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A NOTE ON THE ASYMPTOTIC NORMALITY OF SEQUENTIAL DENSITY ESTIMATORS

By

EIICHI ISOGAI

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Summary. Let $f_n(x)$ be the recursive kernel estimators of an unknown density function $f(x)$ at a given point x. Also, let $N(t)(t>0)$ be a family of positive integer-valued random variables. We consider the sequential estimators $f_{N(t)}(x)$. In this paper, under certain regularity conditions on $N(t)$ we shall show that $(N(t)h_{N(t)}^p)^{1/2}(f_{N(t)}(x)-f(x))$ is asymptotically normally distributed as t tends to infinity. Our conditions on $N(t)$ generalize those given by Carroll [2], Stute [9] and Isogai [6].

1. Introduction

Let $X_1, X_2, X_3, \cdots$ be a sequence of independent identically distributed p-dimensional random vectors, defined on a probability space $(\Omega, \mathcal{F}, P)$, and let f denote their common probability density function with respect to Lebesgue measure on the p-dimensional Euclidean space R^p. Further, let $N(t)$ $(t>0)$ be a family of positive integer-valued random variables defined on $(\Omega, \mathcal{F}, P)$ (i.e., stopping rules). Many authors have investigated the problem of nonparametric density estimation based on $X_1, \cdots, X_n$. This problem was summarized in the books of Prakasa Rao [7] and Devroye and Györfi [4]. On the other hand, there are many situations in practice where the number of observations required to compute density estimators is random. The problem of sequential density estimation based on random number of observations also has been studied (for example, see Srivastava [8], Carroll [2], Isogai [5] and [6] and Stute [9]). The above authors, Davies and Wegman [3] and Wegman and Davies [10] proposed stopping rules and investigated the asymptotic properties of the sequential density estimators and the stopping rules.

In this paper we consider the following sequential density estimators proposed by Isogai [6]:

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$$f_{N(t)}(x)=\sum_{j=1}^{N(t)} a_j\beta_{jN(t)}K_j(x, X_j)+\beta_{0N(t)}K(x), \tag{1.1}$$

where

$$K_n(x, y)=h_n^{-p}K((x-y)/h_n) \quad \text{for} \quad x, y\in R^p, \tag{1.2}$$

K is a bounded, integrable, real-valued Borel measurable function on R^p and $\{h_n\}$ with $h_0=h_1$ is a nonincreasing sequence of positive numbers converging to zero,

$$a_n=a/n \quad \text{for any fixed} \quad a\in(0, 1] \tag{1.3}$$

and

$$\beta_{mn}=\begin{cases} \prod_{j=m+1}^{n}(1-a_j) & \text{if} \quad n>m\geqq 0, \\ 1 & \text{if} \quad n=m\geqq 0. \end{cases} \tag{1.4}$$

The aim of this paper is to show that, under certain regularity conditions on $N(t)$, $(N(t)h_{N(t)}^p)^{1/2}(f_{N(t)}(x)-f(x))$ is asymptotically normally distributed as t tends to infinity. The results like this were shown by Carroll [2], Stute [9] and Isogai [5], [6], but our conditions on $N(t)$ generalize those given by them. This paper consists of three sections. In Section 2 the main theorem is given. In the last section the theorem is proved.

2. Main Result

In this section we shall make some preparations and give the main theorem. Set

$$\gamma_1=1 \quad \text{and} \quad \gamma_n=\sum_{j=2}^{n}(1-a_j) \quad \text{for} \quad n\geqq 2,$$

where a_n is defined in (1.3). It is known in [6] that

$$\beta_{mn}=\gamma_n\gamma_m^{-1} \quad \text{for} \quad n\geqq m\geqq 1 \tag{2.1}$$

and

$$\beta_{mn}\sim m^a n^{-a} \quad \text{as} \quad n\geqq m\longrightarrow\infty, \tag{2.2}$$

where "$\sim$" means the asymptotic equivalence. Throughout this paper the function K in Section 1 is assumed to satisfy the following.

Condition K:

$$\int_{R^p}K(u)du=1, \qquad \int_{R^p}\|u\|^2|K(u)|du<\infty,$$

$$\int_{R^p}u_iK(u)du=0 \quad \text{for} \quad i=1, \cdots, p \quad \text{with} \quad u=(u_1, \cdots, u_p),$$

$$\|u\|^p|K(u)|\longrightarrow 0 \quad \text{as} \quad \|u\|\longrightarrow\infty,$$

where $\|\cdot\|$ denotes the Euclidean norm on R^p.

Assume the sequence $\{h_n\}$ in Section 1 satisfies the following.

Condition H: For a of (1.3)

(H1) $nh_n^p \uparrow \infty$ as $n \to \infty$,

(H2) $n^{1-2a}h_n^p \to 0$ as $n \to \infty$,

(H3) $n^{1-2a}h_n^p \sum_{j=1}^n j^{2(a-1)}h_j^{-p} \to \beta$ as $n \to \infty$ for some constant $\beta>0$,

(H4) $n^{3/2-3a}h_n^{3p/2}\sum_{j=1}^n j^{3(a-1)}h_j^{-2p} \to 0$ as $n \to \infty$,

(H5) $(n^{1-2a}h_n^p)^{1/2}\sum_{j=1}^n j^{a-1}h_j^2 \to 0$ as $n \to \infty$,

(H6) For any $\varepsilon>0$ there exists a positive constant $\delta=\delta(\varepsilon)$ such that $|(n/m)-1|<\delta$ implies $|(h_n/h_m)-1|<\varepsilon$.

Example. Let $h_n=n^{-r/p}$ with $\max\{p/(p+4), 1-2a\}<r<1$. Then $\{h_n\}$ satisfies (H1)$\sim$(H6) with $\beta=(2a+r-1)^{-1}$. We shall give two definitions concerning conditions on f and $N(t)$.

Definition 1. Let g be a real-valued function on R^p. We say that the function g belongs to the class $\mathfrak{M}_p$ (abbreviated as $g\in\mathfrak{M}_p$) if there exist bounded, continuous second partial derivatives $\partial^2 g(x)/\partial x_i \partial x_j$ on R^p for all $i, j=1, \cdots, p$.

Definition 2. A family of positive integer-valued random variables $N(t)(t>0)$ is said to satisfy Condition A if there exist a positive random variable θ on $(\Omega, \mathcal{F}, P)$ and a family of positive numbers $\tau(t)$ $(t>0)$ with $\tau(t)\to\infty$ as $t\to\infty$ such that $N(t)/\tau(t)\xrightarrow[p]{}\theta$ as $t\to\infty$ (in probability).

Remark 1. The stopping rules considered by Carroll [2], Isogai [5], [6] and Stute [9] satisfy Condition A. We shall now give the main theorem of this paper.

Theorem. *Assume $f\in\mathfrak{M}_p$. If $N(t)$ satisfies Condition* A, *then for each point x with $f(x)>0$*

$$(N(t)h_{N(t)}^p)^{1/2}(f_{N(t)}(x)-f(x)) \xrightarrow[L]{} N(0, \sigma^2(x)) \quad as \quad t\to\infty \ (in\ law),$$

where

$$\sigma^2(x)=a^2\beta f(x)\int_{R^p}K^2(u)du\,.$$

Remark 2. Let θ in Condition A be a positive discrete random variable, that is, there exists a sequence of positive numbers $l_k(k=1, 2, \cdots)$ (k may be finite or infinite) such that $\sum_{k=1}^{\infty} p_k=1$ where $p_k=P\{\theta=l_k\}>0$. Then this theorem was proved by Isogai [6].

3. Proof of Theorem

Throughout this section all of the conditions in Theorem are assumed to be satisfied. Let $[b]$ denote the largest integer not greater than b. Note that $b-1<[b]\leqq b$. For notational simplicity let N and τ denote $N(t)$ and $\tau(t)$ respectively unless otherwise specified. For any fixed x set

$$\begin{aligned} &Z_n=K_n(x, X_n)-EK_n(x, X_n), &&\delta_n=EK_n(x, X_n)-f(x), \\ &S_n=\textstyle\sum_{j=1}^{n} a_j\beta_{jn}\{K_j(x, X_j)-f(x)\}, &&V_n=(nh_n^p)^{1/2}S_n \quad \text{for } n\geqq 1, \end{aligned} \tag{3.1}$$

where $S_0=V_0=0$ and K_n is as defined by (1.2). Then it is clear from (1.1) that

$$(Nh_N^p)^{1/2}(f_N(x)-f(x))=V_N+(Nh_N^p)^{1/2}\beta_{0N}(K(x)-f(x)) \quad \text{for } t>0. \tag{3.2}$$

One can easily verify that Lemma 3 of [1] holds for independent random variables, that is,

Lemma 1. *Let $\{Y_n\}$ be a sequence of independent random variables. Also, let $\{k_n\}$ and $\{m_n\}$ be sequences of positive integers tending to infinity, and A_n an event depending only on $Y_{k_n}, \cdots, Y_{m_n}$. If A is any event, then*

$$\limsup_{n\to\infty} P(A_n\mid A)=\limsup_{n\to\infty} P(A_n),$$

where the conditional probability $P(A_n\mid A)=P(A)$ if $P(A)=0$.

The following lemma is found in [6].

Lemma 2. *Let $\{Y_n\}$ be a sequence of independent random variables such that putting $S_n=(1/B_n)\sum_{j=1}^{n} Y_j$ with $B_n>0$ being a constant tending to infinity, the random variable S_n converges in law to some random variable with the distribution function F. Then for any event A with $P(A)>0$ the conditional probability $P\{S_n\leqq x\mid A\}$ tends to $F(x)$ for every continuity point x of F.*

In the same manner as (A.9) in [6] we obtain

Lemma 3. *Let C be an arbitrary given positive constant. Then for any $\varepsilon>0$ there exists a positive integer $n_0=n_0(\varepsilon, C)$ such that for all $m\geqq n\geqq n_0$ with $m/n\leqq C$*

$$\sqrt{mh_m^p}\max_{n\leq i\leq m} \left|\textstyle\sum_{q=1}^{i} a_q\beta_{qi}\delta_q\right|<\varepsilon.$$

Lemma 4. *Under the assumptions of Theorem*

$$(N(t)h^p_{N(t)})^{1/2}(S_{N(t)}-S_{[\theta\tau(t)]}) \xrightarrow[P]{} 0 \quad as \quad t\to\infty .$$

Proof. Let any ε, $\eta\in(0, 1)$ be fixed. First fix a sufficiently large constant $C>0$ and a sufficiently small constant $c>0$, both of which do not depend on ε and η. Here let c^{-1} be far larger than C. Further let constants be denoted as the same notation C and/or c, unless otherwise stated. Choose a positive constant G satisfying

$$1-\Phi(G)+\Phi(-G)<c\eta , \tag{3.3}$$

where Φ denotes the distribution function of the standard normal random variable $N(0, 1)$. Taking account of the positivity of θ, choose an integer $m\geqq 1$ and a constant $\rho\in(0, 1/2)$ depending only on ε and η such that

$$P\{\theta<(m-1)/2^m\}+P\{\theta\geqq m\}<\eta/4 \quad \text{and} \quad m>\max(C\varepsilon^{-2}\eta^{-1},\ CG\varepsilon^{-1}) \tag{3.4}$$

and

$$0<\rho<\min(c\varepsilon^2\eta,\ C^{-1}G^{-1}\varepsilon). \tag{3.5}$$

Let an event A_{km} be denoted as

$$A_{km}=\{(k-1)/2^m\leqq\theta<k/2^m\} \qquad \text{for} \quad m\leqq k\leqq m2^m .$$

It follows from Condition A that

$$P\{|N-[\theta\tau]|\geqq\rho[\theta\tau]\}<\eta/4 \qquad \text{for large } t. \tag{3.6}$$

By (3.4) and (3.6) we get

$$\begin{aligned} &P\{\sqrt{Nh^p_n}\,|S_N-S_{[\theta\tau]}|>\varepsilon\} \\ &\quad\leqq \textstyle\sum_{k=m}^{m2^m} P\{\sqrt{Nh^p_n}\,|S_N-S_{[\theta\tau]}|>\varepsilon,\ |N-[\theta\tau]|<\rho[\theta\tau],\ A_{km}\}+\eta/2 \end{aligned} \tag{3.7}$$

for large t. Fix any $k=m, \cdots, m2^m$. Set

$$\begin{aligned} n_1&=n_1(t)=[(k-1)\tau/2^m], & n_2&=n_2(t)=[k\tau/2^m], \\ m_1&=m_1(t)=[(1-\rho)n_1], & m_2&=m_2(t)=[(1+\rho)n_2]. \end{aligned} \tag{3.8}$$

It is easily verified that

$$\begin{aligned} &\lim_{t\to\infty} m_1/n_1=1-\rho, \qquad \lim_{t\to\infty} n_1/n_2=1-k^{-1}, \\ &\lim_{t\to\infty} \beta_{m_1n_1}=(1-\rho)^\alpha\geqq 1-\rho, \\ &\lim_{t\to\infty} \beta_{n_1n_2}=(1-k^{-1})^\alpha\geqq 1-m^{-1}. \end{aligned} \tag{3.9}$$

It follows from (3.8) that

$$I_k(t)\equiv P\{\sqrt{Nh_n^p}\,|S_N-S_{[\theta\tau]}|>\varepsilon,\ |N-[\theta\tau]|<\rho[\theta\tau],\ A_{km}\}$$

$$\leqq P\{\sqrt{ih_i^p}\,|S_i-S_j|>\varepsilon \text{ for some } m_1\leqq i\leqq m_2 \text{ and some } n_1\leqq j\leqq n_2,\ A_{km}\}$$

$$\text{(3.10)}\quad \leqq P(A_{km})[P\{\sqrt{ih_i^p}\,|S_i-S_j|>\varepsilon \text{ for some } m_1\leqq i<n_1 \text{ and some } n_1\leqq j\leqq n_2\,|\,A_{km}\}$$

$$+P\{\sqrt{ih_i^p}\,|S_i-S_j|>\varepsilon \text{ for some } n_1\leqq i,\ j\leqq n_2\,|\,A_{km}\}$$

$$+P\{\sqrt{ih_i^p}\,|S_i-S_j|>\varepsilon \text{ for some } n_2<i\leqq m_2 \text{ and some } n_1\leqq j\leqq n_2\,|\,A_{km}\}]$$

$$\equiv P(A_{km})(J_1+J_2+J_3),\ \text{say.}$$

First we shall estimate the term J_1. By the monotonicity of nh_n^p we get

$$J_1\leqq P\{\sqrt{n_1h_{n_1}^p}\max_{m_1<i\leqq n_1}|S_i-S_{m_1}|>\varepsilon/3\,|\,A_{km}\}$$

$$\text{(3.11)}\qquad +P\{\sqrt{n_2h_{n_2}^p}\max_{n_1<j\leqq n_2}|S_j-S_{n_1}|>\varepsilon/3\,|\,A_{km}\}$$

$$\equiv J_{11}+J_{12},\quad \text{say.}$$

It follows from (2.1), (3.1) and the monotonicity of γ_n that for $i>j$

$$\text{(3.12)}\qquad |S_i-S_j|\leqq(\gamma_j-\gamma_i)|\textstyle\sum_{q=1}^{j} a_q\gamma_q^{-1}Z_q|+\gamma_i|\sum_{q=j+1}^{i} a_q\gamma_q^{-1}Z_q|$$

$$+|\textstyle\sum_{q=1}^{i} a_q\beta_{qi}\delta_q|+|\sum_{q=1}^{j} a_q\beta_{qj}\delta_q|.$$

Since $\lim_{t\to\infty} m_1=\infty$, by (3.9) and Lemma 3 we have

$$\sqrt{n_1h_{n_1}^p}\max_{m_1\leqq i\leqq n_1}|\textstyle\sum_{q=1}^{i} a_q\beta_{qi}\delta_q|<\varepsilon/4 \qquad \text{for large } t.$$

Thus by the use of this relation and (3.12) we obtain that for large t

$$J_{11}\leqq P\{\sqrt{n_1h_{n_1}^p}\,[(\gamma_{m_1}-\gamma_{n_1})|\textstyle\sum_{q=1}^{m_1} a_q\gamma_q^{-1}Z_q|+\max_{m_1<i\leqq n_1}\gamma_i|\sum_{q=m_1+1}^{i} a_q\gamma_q^{-1}Z_q|]$$

$$+2\sqrt{n_1h_{n_1}^p}\max_{m_1\leqq i\leqq n_1}|\textstyle\sum_{q=1}^{i} a_q\beta_{qi}\delta_q|>\varepsilon\,|\,A_{km}\}$$

$$\text{(3.13)}\qquad \leqq P\{\sqrt{n_1h_{n_1}^p}(\gamma_{m_1}-\gamma_{n_1})|\textstyle\sum_{q=1}^{m_1} a_q\gamma_q^{-1}Z_q|>\varepsilon/4\,|\,A_{km}\}$$

$$+P\{\sqrt{n_1h_{n_1}^p}\max_{m_1<i\leqq n_1}\gamma_i|\textstyle\sum_{q=m_1+1}^{i} a_q\gamma_q^{-1}Z_q|>\varepsilon/4\,|\,A_{km}\}$$

$$\equiv J_{111}+J_{112},\quad \text{say.}$$

It follows from Lemma 1 that

$$\text{(3.14)}\qquad \limsup_{t\to\infty} J_{112}=\limsup_{t\to\infty} P\{\sqrt{n_1h_{n_1}^p}\max_{m_1<i\leqq n_1}\gamma_i|\textstyle\sum_{q=m_1+1}^{i} a_q\gamma_q^{-1}Z_q|>\varepsilon/4\}.$$

Using the inequality of (A.17) in [6], (3.5) and (3.9) we have

$$\text{R.H.S of (3.14)}\leqq C\varepsilon^{-2}\limsup_{t\to\infty}\{(n_1/m_1)^2(1-m_1/n_1)\}<4C\varepsilon^{-2}\rho.$$

Hence by (3.5) we obtain

(3.15) $$\limsup_{t\to\infty} J_{112}<c\eta .$$

Put

$$U_n=a_n\gamma_n^{-1}Z_n,\ W_n=\sum_{q=1}^n U_q \quad \text{and} \quad w_n^2=\sum_{q=1}^n a_q^2\gamma_q^{-2}EZ_q^2 .$$

Then in (2.5) and (2.6) of Isogai [6] it is given that

(3.16) $$w_n^2 \sim \sigma^2(x)(nh_n^p\gamma_n^2)^{-1} \quad \text{as} \quad n\to\infty$$

and

(3.17) $$W_n/w_n \xrightarrow[L]{} N(0,1) \quad \text{as} \quad n\to\infty .$$

By virtue of (3.9), (3.16) and the monotonicity of h_n we have

$$\sqrt{n_1h_{n_1}^p}(\gamma_{m_1}-\gamma_{n_1})w_{m_1}\leqq C\rho \qquad \text{for all} \quad t>0.$$

Hence it follows from this relation, Lemma 2, (3.3), (3.5) and (3.17) that

$$\limsup_{t\to\infty} J_{111}\leqq \limsup_{t\to\infty} P\{|W_{m_1}/w_{m_1}|>C^{-1}\varepsilon\rho^{-1}|A_{km}\}$$
$$\leqq \lim_{t\to\infty} P\{|W_{m_1}/w_{m_1}|>G|A_{km}\}=\lim_{t\to\infty} P\{|W_{m_1}/w_{m_1}|>G\}<c\eta ,$$

which, together with (3.13) and (3.15), implies that

(3.18) $$\limsup_{t\to\infty} J_{11}<c\eta .$$

Replace m_1 and n_1 in J_{11} of (3.11) by n_1 and n_2 in J_{12} of (3.11), respectively. Then in the same manner as (3.18) we obtain

(3.19) $$\limsup_{t\to\infty} J_{12}<c\eta .$$

Thus, combining (3.11), (3.18) and (3.19) we have

(3.20) $$\limsup_{t\to\infty} J_1<c\eta .$$

By the same argument as (3.20) we get

$$\limsup_{t\to\infty} J_2<c\eta \quad \text{and} \quad \limsup_{t\to\infty} J_3<c\eta ,$$

which, together with (3.7), (3.10) and (3.20), yields that

$$\limsup_{t\to\infty} P\{\sqrt{Nh_N^p}\,|S_N-S_{[\theta\tau]}|>\varepsilon\}\leqq c\eta\sum_{k=m}^{m2^m} P(A_{km})+\eta/2\leqq c\eta+\eta/2<\eta .$$

Therefore, as $\eta\to 0$ we have the lemma. This completes the proof.

We are now in the position to prove the theorem. It is given in (2.2) of Isogai [6] that $\gamma_n\leqq Ln^{-a}$ for all n with some constant $L>0$. Since by Condition A $N\xrightarrow[p]{}\infty$ as $t\to\infty$, using (H2) and the above inequality we get that

$\sqrt{Nh_N^p}\,\beta_{0N} \underset{p}{\to} 0$ as $t\to\infty$. Thus, by taking account of (3.2), in order to prove the theorem it is sufficient to show

$$V_N \underset{L}{\longrightarrow} N(0, \sigma^2(x)) \quad \text{as} \quad t\to\infty. \tag{3.21}$$

Clearly,

$$V_N = V_{[\theta\tau]} + \sqrt{Nh_N^p}\,(S_N - S_{[\theta\tau]}) + V_{[\theta\tau]}\{(Nh_N^p/[\theta\tau]h_{[\theta\tau]}^p)^{1/2} - 1\}. \tag{3.22}$$

It follows from Condition A that $N/[\theta\tau] \underset{p}{\to} 1$ as $t\to\infty$, which, together with (H6), yields that $Nh_N^p/[\theta\tau]h_{[\theta\tau]}^p \underset{p}{\to} 1$ as $t\to\infty$. Hence, if

$$V_{[\theta\tau]} \underset{L}{\longrightarrow} N(0, \sigma^2(x)) \quad \text{as} \quad t\to\infty, \tag{3.23}$$

then

$$V_{[\theta\tau]}\{(Nh_N^p/[\theta\tau]h_{[\theta\tau]}^p)^{1/2} - 1\} \underset{P}{\longrightarrow} 0 \quad \text{as} \quad t\to\infty.$$

Thus, by this result, (3.22) and Lemma 4, in order to show (3.21) it is sufficient to prove (3.23). In the remainder we shall show (3.23). Let F be the distribution function of $N(0, \sigma^2(x))$. If

$$\lim_{t\to\infty} P\{V_{[\theta\tau]} \leqq y\} = F(y) \qquad \text{for any } y, \tag{3.24}$$

then (3.23) holds. Hence it suffices to prove (3.24). Let any $\eta>0$ and any y be fixed. Fix a sufficiently small constant $c>0$, not depending on η. Let $G=G(\eta)>0$ be a constant such that $1-F(G)+F(-G)<c\eta$. Choose a sufficiently large integer $m=m(\eta)$ such that all the relations depending on m hold and that $P\{\theta<(m-1)/2^m\}+P\{\theta\geqq m\}<c\eta$, and a constant $\varepsilon=\varepsilon(\eta)>0$ satisfying that $|F(y+i\varepsilon)-F(y)|<\eta/4$ for $i=\pm 1$. Set

$$\mu_m = \textstyle\sum_{k=1}^{\infty} (k/2^m) I((k-1)/2^m \leqq \theta < k/2^m), \tag{3.25}$$

where $I(A)$ denotes the indicator function of A. Note that μ_m is a positive discrete random variable. Thus, from (3.4) of [6] we have

$$V_{[\mu_m\tau]} \underset{L}{\longrightarrow} N(0, \sigma^2(x)) \quad \text{as} \quad t\to\infty. \tag{3.26}$$

Since by virtue of Lemma 1 of [1]

$$\begin{aligned} &|P\{V_{[\theta\tau]} \leqq y\} - F(y)| \\ &\leqq |P\{V_{[\mu_m\tau]} \leqq y+\varepsilon\} - F(y+\varepsilon)| + |F(y+\varepsilon)-F(y)| + |P\{V_{[\mu_m\tau]} \leqq y-\varepsilon\} \\ &\quad - F(y-\varepsilon)| + |F(y-\varepsilon)-F(y)| + P\{|V_{[\theta\tau]} - V_{[\mu_m\tau]}| > \varepsilon\}, \end{aligned}$$

it follows from (3.26) that

$$\limsup_{t\to\infty} |P\{V_{[\theta\tau]} \leqq y\} - F(y)| \leqq \limsup_{t\to\infty} P\{|V_{[\theta\tau]} - V_{[\mu_m\tau]}| > \varepsilon\} + \eta/2. \tag{3.27}$$

Now, we shall estimate the first term of the right-hand side of (3.27). Using the same relation as (3.22), we get

$$\begin{aligned} P\{|V_{[\theta\tau]}-V_{[\mu_m\tau]}|>\varepsilon\} &\leqq P\{([\theta\tau]h^p_{[\theta\tau]})^{1/2}|S_{[\theta\tau]}-S_{[\mu_m\tau]}|>\varepsilon/2\} \\ &\quad +P\{|(([\theta\tau]h^p_{[\theta\tau]}/[\mu_m\tau]h^p_{[\mu_m\tau]})^{1/2}-1)V_{[\mu_m\tau]}|>\varepsilon/2\} \\ &\equiv I_1+I_2, \quad \text{say}. \end{aligned} \tag{3.28}$$

Set $n_1=[[k\tau/2^m]-\tau/2^m]$ and $n_2=[k\tau/2^m]$ for fixed $k\geqq m$. By (3.25) we note that $\mu_m=k/2^m$ if $(k-1)/2^m\leqq\theta<k/2^m$ and that $\theta<\mu_m$. Then, taking account of the monotonicity of nh^p_n, we have

$$\begin{aligned} I_1 &\leqq \textstyle\sum_{k=m}^{m2^m} P\{\sqrt{n_2h^p_{n_2}}|S_{n_2}-S_i|>\varepsilon/2 \text{ for some } n_1\leqq i\leqq n_2|A_{km}\}P(A_{km})+c\eta \\ &\leqq \textstyle\sum_{k=m}^{m2^m} P\{\sqrt{n_2h^p_{n_2}}\max_{n_1<i\leqq n_2}|S_i-S_{n_1}|>\varepsilon/4|A_{km}\}P(A_{km})+c\eta, \end{aligned}$$

where A_{km} is as in the proof of Lemma 4. Hence, following a similar argument as (3.18), we can prove that

$$I_1<c\eta \quad \text{for sufficiently large } t. \tag{3 29}$$

Lastly, we shall estimate I_2. By (3.26) and the property of G we have $P\{|V_{[\mu_m\tau]}|>G\}<c\eta$ for large t. Thus, we obtain that for large t

$$I_2\leqq P\{|V_{[\mu_m\tau]}|>G\}+I_{21}<c\eta+I_{21}, \tag{3.30}$$

where

$$I_{21}=P\{|([\theta\tau]h^p_{[\theta\tau]}/[\mu_m\tau]h^p_{[\mu_m\tau]})^{1/2}-1| > \varepsilon/2G\}.$$

Put $\eta_1=\min\{(\varepsilon/2G)^2, 1\}$. Using the fact that $\theta<\mu_m$ and the simple inequality that $(\sqrt{a}-1)^2\leqq|a-1|$ for all $a\geqq0$, we have

$$\begin{aligned} I_{21} &\leqq P\{|(h_{[\theta\tau]}/h_{[\mu_m\tau]})^p-1||[\theta\tau]/[\mu_m\tau]|+|([\theta\tau]/[\mu_m\tau])-1|>\eta_1\} \\ &\leqq P\{|(h_{[\theta\tau]}/h_{[\mu_m\tau]})^p-1|>\eta_1/2\}+P\{|([\theta\tau]/[\mu_m\tau])-1|>\eta_1/2\} \\ &\equiv J_1+J_2, \quad \text{say}. \end{aligned} \tag{3.31}$$

From (H6) there exists a constant $\eta_2=\eta_2(\eta)>0$ such that $|n/m-1|<\eta_2$ implies $|h_n/h_m-1|<\eta_1/2^{p+2}$. Hence, by the use of this relation and the fact that

$$x^p-1=(x-1)\textstyle\sum_{i=0}^{p-1}x^i \quad\text{and}\quad \textstyle\sum_{i=0}^{p-1}(1+\eta_1/2^{p+2})^i<2^p,$$

we have

$$\begin{aligned} J_1 &\leqq P\{|(h_{[\theta\tau]}/h_{[\mu_m\tau]})-1|\textstyle\sum_{i=0}^{p-1}(h_{[\theta\tau]}/h_{[\mu_m\tau]})^i>\eta_1/2, \\ &\qquad |(h_{[\theta\tau]}/h_{[\mu_m\tau]})-1|<\eta_1/2^{p+2}\}+J_3=J_3, \end{aligned} \tag{3.32}$$

where

$$J_3 = P\{|([\theta\tau]/[\mu_m\tau])-1| \geqq \eta_2\}.$$

Set $\eta_3 = \min(\eta_1/2, \eta_2)$. It follows from (3.31) and (3.32) that

$$I_{21} \leqq 2\sum_{k=m}^{m2^m} P\{|([\theta\tau]/[\mu_m\tau])-1| \geqq \eta_3, A_{km}\} + c\eta. \tag{3.33}$$

Since on the event A_{km} for $k \geqq m$

$$\begin{aligned} |([\theta\tau]/[\mu_m\tau])-1| &< 1-(\theta\tau-1)/(\mu_m\tau) < (2^{-m}+\tau^{-1})/\mu_m \\ &\leqq m^{-1}(1+2^m\tau^{-1}) < \eta_3 \qquad \text{for large } t, \end{aligned}$$

it follows from (3.33) that $I_{21} < c\eta$ for large t. Thus, from this result, (3.28), (3.29) and (3.30) we have

$$\limsup_{t\to\infty} P\{|V_{[\theta\tau]} - V_{[\mu_m\tau]}| > \varepsilon\} < \eta/2.$$

Hence, by this relation and (3.27), (3.24) holds. Therefore the theorem was proved.

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Department of Mathematics
Faculty of Science
Niigata University
Niigata 950-21
Japan