YOKOHAMA MATHEMATICAL
JOURNAL VOL. 39, 1992

COMPLETE HYPERSURFACES IN A 4-DIMENSIONAL UNIT SPHERE WITH CONSTANT MEAN CURVATURE

By

QING-MING CHENG

(Received January 16, 1991)

1. Introduction

Let M be an n-dimensional hypersurface in a unit sphere $S^{n+1}(1)$. If M is compact, minimal and $0 \leqq S \leqq n$, then Simons [9] proved $S=0$ or $S=n$, where S is the square of length of second fundamental form. Chern, do Carmo and Kobayashi [5] proved that Clifford tori are the only minimal hypersurfaces with $S=n$. Peng and Terng [8] studied the case $S=$constant and $n=3$, and proved if $S>3$, then $S \geqq 6$. In Otsuki's examples of minimal hypersurfaces in $S^{n+1}(1)$, Hu proved that there exist complete and non-compact minimal hypersurfaces in $S^{n+1}(1)$. Hence it is interesting to research the complete minimal hypersurfaces in $S^{n+1}(1)$. The author [2] and [3] generalized Chern, do Carmo and Kobayashi's Theorem and Peng and Terng's Theorem to complete case. The author and Nakagawa [4] also studied the complete hypersurfaces with constant mean curvature in $S^{n+1}(1)$.

In this paper, we generalize the result in [3] due to the author to complete hypersurfaces with constant mean curvature in $S^4(1)$.

2. Preliminaries

Let M be an n-dimensional immersed hypersurface in an $n+1$-dimensional unit sphere $S^{n+1}(1)$. We choose a local field of orthonormal frames $e_1, \cdots, e_{n+1}$ in $S^{n+1}(1)$ such that, restricted to M, the vectors $e_1, \cdots, e_n$ are tangent to M. We use the following convention on the range of indices unless otherwise stated: $A, B, C, \cdots = 1, 2, \cdots, n+1$; $i, j, k, \cdots = 1, 2, \cdots, n$. We agree that the repeated indices under a summation sign without indication are summed over the respective ranges. With respect to the frame field of $S^{n+1}(1)$ chosen above, let $\omega_1, \cdots, \omega_{n+1}$ be the dual frame. Then structure equations of $S^{n+1}(1)$ are given by

$$d\omega_A = \sum \omega_{AB} \wedge \omega_B, \qquad \omega_{AB} + \omega_{BA} = 0,$$

$$d\omega_{AB}=\sum\omega_{AC}\wedge\omega_{CB}+\Omega_{AB},$$

$$\Omega_{AB}=-(1/2)\sum K_{ABCD}\omega_C\wedge\omega_D.$$

Restricting these forms to M, we have the structure equations of M.

(2.1) $$\omega_{n+1}=0,$$

(2.2) $$\omega_{n+1\,i}=\sum h_{ij}\omega_j, \qquad h_{ij}=h_{ji},$$

(2.3) $$d\omega_i=\sum\omega_{ij}\wedge\omega_j, \qquad \omega_{ij}+\omega_{ji}=0,$$

(2.4) $$d\omega_{ij}=\sum\omega_{ik}\wedge\omega_{kj}-(1/2)\sum R_{ijkl}\omega_k\wedge\omega_l,$$

(2.5) $$R_{ijkl}=(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})+(h_{ik}h_{jl}-h_{il}h_{jk}).$$

The symmetric 2-form

$$h=\sum h_{ij}\omega_i\omega_j$$

and the scalar

(2.6) $$H=\sum h_{ii}$$

are called the second fundamental form and the mean curvature of M respectively. If $H=0$, then M is said to be minimal. Define h_{ijk}, h_{ijkl} and h_{ijkls} by, respectively,

(2.7) $$\sum h_{ijk}\omega_k=dh_{ij}+\sum h_{mj}\omega_{mi}+\sum h_{im}\omega_{mj},$$

(2.8) $$\sum h_{ijkl}\omega_l=dh_{ijk}+\sum h_{mjk}\omega_{mi}+\sum h_{imk}\omega_{mj}+\sum h_{ijm}\omega_{mk},$$

(2.9) $$\sum h_{ijkls}\omega_s=dh_{ijkl}+\sum h_{mjkl}\omega_{mi}+\sum h_{imkl}\omega_{mj}+\sum h_{ijml}\omega_{mk}+\sum h_{ijkm}\omega_{ml}.$$

Exterior differentiating (2.2) and using the structure equations, we obtain

$$\sum h_{ijk}\omega_j\wedge\omega_k=0.$$

Hence

(2.10) $$h_{ijk}=h_{ikj}.$$

Similarly, we obtain, by differentiating (2.7) and (2.8) respectively,

(2.11) $$h_{ijkl}-h_{ijlk}=\sum h_{mj}R_{mikl}+\sum h_{im}R_{mjkl},$$

(2.12) $$h_{ijkls}-h_{ijksl}=\sum h_{mjk}R_{mils}+\sum h_{imk}R_{mjls}+\sum h_{ijm}R_{mkls}.$$

For any point $p\in M$, we can choose a frame field $e_1, \cdots, e_n$ so that

(2.13) $$h_{ij}=\lambda_i\delta_{ij}.$$

Let

(2.14) $$\mu_i=H/n-\lambda_i,$$

(2.15) $$B_k=\sum\mu_i^k, \qquad f_k=\sum\lambda_i^k.$$

Then

(2.16) $$B_1=0, \quad B_2=S-H^2/n, \quad B_3=(3HS)/n-(2H^3)/n^2-f_3.$$

If both S and H are constant, by a simple and direct calculation, we can obtain

(2.17) $$-Hf_3=S(n-S)-H^2+\sum h_{ijk}^2,$$

(2.18) $$(1/2)\Delta\sum h_{ijk}^2=(2n+3-S)\sum h_{ijk}^2-3(\sum\lambda_i^2h_{ijk}^2-2\sum\lambda_i\lambda_jh_{ijk}^2)+3H\sum\lambda_ih_{ijk}^2+\sum h_{ijkl}^2,$$

(2.19) $$-H\Delta f_3=\Delta\sum h_{ijk}^2.$$

Especially, when $n=3$, we can also get

(2.20) $$\Delta f_3=6\sum\lambda_ih_{ijk}^2-3Sf_3+3Hf_4+9f_3-3HS,$$

(2.21) $$\Delta f_4=-4Sf_4-4Hf_3+4Hf_5+12f_4+8\sum\lambda_i^2h_{ijk}^2+4\sum\lambda_i\lambda_jh_{ijk}^2,$$

(2.22) $$f_4=-H^2S+H^4/6+4Hf_3/3+S^2/2,$$

(2.23) $$f_5=(5Sf_3+5H^2f_3+H^5-5H^3S)/6.$$

(2.24) $$S^3-11H^2S^2/6-6S^2+SH^4+9S+6H^2S-H^6/6-3H^2-4H^4/3$$
$$=(S-H^2/3)(S-3H^2/4-3+\sqrt{H^4+8H^2}/4)(S-3H^2/4-3-\sqrt{H^4+8H^2}/4).$$

Since $\sum\mu_i=0$, we have

(2.25) $$(-1/\sqrt{6})(S-H^2/3)^{3/2}\leqq B_3\leqq(1/\sqrt{6})(S-H^2/3)^{3/2},$$

and equality is reached if and only if two of μ_1, μ_2 and μ_3 are equal. (cf. Okumura [6])

Lemma 1. *Let M be a 3-dimensional hypersurface in $S^4(1)$ with constant mean curvature. If S=constant, then*

(2.26) $$\frac{1}{3}\Delta\sum h_{ijk}^2=2H\sum\mu_ih_{ijk}^2-\Big(S-\frac{2}{3}H^2-3\Big)\sum h_{ijk}^2+\Big(S-\frac{1}{3}H^2\Big)\Big(S-\frac{3}{4}H^2-3+\frac{1}{4}\sqrt{H^4+8H^2}\Big)\Big(S-\frac{3}{4}H^2-3-\frac{1}{4}\sqrt{H^4+8H^2}\Big),$$

(2.27) $$\sum h_{ijkl}^2=-\frac{5}{3}\sum h_{ijk}^2(\mu_i+\mu_j+\mu_k)^2+8H\sum\mu_ih_{ijk}^2+\frac{3}{2}\Big(S-\frac{2}{3}H^2-3\Big)\sum h_{ijk}^2+\frac{3}{2}\Big(S-\frac{1}{3}H^2\Big)\Big(S-\frac{3}{4}H^2-3+\frac{1}{4}\sqrt{H^4+8H^2}\Big)\Big(S-\frac{3}{4}H^2-3-\frac{1}{4}\sqrt{H^4+8H^2}\Big).$$

Proof. By (2.19) and (2.22), we get

(2.28) $$\Delta f_4=\frac{4}{3}H\Delta f_3=-\frac{4}{3}\Delta\sum h_{ijk}^2.$$

According to (2.21) and (2.28), we obtain

(2.29) $$2\sum\lambda_i^2h_{ijk}^2+\sum\lambda_i\lambda_jh_{ijk}^2=-\frac{1}{3}\Delta\sum h_{ijk}^2+Sf_4+Hf_3-Hf_5-3f_4.$$

(2.14) implies

(2.30) $$\sum\lambda_i^2h_{ijk}^2+2\sum\lambda_i\lambda_jh_{ijk}^2=\frac{H^2}{3}\sum h_{ijk}^2-2H\sum\mu_ih_{ijk}^2+\frac{1}{3}\sum(\mu_i+\mu_j+\mu_k)^2h_{ijk}^2.$$

From (2.14), (2.19) and (2.20), we have

(2.31) $$2H\sum\mu_ih_{ijk}^2=\frac{2}{3}H^2\sum h_{ijk}^2+\frac{1}{3}\Delta\sum h_{ijk}^2-HSf_3-H^2S+H^2f_4+3Hf_3.$$

Hence,

$$\begin{aligned}\frac{1}{3}\Delta\sum h_{ijk}^2&=2H\sum\mu_ih_{ijk}^2-\frac{2}{3}H^2\sum h_{ijk}^2+HSf_3+H^2S-H^2f_4-3Hf_3\\
&=2H\sum\mu_ih_{ijk}^2-\frac{2}{3}H^2\sum h_{ijk}^2+H^2S+(S-3)Hf_3\\
&\quad-H^2\Big(-H^2S+\frac{1}{6}H^4+\frac{4}{3}Hf_3+\frac{1}{2}S^2\Big)\qquad\text{(by (2.22))}\\
&=2H\sum\mu_ih_{ijk}^2-\frac{2}{3}H^2\sum h_{ijk}^2+H^4S+H^2S-\frac{1}{6}H^6-\frac{1}{2}H^2S^2+\Big(S-3-\frac{4}{3}H^2\Big)Hf_3\\
&=2H\sum\mu_ih_{ijk}^2-\frac{2}{3}H^2\sum h_{ijk}^2+H^4S+H^2S-\frac{1}{6}H^6-\frac{1}{2}H^2S^2\\
&\quad-\Big(S-3-\frac{4}{3}H^2\Big)[S(3-S)-H^2+\sum h_{ijk}^2]\qquad\text{(by (2.17))}\\
&=2H\sum\mu_ih_{ijk}^2-\Big(S-\frac{2}{3}H^2-3\Big)\sum h_{ijk}^2+H^4S+H^2S-\frac{1}{6}H^6-\frac{1}{2}H^2S^2\\
&\quad-\Big(S-3-\frac{4}{3}H^2\Big)[S(3-S)-H^2]\\
&=2H\sum\mu_ih_{ijk}^2-\Big(S-\frac{2}{3}H^2-3\Big)\sum h_{ijk}^2\\
&\quad+\Big(S-\frac{1}{3}H^2\Big)\Big(S-\frac{3}{4}H^2-3+\frac{1}{4}\sqrt{H^4+8H^2}\Big)\Big(S-\frac{3}{4}H^2-3-\frac{1}{4}\sqrt{H^4+8H^2}\Big)\\
&\qquad\text{(by (2.24)),}\end{aligned}$$

that is, (2.26) is valid.

On the other hand, according to (2.18), we have

$$
\begin{aligned}
(2.32)\quad \sum h^2_{ijkl}={}&\frac{1}{2}\Delta\sum h^2_{ijk}-5\Big(\sum\lambda_i^2h^2_{ijk}+2\sum\lambda_i\lambda_jh^2_{ijk}\Big)+4\Big(2\sum\lambda_i^2h^2_{ijk}+\\
&\sum\lambda_i\lambda_jh^2_{ijk}\Big)+3H\sum\mu_ih^2_{ijk}+(S-9-H^2)\sum h^2_{ijk}\\
={}&\frac{1}{2}\Delta\sum h^2_{ijk}-5\Big[\frac{H^2}{3}\sum h^2_{ijk}-2H\sum\mu_ih^2_{ijk}+\frac{1}{3}\sum(\mu_i+\mu_j+\mu_k)^2h^2_{ijk}\Big]\\
&+4\Big(-\frac{1}{3}\Delta\sum h^2_{ijk}+Sf_4+Hf_3-Hf_5-3f_4\Big)+3H\sum\mu_ih^2_{ijk}+(S-9-H^2)\sum h^2_{ijk}\\
&\qquad\qquad\text{(by (2.29) and (2.30))}\\
={}&-\frac{5}{3}\sum(\mu_i+\mu_j+\mu_k)^2h^2_{ijk}+13H\sum\mu_ih^2_{ijk}-\frac{5}{6}\Delta\sum h^2_{ijk}\\
&+\Big(S-9-\frac{8}{3}H^2\Big)\sum h^2_{ijk}+4Sf_4+4Hf_3-4Hf_5-12f_4\\
={}&-\frac{5}{3}\sum(\mu_i+\mu_j+\mu_k)^2h^2_{ijk}+13H\sum\mu_ih^2_{ijk}+4(S-3)f_4+4Hf_3-4Hf_5\\
&-\frac{5}{2}\Big[2H\sum\mu_ih^2_{ijk}-\Big(S-\frac{2}{3}H^2-3\Big)\sum h^2_{ijk}+\Big(S-\frac{1}{3}H^2\Big)\Big(S-\frac{3}{4}H^2-3\\
&+\frac{1}{4}\sqrt{H^4+8H^2}\Big)\Big(S-\frac{3}{4}H^2-3-\frac{1}{4}\sqrt{H^4+8H^2}\Big)\Big]+\Big(S-9-\frac{8}{3}H^2\Big)\sum h^2_{ijk}\\
&\qquad\qquad\text{(by (2.26))}\\
={}&-\frac{5}{3}\sum(\mu_i+\mu_j+\mu_k)^2h^2_{ijk}+8H\sum\mu_ih^2_{ijk}\\
&-\frac{5}{2}\Big(S-\frac{1}{3}H^2\Big)\Big(S-\frac{3}{4}H^2-3+\frac{1}{4}\sqrt{H^4+8H^2}\Big)\Big(S-\frac{3}{4}H^2-3-\frac{1}{4}\sqrt{H^4+8H^2}\Big)\\
&+4(S-3)\Big[-H^2S+\frac{1}{6}H^4+\frac{4}{3}Hf_3+\frac{1}{2}S^2\Big]+4Hf_3\\
&-4H\Big(\frac{5}{6}f_3S+\frac{5}{6}H^2f_3+\frac{1}{6}H^5-\frac{5}{6}H^3S\Big)+\Big(\frac{7}{2}S-\frac{13}{3}H^2-\frac{33}{2}\Big)\sum h^2_{ijk}\\
&\qquad\qquad\text{(by (2.22) and (2.23))}\\
={}&-\frac{5}{3}\sum(\mu_i+\mu_j+\mu_k)^2h^2_{ijk}+8H\sum\mu_ih^2_{ijk}\\
&-\frac{5}{2}\Big(S-\frac{1}{3}H^2\Big)\Big(S-\frac{3}{4}H^2-3+\frac{1}{4}\sqrt{H^4+8H^2}\Big)\Big(S-\frac{3}{4}H^2-3-\frac{1}{4}\sqrt{H^4+8H^2}\Big)\\
&+\Big(\frac{7}{2}S-\frac{13}{3}H^2-\frac{33}{2}\Big)\sum h^2_{ijk}+4(S-3)\Big(-H^2S+\frac{1}{6}H^4+\frac{1}{2}S^2\Big)\\
&-4H\Big(\frac{1}{6}H^5-\frac{5}{6}H^3S\Big)+\Big(2S-12-\frac{10}{3}H^2\Big)\Big(S(S-3)+H^2-\sum h^2_{ijk}\Big)\quad\text{(by (2.17))}
\end{aligned}
$$

$$=-\frac{5}{3}\sum(\mu_i+\mu_j+\mu_k)^2h_{ijk}^2+8H\sum\mu_ih_{ijk}^2$$

$$-\frac{5}{2}\left(S-\frac{1}{3}H^2\right)\left(S-\frac{3}{4}H^2-3+\frac{1}{4}\sqrt{H^4+8H^2}\right)$$

$$\times\left(S-\frac{3}{4}H^2-3-\frac{1}{4}\sqrt{H^4+8H^2}\right)+\frac{3}{2}\left(S-\frac{2}{3}H^2-3\right)\sum h_{ijk}^2$$

$$+4\left(S^3-6S^2+9S-\frac{11}{6}H^2S^2+6H^2S+H^4S-\frac{H^6}{6}-3H^2-\frac{4}{3}H^4\right)$$

$$=-\frac{5}{3}\sum(\mu_i+\mu_j+\mu_k)^2h_{ijk}^2+8H\sum\mu_ih_{ijk}^2+\frac{3}{2}\left(S-\frac{2}{3}H^2-3\right)\sum h_{ijk}^2$$

$$+\frac{3}{2}\left(S-\frac{1}{3}H^2\right)\left(S-\frac{3}{4}H^2-3+\frac{1}{4}\sqrt{H^4+8H^2}\right)\left(S-\frac{3}{4}H^2-3-\frac{1}{4}\sqrt{H^4+8H^2}\right)$$

(by (2.24)).

Thus (2.27) is true.

Lemma 2 (cf. [1]). *Let M be a hypersurface in $S^{n+1}(1)$ with constant principal curvatures. Let $\lambda_1, \lambda_2, \cdots, \lambda_p$ be the distinct principal curvatures of M and $m_1, \cdots, m_p$ their multiplicities. Then*

$$\sum_{j\neq i} m_j\frac{1+\lambda_i\lambda_j}{\lambda_i-\lambda_j}=0.$$

Especially, when $n=3$, we have

$$S=H^2/3,\ 3H^2/4-\sqrt{H^4+8H^2}/4+3,\ 3H^2/4+\sqrt{H^4+8H^2}/4+3 \text{ or } H^2+6.$$

Lemma 3 (cf. Omori [7] or Yau [10]). *Let M be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below. Let F be a C^2-function bounded from below on M, then there exists a sequence $\{p_m\}$ in M such that*

$$\lim_{m\to\infty} F(p_m)=\inf F,\quad \lim_{m\to\infty}|\nabla F(p_m)|=0 \quad\text{and}\quad \lim_{m\to\infty}\inf \Delta F(p_m)\geqq 0.$$

3. Theorems and their proofs

Theorem 1. *Let M be a complete hypersurface in $S^4(1)$ with constant mean curvature. If S=constant and*

$$3H^2/4-\sqrt{H^4+8H^2}/4+3<S\leqq 3H^2/4+\sqrt{H^4+8H^2}/4+3,$$

then

$$S=3H^2/4+\sqrt{H^4+8H^2}/4+3.$$

Theorem 2. *Let M be a complete hypersurface in $S^4(1)$ with constant mean* *curvature. If S=constant and*

$$3H^2/4+\sqrt{H^4+8H^2}/4+3<S\leqq H^2+6\,,$$

then $S=H^2+6$ and M is an isoparametric hypersurface if $H\neq 0$.

Corollary. *Let M be a complete hypersurface in $S^4(1)$ with constant mean curvature. If S=constant and $S\leqq H^2+6$, then $S=H^2/3$, $3H^2/4-\sqrt{H^4+8H^2}/4+3$, $3H^2/4+\sqrt{H^4+8H^2}/4+3$ or H^2+6.*

Proof. According to Theorem 1, Theorem 2 and the result due to the author and Nakagawa [4], Corollary is valid obviously.

Since H is constant, we may suppose $H\neq 0$. In fact, if $H=0$, then M is minimal. From the result in [3] due to the author, we know that Theorems 1 and 2 are true. Without loss of generality, we may assume $H>0$.

Proof of Theorem 1. If f_3=constant, then M is a hypersurface in $S^4(1)$ with constant principal curvatures. Hence Theorem 1 is valid from Lemma 2. Next we will only consider the case $f_3\neq$constant. Since S is constant, we know that f_3, $\sum h_{ijk}^2$ and $\sum h_{ijkl}^2$ are bounded from (2.17) and (2.27). And the Ricci curvature of M is bounded from below. Let $F=\sum h_{ijk}^2$. We can apply Lemma 3 to F. Hence there exists a sequence $\{p_m\}$ in M such that

$$\lim_{m\to\infty} F(p_m)=\inf F,\quad \lim_{m\to\infty} |\nabla F(p_m)|=0 \quad\text{and}\quad \liminf_{m\to\infty} \Delta F(p_m)\geqq 0\,,$$

that is,

$$\lim_{m\to\infty}\sum h_{ijk}^2(p_m)=\inf\sum h_{ijk}^2,\quad \lim_{m\to\infty}|\nabla\sum h_{ijk}^2|(p_m)=0\,, \tag{3.1}$$
$$\liminf_{m\to\infty}\Delta\sum h_{ijk}^2(p_m)\geqq 0\,.$$

According to (2.16) and (2.17), we obtain

$$\lim_{m\to\infty} f_3(p_m)=\sup f_3,\quad \lim_{m\to\infty}|\nabla f_3|=0,\qquad (\text{by } H>0) \tag{3.2}$$

$$\lim_{m\to\infty} B_3(p_m)=\inf B_3,\quad \lim_{m\to\infty}|\nabla B_3|=0\,. \tag{3.3}$$

If $\lim_{m\to\infty} B_3(p_m)=(1/\sqrt{6})(S-H^2/3)^{3/2}$, then $B_3=(1/\sqrt{6})(S-H^2/3)^{3/2}$ from (2.25) and (3.3). Hence f_3 is constant. This is a contradiction. Thus

$$\lim B_3(p_m)<(1/\sqrt{6})(S-H^2/3)^{3/2}\,. \tag{3.4}$$

Next we will prove $\lim B_3(p_m)=-(1/\sqrt{6})(S-H^2/3)^{3/2}$. In fact,

if $\lim B_3(p_m)>-(1/\sqrt{6})(S-H^2/3)^{3/2}$, then

(3.5) $$|\lim B_3(p_m)|<(1/\sqrt{6})(S-H^2/3)^{3/2}.$$

Because λ_i, h_{ijk} and h_{ijkl} are bounded from S=constant, we may suppose

(3.6) $$\lim_{m\to\infty} \lambda_i(p_m)=\lambda_i^\circ,$$

(3.7) $$\lim_{m\to\infty} h_{ijk}(p_m)=h_{ijk}^\circ,$$

(3.8) $$\lim_{m\to\infty} h_{ijkl}(p_m)=h_{ijkl}^\circ,$$

by taking subsequence of $\{p_m\}$ if necessary. Hence

(3.9) $$\lim \mu_i(p_m)=H/3-\lambda_i^\circ=\mu_i^\circ.$$

Thus, from (3.5), we have

$$\sum\mu_i^\circ=0,$$

$$\sum(\mu_i^\circ)^2=S-H^2/3,$$

$$|\sum(\mu_i^\circ)^3|<(1/\sqrt{6})(S-H^2/3)^{3/2}.$$

From $\sum\mu_i=0$, $\sum\mu_i^2=S-H^2/3$, we obtain

(3.10) $$\sum_i h_{iik}=0,$$

(3.11) $$\sum_i \mu_i h_{iik}=0.$$

(3.9) and (3.7) imply

(3.12) $$\sum_i h_{iik}^\circ=0,$$

(3.13) $$\sum_i \mu_i^\circ h_{iik}^\circ=0.$$

According to (3.3), we have

$$\lim|\nabla B_3|=\lim\sum_k(\sum_i \mu_i^2 h_{iik})^2=0.$$

Hence

(3.14) $$\sum_i(\mu_i^\circ)^2 h_{iik}^\circ=0.$$

Since μ_i° are distinct from (3.5), (3.12), (3.13) and (3.14) imply

(3.15) $$h_{iik}^\circ=0 \quad \text{for any } i \text{ and } k.$$

From Lemma 1, (3.1), (3.6), (3.7) and (3.8), we obtain

(3.16) $$(S-2H^2/3-3)\sum h^{\circ 2}_{ijk}$$
$$\leqq 2H\sum \mu^\circ_i h^{\circ 2}_{ijk}+\left(S-\frac{1}{3}H^2\right)(S-3H^2/4-3+\sqrt{H^4+8H^2}/4)$$
$$\times(S-3H^2/4-3-\sqrt{H^4+8H^2}/4),$$

(3.17) $$\sum h^{\circ 2}_{ijkl}=-\frac{5}{3}\sum(\mu^\circ_i+\mu^\circ_j+\mu^\circ_k)^2 h^{\circ 2}_{ijk}+8H\sum\mu^\circ_i h^{\circ 2}_{ijk}+\frac{3}{2}\left(S-\frac{2}{3}H^2-3\right)\sum h^{\circ 2}_{ijk}$$
$$+\frac{3}{2}\left(S-\frac{1}{3}H^2\right)(S-3H^2/4-3+\sqrt{H^4+8H^2}/4)(S-3H^2/4-3-\sqrt{H^4+8H^2}/4).$$

From (3.15) and $\sum\mu^\circ_i=0$, we have

(3.18) $$\sum(\mu^\circ_i+\mu^\circ_j+\mu^\circ_k)^2 h^{\circ 2}_{ijk}=0\,,\qquad \sum\mu^\circ_i h^{\circ 2}_{ijk}=0\,.$$

On the other hand,

$$\sum h^2_{ijkl}\geqq 3\sum_{i\neq j} h^2_{ijij}=3\sum_{i\neq j}(h_{ijij}-(1/2)t_{ij})^2+(3/4)\sum_{i\neq j}t^2_{ij}\,,$$

where $t^2_{ij}=(h_{ijij}-h_{jiji})^2=(\lambda_i-\lambda_j)^2(1+\lambda_i\lambda_j)^2$. Hence

(3.19) $$\sum h^{\circ 2}_{ijkl}\geqq(3/4)\sum(\lambda^\circ_i-\lambda^\circ_j)^2(1+\lambda^\circ_i\lambda^\circ_j)^2>0\,,$$

since λ°_1, λ°_2 and λ°_3 are distinct. (3.16), (3.17) and (3.18) yield

(3.20) $$\sum h^{\circ 2}_{ijkl}\leqq 0 \qquad \text{from } S>3H^2/4-\sqrt{H^4+8H^2}/4+3\,.$$

(3.19) and (3.20) are a contradiction. Hence

(3.21) $$\lim_{m\to\infty} B_3(p_m)=-(1/\sqrt{6})(S-H^2/3)^{3/2}\,.$$

By (2.16) and (2.17), we obtain

(3.22) $$\sum h^{\circ 2}_{ijk}=S(S-3)+H^2-H\lim_{m\to\infty} f_3(p_m) \text{ (by (3.6), (3.7) and (3.8))}$$
$$=(S-H^2)/3[\sqrt{S-H^2/3}-H/(2\sqrt{6})+\sqrt{(3/8)H^2+3}][\sqrt{S-H^2/3}$$
$$-H/(2\sqrt{6})-\sqrt{(3/8)H^2+3}] \text{ (by (3.21) and (2.16))}.$$

Since $S>3H^2/4-\sqrt{H^4+8H^2}/4+3$, we have

$$\sqrt{S-H^2/3}-H/(2\sqrt{6})+\sqrt{(3/8)H^2+3}>0, \qquad S-H^2/3>0\,.$$

Hence,

$$\sqrt{S-H^2/3}-H/(2\sqrt{6})-\sqrt{(3/8)H^2+3}\geqq 0 \quad \text{(by (3.22))}.$$

Thus $S\geqq 3H^2/4+\sqrt{H^4+8H^2}/4+3$. Therefore,

$$S=3H^2/4+\sqrt{H^4+8H^2}/4+3.$$

This completes the proof of Theorem 1.

To give the proof of Theorem 2, at first, we show that the following two propositions.

Proposition 1. *Let M be a hypersurface in $S^4(1)$ with constant mean curvature. If S=constant,* $\inf B_3\cdot\sup B_3=0$ *and*

$$3H^2/4+\sqrt{H^4+8H^2}/4+3<S\leqq H^2+6,$$

then $S=H^2+6$ and M is an isoparametric hypersurface if $H\neq 0$.

Proposition 2. *Let M be a hypersurface in $S^4(1)$ with constant mean curvature. If S=constant,* $\inf B_3\cdot\sup B_3\neq 0$ *and*

$$3H^2/4+\sqrt{H^4+8H^2}/4+3<S\leqq H^2+6,$$

then $S=H^2+6$ and M is an isoparametric hypersurface if $H\neq 0$.

Proof of Proposition 1. If $\inf B_3=\sup B_3=0$, then $B_3\equiv 0$, that is, $f_3=$ constant. Hence M is a hypersurface with constant principal curvature. Lemma 2 implies that Proposition 2 is true.

Next we only consider the case $f_3\neq$constant. And we will prove that it can not occur. In fact, without loss of generality, we may suppose $\inf B_3=0$. According to Lemma 3, there exists a sequence $\{p_m\}$ in M such that

$$\lim B_3(p_m)=\inf B_3=0,\quad \lim|\nabla B_3(p_m)|=0,\quad \liminf \Delta B_3(p_m)\geqq 0. \tag{3.23}$$

By (2.16), we have

$$\lim f_3(p_m)=HS-2H^3/9,\quad \lim|\nabla f_3(p_m)|=0,\quad \limsup \Delta f_3\leqq 0.$$

Making use of same proof as in Theorem 1, we obtain

$$\begin{aligned}&\lim \lambda_i(p_m)=\lambda_i^\circ,\\ &\lim h_{ijk}(p_m)=h_{ijk}^\circ,\\ &\lim h_{ijkl}(p_m)=h_{ijkl}^\circ.\end{aligned} \tag{3.24}$$

Hence

$$\sum\mu_i^\circ=0,\quad \sum(\mu_i^\circ)^2=B=(S-H^2/3)\quad\text{and}\quad \sum(\mu_i^\circ)^3=0. \tag{3.25}$$

Thus

$$\mu_1^\circ=-\sqrt{B/2},\quad \mu_2^\circ=0\quad\text{and}\quad \mu_3^\circ=\sqrt{B/2}. \tag{3.26}$$

Here we assume $\mu_1 \leqq \mu_2 \leqq \mu_3$. Since μ_1°, μ_2° and μ_3° are distinct, by the same proof as in Theorem 1, we can get

$$h_{iik}^\circ = 0 \quad \text{for any } i \text{ and } k, \tag{3.27}$$

$$\sum \mu_i^\circ h_{jik}^{\circ 2} = 0\,. \tag{3.28}$$

Because $H\Delta B_3 = \Delta \sum h_{ijk}^{\circ 2}$ from (2.16) and (2.17), (2.26), (3.23) and $H>0$ imply

$$\begin{aligned} 0 \leqq &-(S-2H^2/3-3)\sum h_{ijk}^{\circ 2} \\ &+(3/2)(S-H^2/3)(S-3H^2/4-3+\sqrt{H^4+8H^2}/4) \\ &\times(S-3H^2/4-3-\sqrt{H^4+8H^2}/4)\,. \end{aligned} \tag{3.29}$$

Since $\lim f_3(p_m) = HS - 2H^3/9$, we obtain

$$\sum h_{ijk}^{\circ 2} = (S-H^2/3)(S-2H^2/3-3) \qquad (\text{by } (2.17))\,. \tag{3.30}$$

(3.29) and (3.30) imply

$$\begin{aligned} 0 &\leqq -(S-H^2/3)(S-2H^2/3-3)^2 + (S-H^2/3)[(S-3H^2/4-3)^2 - (H^4+8H^2)/16] \\ &= (S-H^2/3)[-(S-2H^2/3-3)H^2/6 + H^4/12^2 - (H^4+8H^2)/16] \\ &\leqq -(S-H^2/3)[(S-2H^2/3-3)H^2/6 + H^2/2] < 0\,. \end{aligned}$$

This is a contradiction. Hence, Proposition 1 is valid.

Proof of Proposition 2. If $f_3 = \text{constant}$, then M is a hypersurface with constant principal curvature since H and S are constant. Hence Proposition 2 is valid from Lemma 2. Next we only consider the case $f_3 \neq \text{constant}$. And we prove that it can not occur. In fact, since $\inf B_3 \cdot \sup B_3 \neq 0$, we have

(1) If $\inf B_3 \cdot \sup B_3 < 0$, then, from the continuation of B_3, we have that there exists a point $p \in M$ such that

$$B_3(p) = 0\,. \tag{3.31}$$

(2) If $\inf B_3 \cdot \sup B_3 > 0$, then $\inf B_3$ and $\sup B_3$ are same sign. We shall prove that this cases doe not occur. In fact, without loss of generality, we may assume $\inf B_3 > 0$. According to (2.25), we have

$$0 < \inf B_3 < (1/\sqrt{6})(S-H^2/3)^{3/2}\,. \tag{3.32}$$

We apply Lemma 3 to B_3, then

$$\lim B_3(p_m) = \inf B_3, \quad \lim|\nabla B_3(p_m)| = 0, \quad \liminf \Delta B_3(p_m) \geqq 0\,. \tag{3.33}$$

Therefore,

$$\lim f_3(p_m) = \sup f_3, \quad \lim|\nabla f_3(p_m)| = 0, \quad \limsup \Delta f_3(p_m) \leqq 0\,. \tag{3.34}$$

By the same proof as in Theorem 1, we can get

$$\lim \mu_i(p_m)=\mu_i^\circ , \tag{3.35}$$

$$\lim h_{ijk}(p_m)=h_{ijk}^\circ , \tag{3.36}$$

$$\mu_1^\circ+\mu_2^\circ+\mu_3^\circ=0, \tag{3.37}$$

$$\mu_1^{\circ 2}+\mu_2^{\circ 2}+\mu_3^{\circ 2}=B, \tag{3.38}$$

$$\mu_1^{\circ 3}+\mu_2^{\circ 3}+\mu_3^{\circ 3}=\inf B_3 . \tag{3.39}$$

(3.32), (3.33), (3.38) and (3.39) imply that μ_1°, μ_2° and μ_3° are distinct. By the same proof as in Theorem 1, we can obtain

$$h_{iik}^\circ=0 \qquad \text{for any } i \text{ and } k, \tag{3.40}$$

$$\sum \mu_i^\circ h_{ijk}^{\circ 2}=0 . \tag{3.41}$$

By (2.14), (2.20) and (2.22), we get

$$\begin{aligned} \Delta f_3 &= -6\sum \mu_i h_{ijk}^2+2H\sum h_{ijk}^2-3Sf_3-3HS+9f_3 \\ &\quad +3H\left(-H^2S+\frac{1}{6}H^4+\frac{4}{3}Hf_3+\frac{1}{2}S^2\right) \\ &= -6\sum \mu_i h_{ijk}^2-3\left(S-\frac{2}{3}H^2-3\right)f_3+H\left(\frac{7}{2}S^2-9S+2H^2-3H^2S+\frac{H^4}{2}\right) \\ &\qquad\qquad \text{(by (2.17))} \\ &\geqq -6\sum \mu_i h_{ijk}^2-3\left(S-\frac{2}{3}H^2-3\right)\left(HS-\frac{2H^3}{9}\right) \\ &\quad +H\left(\frac{7}{2}S^2-9S+2H^2-3H^2S+\frac{1}{2}H^4\right) \quad \text{(by (2.16) and (3.32))}. \end{aligned} \tag{3.42}$$

(3.34), (3.41) and (3.42) imply

$$0\geqq H(S-H^2/3)^2>0 .$$

This is impossible. Hence (2) can not occur.

Next we will prove that (1) can not also occur. From (1) we know $B_3(p)=0$, that is,

$$f_3(p)=HS-2H^3/9 . \tag{3.43}$$

At point p, we have

$$\mu_1+\mu_2+\mu_3=0 ,$$

$$\mu_1^2+\mu_2^2+\mu_3^2=B=S-H^2/3 ,$$

$$\mu_1^3+\mu_2^3+\mu_3^3=0 .$$

Hence

(3.44) $$\mu_1=-\sqrt{B/2},\quad \mu_2=0,\qquad \mu_3=\sqrt{B/2}.$$

(3.45) $$\lambda_1=H/3+\sqrt{B/2},\qquad \lambda_2=H/3,\qquad \lambda_3=H/3-\sqrt{B/2}.$$

According to $\sum\lambda_i=H$ and $\sum\lambda_i^2=S$, we have

$$\sum h_{iik}=0,\qquad \sum\lambda_i h_{iik}=0.$$

Hence,

(3.46) $$h_{11k}=h_{33k},\quad h_{22k}=-2h_{11k}\qquad \text{for } k=1,2,3.$$

On the other hand,

(3.47) $$\sum h_{ijk}^2=B(S-2H^2/3-3)\qquad (\text{by (2.16) and } B_3(p)=0).$$

From (3.46), we get

$$\sum h_{ijk}^2=6h_{123}^2+16h_{111}^2+(5/2)h_{222}^2+16h_{333}^2=B(S-2H^2/3-3).$$

Therefore,

(3.48) $$\begin{aligned}\sum h_{ij2}^2&=2h_{123}^2+8h_{111}^2+(3/2)h_{222}^2+8h_{333}^2\\ &\geqq(1/3)\sum h_{ijk}^2=(1/3)B(S-2H^2/3-3),\end{aligned}$$

(3.49) $$\begin{aligned}\sum_{i\neq j}t_{ij}^2&=\sum(\lambda_i-\lambda_j)^2(1+\lambda_i\lambda_j)^2\\ &=2B(S^2/2-H^2S/2+4H^4/27-2S+4H^2/3+3)\qquad (\text{by (3.45)}),\end{aligned}$$

(3.50) $$\begin{aligned}\sum_{i\neq j}(h_{ijij}-t_{ij}/2)^2&\geqq 2[(h_{1212}-t_{12}/2)^2+(h_{2323}-t_{23}/2)^2]\\ &=[h_{1212}+h_{2323}-(t_{12}+t_{23})/2]^2+[h_{1212}-h_{2323}-(t_{12}-t_{23})/2]^2\\ &\geqq[h_{1212}-h_{2323}-(t_{12}-t_{23})/2]^2\\ &=[h_{1212}-h_{3232}-t_{23}-(t_{12}-t_{23})/2]^2.\end{aligned}$$

By differentiating $S=\sum h_{ij}^2$, we obtain

$$\sum h_{iikk}\lambda_i+\sum h_{ijk}^2=0\qquad \text{for any } k=1,2,3.$$

Hence

(3.51) $$\sqrt{B/2}(h_{3232}-h_{1212})=\sum h_{ij2}^2\qquad (\text{by (3.45)}),$$

(3.52) $$t_{12}-t_{23}=HB/3,\quad t_{23}=\sqrt{B/2}+H^2\sqrt{B/2}/9-HB/6\qquad (\text{by (3.45)}),$$

(3.53) $$\begin{aligned}\sum_{i\neq j}h_{ijkl}^2&\geqq 3\sum_{i\neq j}(h_{ijij}-t_{ij}/2)^2+(3/4)\sum t_{ij}^2\\ &\geqq(3/2)B(S^2/2-H^2S/2+4H^4/27-2S+4H^2/3+3)\\ &\quad+6B(S/3-1/2-H^2/6)^2\end{aligned}$$

(by (3.49), (3.50), (3.51), (3.52) and (3.48)).

(2.27), (3.47) and (3.53) imply

$$(3/2)B(S^2/2-H^2S/2+4H^4/27-2S+4H^2/3+3)+6B(S/3-1/2-H^2/6)^2$$
$$\leqq(-5/3)\sum(\mu_i+\mu_j+\mu_k)^2h_{ijk}^2+8H\sum\mu_ih_{ijk}^2+(3/2)B(S-2H^2/3-3)^2$$
$$+(3/2)(S-H^2/3)[(S-3H^2/4-3)^2-(H^4+8H^2)/16],$$

that is,

$$0\leqq-(5/3)\sum(\mu_i+\mu_j+\mu_k)^2h_{ijk}^2+8H\sum\mu_ih_{ijk}^2$$
$$+[19S^2/12-(17H^2/6+13)S+37H^4/36+9H^2+21]B. \tag{3.54}$$

(a) If $-(5/3)\sum(\mu_i+\mu_j+\mu_k)^2h_{ijk}^2+8H\sum\mu_ih_{ijk}^2\leqq0$, then

$$19S^2/12-(17H^2/6+13)S+37H^4/36+9H^2+21\geqq0.$$

According to $S>3H^2/4+\sqrt{H^4+8H^2}/4+3$, we obtain $S>H^2+6$. This is a contradiction.

(b) If $-(5/3)\sum(\mu_i+\mu_j+\mu_k)^2h_{ijk}^2+8H\sum\mu_ih_{ijk}^2\geqq0$, then, since

$$-\sum(\mu_i+\mu_j+\mu_k)^2h_{ijk}^2\leqq-12Bh_{333}^2,\qquad 8H\sum\mu_ih_{ijk}^2\leqq32H\sqrt{B/2}h_{333}^2,$$

we obtain

$$0\leqq(5/3)\sum(\mu_i+\mu_j+\mu_k)^2h_{ijk}^2+8H\sum\mu_ih_{ijk}^2 \tag{3.55}$$
$$\leqq(32H-40\sqrt{B/2})\sqrt{B/2}h_{333}^2$$
$$\leqq(1/16)(32H-40\sqrt{B/2})\sqrt{B/2}\sum h_{ijk}^2$$
$$=(1/16)(32H-40\sqrt{B/2})\sqrt{B/2}B(S-2H^2/3-3)$$
$$\leqq B(H^2+B/2-5B/4)(S-2H^2/3-3)$$
$$=B[-3S^2/4+9S/4+7H^2S/4-5H^4/6-15H^2/4].$$

We get, from (3.54) and (3.55),

$$5S^2/6-(13H^2/12+43/4)S+7H^4/36+21H^2/4+21\geqq0. \tag{3.56}$$

Let $F(S)=5S^2/6-(13H^2/12+43/4)S+7H^4/36+21H^2/4+21$. As a function of S, $F(S)$ reaches its minimum at $S=(3/5)(13H^2/12+43/4)$. Without loss of generality, we may suppose

$$3H^2/4+3+\sqrt{H^4+8H^2}/4\leqq(3/5)(13H^2/12+43/4)<H^2+6.$$

Since $F(3H^2/4+3+\sqrt{H^4+8H^2}/4)<0$, $F(H^2+6)<0$ and $F[(3/5)(13H^2/12+43/4)]$ is only a minimum of F, we obtain $F<0$ when

$$3H^2/4+3+\sqrt{H^4+8H^2}/4\leqq S\leqq H^2+6.$$

This is a contradiction. Hence (1) can not also occur. Thus, we complete the proof of Proposition 2.

Proof of Theorem 2. According to Propositions 1 and 2, Theorem 2 is valid obviously.

References

[1] E. Cartan, Familles de surfaces isoparametriques dans les espaces a courbure constante, *Annali di Mat.*, **17** (1938), 177-191.

[2] Q.M. Cheng, A characterization of complete Riemannian manifold minimally immersed in a unit sphere, to appear in *Nagoya Math. J.*.

[3] Q.M. Cheng, Complete minimal hypersurfaces in $S^4(1)$ with constant scalar curvature, *Osaka Math. J.*, **27** (1990), 855-892.

[4] Q.M. Cheng and H. Nakagawa, Totally umbilic hypersurfaces, *Hiroshima Math. J.*, **20** (1990), 1-10.

[5] S.S. Chern, M. do Carmo and S. Kobayashi, Minimal submanifold of a sphere with second fundamental form of constant length, *Functional analysis and related fields*, Springer, 1970, 59-75.

[6] M. Okumura, Hypersurfaces and a pinching problem on the second fundamental tensor, *Amer. J. of Math.*, **97** (1974), 207-213.

[7] H. Omori, Isometric immersion of Riemannian manifolds, *J. Math. Soc. Japan*, **19** (1967), 205-214.

[8] C.K. Peng and C.L. Terng, Minimal hypersurfaces of spheres with constant scalar curvature, *Seminar on minimal submanifolds*, Princeton Univ. Prees, 1983, 177-198.

[9] J. Simons, Minimal varieties in Riemannian manifolds, *Ann of Math.*, **88** (1968), 62-105.

[10] S.T. Yau, Harmonic functions on complete Riemannian manifolds, *Comm. Pure and Appl. Math.*, **28** (1975), 201-228.

Department of Mathematics
Northeast University of Technology
Shenyang, Liaoning, P.R. China