)$ is continuous at the point w_0 and $\varphi(w_0)=0$, we can write that when $\nu \to +\infty$

$$\frac{1}{2\pi i}\int_{\gamma_\rho^\nu(w_0)} K_\nu(w_0, w)\varphi(w)dw = o[\nu l_\nu(w_0)] = o(1).$$

It remains to show that if $\nu \to +\infty$, then

$$\frac{1}{2\pi i}\int_{\Gamma_\rho^\nu(w_0)} L_\nu(w_0, w)\varphi(w)dw = o(1). \tag{5.3}$$

If $\xi_1(\nu)$ and $\xi_2(\nu)$ are the endpoints of the arc $\gamma_\rho^\nu(w_0)$, let us denote

$$\theta_0 = \arg\xi_\alpha(w_0), \qquad \theta_1(\nu) = \arg[\xi_\alpha(w_0) - w_1(\nu)],$$

$$\theta_2(\nu) = \arg[w_2(\nu) - \xi_\alpha(w_0)]$$

where

$$w_k(\nu) = \frac{\{[\xi_k^2(\nu)-\alpha^2]^{1/2} + [\xi_k^2(\nu)-1]^{1/2}\}^2}{1-\alpha^2}, \quad k=1, 2.$$

After some computations we can write that

$$\frac{1}{2\pi i}\int_{\Gamma_\rho^\nu(w_0)} L_\nu(w_0, w)\varphi(w)dw = o\left\{\frac{1}{\nu+1}\int_{\theta_0+\theta_2(\nu)}^{2\pi+\theta_0-\theta_1(\nu)} (\theta-\theta_0)^{-2}\Phi(\theta)d\theta\right\},$$

where $\Phi(\theta)$ is continuous function in a neighbourhood of the point θ_0, and moreover $\Phi(\theta_0)=0$. If we define

$$\Psi(\theta) = \int_{\theta_0}^{\theta} \Phi(\theta)d\theta,$$

then Ψ is differentiable on $[\theta_0, \delta]$ for every $\delta>0$ small enough and moreover $\Psi(\theta) = o[\theta-\theta_0]$, when $\theta \to \theta_0$. Hence

$$\frac{1}{2\pi i}\int_{\Gamma_\rho^\nu(w_0)} L_\nu(w_0, w)\varphi(w)dw = o\left\{\frac{1}{\nu+1}\left[\frac{\Psi(2\pi+\theta_0-\theta_1(\nu))}{(2\pi-\theta_1(\nu))^2}\right.\right.$$

$$\left.\left. - \frac{\Psi(\theta_0+\theta_2(\nu))}{\theta_2^2(\nu)}\right] + \frac{2}{\nu+1}\int_{\theta_0+\theta_2(\nu)}^{2\pi+\theta_0-\theta_1(\nu)} (\theta-\theta_0)^{-3}\Psi(\theta)d\theta\right\} = o(1)$$

and thus (5.3) is proved. □

Corollary (Fejer's theorem). *Let* $[(1-\alpha)^{-1}(1+\alpha)]^{1/2} < \rho < +\infty$, $0 \leqq \alpha < 1$ *and* f *be a complex function continuous on* $\bar{G}_\rho$ *and holomorphic in* G_ρ. *Then the series* (3.1) *representing* f *in* G_ρ, *is* (C, 1)*-summable at every point* $w_0 \in \Gamma_\rho$ *with sum* $f(w_0)$.

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