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CONTACT RIEMANNIAN MANIFOLDS SATISFYING $R(X, \xi)\cdot R=0$

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Abstract. Locally symmetric K-contact manifolds, or semi-symmetric Sasakian manifolds, are of constant curvature 1 (cf. [10], [4], [9]). The main purpose of this paper is to extend these results and that of [3] considering semi-symmetric contact Riemannian manifolds satisfying one condition which generalizes the K-contact condition. Finally we extend some result of [4] and [11] considering contact Riemannian manifolds satisfying $R(X,\xi)\cdot S=0$ where S is the Ricci tensor.

1. Introduction

Let M be a contact Riemannian manifold and (ω, g, ϕ, ξ) its contact Riemannian structure. If the characteristic vector field ξ is of Killing, then M is called a K-contact Riemannian manifold. Further, if the curvature tensor R satisfies $R(X, Y)\xi=\omega(Y)X-\omega(X)Y$, then M is called a Sasakian manifold. Tanno [10], generalizing the corresponding result of Okumura [4] for Sasakian manifolds, proved that a locally symmetric K-contact manifold is of constant curvature 1. Takahashi [9] proved that a Sasakian manifold satisfying $R(X, Y)\cdot R=0$ for any X and Y, where $R(X, Y)$ acts on R as a derivation, is of constant curvature 1. However, we know few about geometry of an arbitrary contact Riemannian manifold satisfying $\nabla R=0$ or $R(X, Y)\cdot R=0$.

The Riemannian manifolds satisfying $R(X, Y)\cdot R=0$ are called semi-symmetric spaces and can be considered as a direct generalization of the notion of locally symmetric spaces (cf. [8]). In [3] it is proved that a three-dimensional locally symmetric contact Riemannian manifold is either flat or of constant curvature 1. Moreover recently Blair [1] has classified the locally symmetric contact Riemannian manifolds which are tangent sphere bundles; in particular the tangent sphere bundle of the Euclidean space is a locally symmetric contact Riemannian manifold but it is not of constant curvature.

In this paper we consider semi-symmetric contact Riemannian manifolds satisfying one condition on the operator $l=R(\cdot, \xi)\xi$ which generalizes the K-

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contact condition $l=-\phi^2$. On the other hand the set of all contact Riemannian structures associated to a fixed contact form is huge (in fact it is infinite dimensional), so it is natural to consider on the operator l conditions which generalize the K-contact condition. More precisely our result, which generalizes the above results of [3], [9] and [10], is the following (cf. section 3):

Let M be a contact Riemannian manifold satisfying $R(X, \xi)\cdot R=0$.

a) If $\dim M>3$ and $l=-k\phi^2$ for some function k defined on M, then either M is Sasakian and of constant curvature 1 or $l=0$;

b) if $\dim M=3$ and $\nabla_\xi l=0$, then either M is flat or is of constant curvature 1. Finally in the section 4 we extend some result of [4] and [11] considering contact Riemannian manifolds satisfying $R(X, \xi)\cdot S=0$, where S is the Ricci tensor.

Remark. After I obtained the results of the present paper, it come to my knowledge the paper [7]. We observe that our Corollary 3.2, taking account of the Remark 2.4, is a considerable improvement of Theorem 1 of [7] where it is shown that if a locally symmetric contact metric manifold has sectional curvature $K(X, \xi)=\text{const.}=c$, then either $c=0$ or M is a Sasakian manifold of constant curvature 1. Also the Theorem 3 of [7] is a consequence of our Theorem 4.1.

2. Some remarks on contact Riemanian manifolds

A contact manifold is a differentiable $(2n+1)$-manifold M equipped with a global 1-form ω such that $\omega\wedge(d\omega)^n\neq 0$ everywhere on M. It has an underlying *contact Riemannian structure*, (ω, g, ϕ, ξ) where ξ is a global vector field (called the *characteristic vector field*), ϕ a global tensor of type (1, 1) and g a Riemannian metric (called *associated metric*). These structure tensors satisfy

$$\omega(\xi)=1, \qquad \phi^2=-I+\omega\otimes\xi, \qquad \omega=g(\xi, \cdot),$$

$$d\omega(X, Y)=g(X, \phi Y), \quad \text{and} \quad g(\phi X, \phi Y)=g(X, Y)-\omega(X)\omega(Y).$$

The tensors g and ϕ are created simultaneously by polarization of $d\omega$ evaluated on a local orthonormal basis of an arbitrary metric on the contact distribution B defined by $\operatorname{Ker}\omega$.

From now on we assume M is a contact Riemannian manifold with contact Riemannian structure (ω, g, ϕ, ξ). Denoting by L and R, Lie differentiation and curvature tensor respectively we define the tensors τ, h and l by

$$\tau=L_\xi g, \quad h=\frac{1}{2}L_\xi\phi \quad \text{and} \quad l(X)=R(X, \xi)\xi.$$

The tensors τ, h and l are symmetric and satisfy (cf. [2])

(2.1)
$$\tau(X, Y)=2g(\phi X, hY),$$
$$\tau(\xi, \cdot)=h(\xi)=l(\xi)=0, \qquad \mathrm{Tr}h=\mathrm{Tr}\tau=0, \qquad h\phi=-\phi h.$$

Moreover the following formulas hold (see for example [1], [2]):

(2.2)
$$\nabla_X\xi=-\phi X-\phi hX,$$

(2.3)
$$\nabla_\xi h=\phi-\phi h^2-\phi l,$$

(2.4)
$$l=\phi l\phi-2(h^2+\phi^2),$$

(2.5)
$$\nabla_\xi\phi=0,$$

where ∇ is the Riemannian connection of g.

If ξ is a Killing vector field with respect to g, then M is said to be *K-contact manifold*. Clearly M is K-contact iff $\tau=0$ (or equivalently $h=0$). Moreover from above formulas it follows easily that M is K-contact if and only if $l=-\phi^2$. If the almost complex structure J on $M\times\boldsymbol{R}$ defined by $J(X, fd/dt)=(\phi X-f\xi, \omega(X)d/dt)$, where f is a real-valued function, is integrable, the contact structure is said to be *normal* and M is called *Sasakian*. A Sasakian manifold may be characterized by $R(X, Y)\xi=\omega(Y)X-\omega(X)Y$. Moreover a Sasakian manifold is of K-contact but the converse holds only when $\dim M=3$.

Now we give some propositions of which we need in the sequel.

Proposition 2.1. *In a contact Riemannian manifold M, the following four conditions are equivalent*:

$$\text{(i)}\ \nabla_\xi h=0, \qquad \text{(ii)}\ \nabla_\xi\tau=0, \qquad \text{(iii)}\ \nabla_\xi l=0, \qquad \text{(iv)}\ l\phi=\phi l.$$

Proof. From (2.1) and (2.5) we have

(ii)
$$(\nabla_\xi\tau)(X, Y)=2g(\phi X, (\nabla_\xi h)Y)$$

and hence (i) is equivalent to (ii). Assuming (i), from (2.3) we have

$$\phi^2+h^2=-l.$$

Differentiating this equation with respect to ξ, using (2.5), we have

$$-\nabla_\xi l=\nabla_\xi h^2=(\nabla_\xi h)\cdot h+h\cdot(\nabla_\xi h)=0.$$

Now assume (iii). Then differentiating (2.4) and (2.3) with respect to ξ, using (2.5), we have

$$\nabla_\xi h^2=0=\nabla_\xi\nabla_\xi h.$$

Consequently

$$2(\nabla_\xi h)^2=\nabla_\xi\{h\cdot\nabla_\xi h+(\nabla_\xi h)\cdot h\}=\nabla_\xi\nabla_\xi h^2=0$$

and hence, since $\nabla_\xi h$ is symmetric, we get $\nabla_\xi h=0$.

Finally, from (2.3) and (2.4), one obtains

$$2\nabla_{\xi}h=l\phi-\phi l \tag{2.6}$$

and hence (i) is equivalent to (iv).

Proposition 2.2. *In a contact Riemannian manifold M, the following three conditions are equivalent:*

(i) $$l=-k\phi^2,$$

(ii) $$\nabla_{\xi}l=0 \text{ and } h^2=(k-1)\phi^2,$$

(iii) $$K(\xi, X)=k=K(\xi, Y) \text{ for any } X, Y \text{ in } B,$$

where K denotes the sectional curvature and k is a function on M.

Moreover, when $\dim M=3$, *the condition* (ii) *becomes simply* $\nabla_{\xi}l=0$.

Proof. From (2.3) and (2.6), using Prop. 2.1, one obtains that (i) is equivalent to (ii). Now assume (iii). Fixed an arbitrary Z in B, $|Z|=1$, let $\{\xi, Z, E_i\}_i$ be an orthonormal basis. Since $X_i=(Z+E_i)/\sqrt{2}$ and $Y_i=(Z-E_i)/\sqrt{2}$ are in B with $|X_i|=|Y_i|=1$, we have $g(lX_i, X_i)=g(lY_i, Y_i)$, from which, because l is symmetric, we get $g(lZ, E_i)=0$ for any i. On the other hand $g(lZ, \xi)=0$, therefore will be $lZ=kZ$ for any Z in B and for some function k on M. Consequently $lX=-k\phi^2X$ for any X in TM. So we have (i). The implication (i)$\Rightarrow$(iii) is trivial. Finally, if $\dim M=3$, the second part of the condition (ii) is always satisfied. In fact, from (3.4) of [6] we have

$$(\nabla_{\xi}\tau)(X, X)=|X|^2\{K(\xi, \phi X)-K(\xi, X)\} \quad \text{for any } X \text{ in } B$$

and, since $\dim B=2$, the condition (iii) is equivalent to the condition

$$K(\xi, X)=K(\xi, \phi X) \quad \text{for any } X \text{ in } B.$$

Therefore in dimension 3, the condition $\nabla_{\xi}\tau=0$ (and hence $\nabla_{\xi}l=0$) is equivalent to (iii).

Proposition 2.3 ([6] p. 97). *Let M be a 3-dimensional contact Riemannian manifold. Tnen* $S(\xi, \cdot)|_B=0$ *if and only if the Ricci tensor is given by*

$$S=-\frac{1}{2}\nabla_{\xi}\tau+\left(\frac{r}{2}-1+\frac{1}{8}|\tau|^2\right)g-\left(\frac{r}{2}-3+\frac{3}{8}|\tau|^2\right)\omega\otimes\omega$$

where r denotes the scalar curvature.

Remark 2.4. By a proof similar to the proof of Prop. 2.2, it is not difficult to see that the following three properties are equivalent:

(i) $$h^2=(k-1)\phi^2,$$

(ii) $$l=-k\phi^2+\phi\nabla_{\xi}h,$$

(iii) $$K(X, \xi)+K(\phi X, \xi)=2k \text{ for any } X \text{ in } B,$$

where k is a function on M,

Remark 2.5. Given a contact Riemannian structure (ω, g, ϕ, ξ), by a *B-homothetic deformation* (often called a *D-homothetic deformation*) [12], we mean a change of the structure tensors of the form

$$\tilde{\omega}=a\omega, \qquad \tilde{g}=ag+a(a-1)\omega\otimes\omega, \qquad \tilde{\xi}=\frac{1}{a}\xi, \qquad \tilde{\phi}=\phi$$

where a is a positive constant. Then the operator l transforms in the following manner

$$\tilde{l}=l+((1-a^2)/a^2)\phi\nabla_\xi h+((a^2-1)/a^2)h^2+(2(a-1)/a^2)h\,. \tag{2.7}$$

In fact, computing $\tilde{h}$ and $\tilde{\nabla}_{\tilde{\xi}}\tilde{h}$ we get

$$\tilde{h}=\frac{1}{a}h \quad \text{and} \quad \tilde{\nabla}_{\tilde{\xi}}\tilde{h}=(2(1-a)/a^2)\phi h+(1/a^2)\nabla_\xi h\,, \tag{2.8}$$

then by (2.3) we have (2.7).

Now as an application of the formula (2.7), we can see that there exist contact Riemannian structures which satisfy the conditions of Remark 2.4 but with $K(X, \xi)$ not constant. In fact if assume $l=0$ then, using (2.8), (2.7) becomes

$$\tilde{l}=-k\tilde{\phi}^2+\tilde{\phi}\tilde{\nabla}_{\tilde{\xi}}\tilde{h}\,,$$

where $k=1-1/a^2$. So we obtain the condition (ii) of Remark 2.4; moreover $l=0$ implies that the eigenvalues of h are ± 1 and denoting by $[\pm 1]$ the corresponding eigenspaces, from (2.7) we have

$$\tilde{K}(X, \tilde{\xi})=(a-1)^2/a^2 \qquad \text{for } X\in[-1],$$

and

$$\tilde{K}(X, \tilde{\xi})=(a+3)(a-1)/a^2 \qquad \text{for } X\in[+1]\,,$$

therefore $\tilde{K}(X, \tilde{\xi})$ is not constant for X in $\tilde{B}=B$.

3. Semi-symmetric contact Riemannian manifolds

In this section we consider semi-symmetric contact Riemannianm manifolds (i.e. $R(X, Y)\cdot R=0$) and give the main results of the paper.

Theorem 3.1. *Let M be a contact Riemannian manifold with $R(X, \xi)\cdot R=0$.*
a) *If* $\dim M>3$ *and moreover holds one of the following conditions:*
(i) $l=-k\phi^2$ *for some function k on M;*
(ii) $\nabla_\xi l=0$ *and $h^2=(k-1)\phi^2$ for some function k on M;*
(iii) $K(\xi, X)=k=K(\xi, Y)$ *for any X, Y in B and for some function k on M;*
then either M is Sasakian and of constant sectional curvature 1 or $l=0$.

b) *If* $\dim M=3$ *and* $\nabla_\xi l=0$, *then either* M *is flat or it is of constant sectional curvature* 1.

The condition $\nabla R=0$ implies $R(X, Y)\cdot R=0$ and also, since $\nabla_\xi \xi=0$, $\nabla_\xi l=0$. So, from Theorem 3.1 we get the following

Corollary 3.2. *Let* M *be a locally symmetric contact Riemannian manifold.*

a) *If* $\dim M=3$, *then either* M *is flat or it is of constant sectional curvature* 1.

b) *If* $\dim M>3$ *and* $h^2=(k-1)\phi^2$ *for some function* k *on* M, *then either* $l\equiv 0$ *or* M *is Sasakian and of constant sectional curvature* 1.

Proof (of Theorem 3.1). Assume

$$R(X, \xi)\cdot R=0 \quad \text{for any } X. \tag{3.1}$$

Moreover, by Proposition 2.2, we can assume

$$lX=-k\phi^2 X=k\{X-\omega(X)\xi\} \tag{3.2}$$

for any X and for some function k on M. We assume $l\not\equiv 0$, i.e. $k\not\equiv 0$. In (3.1) $R(X, \xi)$ acts as a derivation, so we have

$$\begin{aligned} R(X, \xi)R(Y, Z)V-R(R(X, \xi)Y, Z)V& \\ -R(Y, R(X, \xi)Z)V-R(Y, Z)R(X, \xi)V&=0 \end{aligned} \tag{3.3}$$

for any X, Y, Z, V. Putting $Z=V=\xi$ in (3.3), we get

$$R(X, \xi)lY-lR(X, \xi)Y-R(Y, lX)\xi-R(Y, \xi)lX=0$$

which, using (3.2), becomes

$$kR(X, Y)\xi+kR(\xi, Y)X=k^2\{\omega(Y)X-2\omega(X)Y+g(X, Y)\xi\}. \tag{3.4}$$

Moreover, using the first Bianchi's idendity, the (3.4) yields

$$2kR(Y, X)\xi+kR(\xi, Y)X=k^2\{\omega(X)Y-2\omega(Y)X+g(X, Y)\xi\}. \tag{3.5}$$

Substracting (3.5) from (3.4), we get

$$kR(X, Y)\xi=k^2\{\omega(Y)X-\omega(X)Y\}. \tag{3.6}$$

Now, putting $V=\xi$ in (3.3) and multiplying for k, we have

$$\begin{aligned} kR(X, \xi)R(Y, Z)\xi-kR(R(X, \xi)Y, Z)\xi& \\ -kR(Y, R(X, \xi)Z)\xi-kR(Y, Z)lX&=0, \end{aligned}$$

from which, using two times (3.2) and (3.6), we obtain

$$k^2R(Y, Z)X=k^3\{g(X, Z)Y-g(X, Y)Z\}. \tag{3.7}$$

Since the function k is not identically zero, $k(x)\neq 0$ holds at some point x, then it holds on some open neighborhood U of x. So, from (3.7), we have $R(Y, Z)X=k\{g(X, Z)X-g(X, Y)Z\}$ on U and by Schur's theorem k will be constant on U and thus on M. Therefore M is of sectional curvature k=const. $\neq 0$. Then by [5] and [3], we can conclude that M is Sasakian with k=const. $=1$.

Taking account of the Prop. 2.2, remain to consider the case $\dim M=3$ with $l\equiv 0$. In this case the (3.1) is equivalent to

$$R(X, \xi)\cdot S=0 \tag{3.8}$$

where S is the Ricci tensor. From (3.8) we have

$$S(R(X, \xi)Y, \xi)=0 \qquad \text{for any } X \text{ and } Y. \tag{3.9}$$

Moreover

$$\begin{aligned} R(X, \xi)Y=&S(\xi, Y)X+\omega(Y)Q(X)-S(X, Y)\xi-g(X, Y)Q(\xi) \\ &-(r/2)\{\omega(Y)X-g(X, Y)\xi\} \end{aligned} \tag{3.10}$$

where Q is the Ricci operator and r the scalar curvature. Now we show that

$$S=(r/2)(g-\omega\otimes\omega). \tag{3.11}$$

Put $\sigma=S(\xi, \cdot)|_B$. If $\sigma=0$, since also $l=0$, (3.11) follows easily from Prop. 2.3. If suppose $\sigma_x\neq 0$ in some point x, then we can consider a vector $E\in T_x(M)$ such that $|E|=1$, $\sigma_x(E)=0$ and $\sigma_x(\phi E)=|\sigma|$. Then from (3.10) we have

$$R(E, \xi)E=-S(E, E)\xi-Q(\xi)+(r/2)\xi$$

which together with (3.9) gives

$$S(Q(\xi), \xi)=S(\xi, \xi)\{(r/2)-S(E, E)\}$$

where $S(\xi, \xi)=\mathrm{Tr}\,l=0$. Consequently

$$|\sigma|^2=S(\xi, |\sigma|\phi E)=S(\xi, Q(\xi))=0.$$

So we have (3.11) which together with (3.10) gives

$$R(X, \xi)Y=0 \qquad \text{for any } X \text{ and } Y,$$

and by the first Bianchi's identity

$$R(X, Y)\xi=0 \qquad \text{for any } X \text{ and } Y.$$

So by the proof of a theorem of Blair (cf. [2] p. 121) we obtain that M is locally the Riemannian product of a flat 2-dimensional manifold and an 1-dimensional manifold. Therefore M is locally flat.

Question. *In dimension* 3 *we have proved that* $R(X, \xi)\cdot R=0$ *and* $l=0$ *imply* $R(X, Y)\xi=0$. *Now, in arbitrary dimension, we pose the following question*:

$$R(X, \xi)\cdot R=0 \text{ (or } \nabla R=0) \quad \text{and} \quad l=0 \overset{?}{\Longrightarrow} R(X, Y)\xi=0.$$

Note that, when $\dim M=2n+1>3$ and $R(X, Y)\xi=0$, a theorem of Blair ([2] p. 121) says that M is locally the tangent sphere bundle of a flat manifold, i.e. locally the Riemannian product $E^{n+1}\times S^n(4)$ of the Euclidean space with the sphere of constant sectional curvature 4.

4. Contact Riemannian manifolds with $R(X, \xi)\cdot S=0$

Let M be a contact Riemannian manifold of dimension $2n+1>3$. Tanno [11] (resp. Okumura [4]) proved that if the following two conditions hold:

a) (M, g) is Einstein (resp. $\nabla S=0$),

b) ξ belongs to the k-nullity distribution for some real number k (resp. $k=1$), i.e. $R(X, Y)\xi=k\{\omega(Y)X-\omega(X)Y\}$,

then M is Einstein-Sasakian.

In this section we extend these results. In fact we consider in a) the more general condition $R(X, \xi)\cdot S=0$ and in b) we consider k as a function on M. More precisely our result is the following.

Theorem 4.1. *Let M be a contact Riemannian manifold. If*

a) $$R(X, \xi)\cdot S=0, \textit{ and}$$

b) $$R(X, Y)\xi=k\{\omega(Y)X-\omega(X)Y\}$$

for some function k on M, then either M is locally isometric to the Riemannian product $E^{n+1}\times S^n(4)$ *or M is a Einstein-Sasakian manifold.*

Proof. If k is identically zero, by a result of Blair ([2] p. 121) we have the first part of the theorem. So assume $k\not\equiv 0$. By b) we have

$$S(X, \xi)=2nk\omega(X) \quad \text{for any } X. \tag{4.1}$$

Moreover a) is equivalent to

$$S(R(X, \xi)Y, Z)=-S(Y, R(X, \xi)Z) \quad \text{for any } X, Y, Z. \tag{4.2}$$

Put $Z=\xi$, by (4.2) and (4.1) we have

$$2nk\omega(R(X, \xi)Y)=-S(Y, lX),$$

i.e.

$$S(Y, lX)=2nkg(Y, lX). \tag{4.3}$$

Since the condition b) implies $l=k(I-\omega\otimes\xi)$, the equation (4.3), using (4.1), becomes

(4.4) $$kS(X, Y)=2nk^2g(X, Y)$$

for any X, Y. Since the function k is not identically zero, $k(x)\neq 0$ holds at some point x, then it holds on some open neighborhood U of x. So we have $S=kg$ on U, then k will be constant on U and hence on M. Therefore M is of Einstein. So applying Theorem 5.2 of [11] we conclude that M is Einstein-Sasakian.

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