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ON THE RATE OF CONVERGENCE FOR SEQUENTIAL DENSITY ESTIMATION

By

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Abstract. The density estimator $f_{T_n}(t)=T_n^{-1}\sum_{j=1}^{T_n}h_j^{-1}K((t-X_j)/h_j)$ is considered, where T_n is a random index. A sharp rate of convergence for the uniform distance between the probability of $f_{T_n}(t)$, when properly normed, and the standard normal distribution is obtained. It is shown that for any $\varepsilon>0$, the optimum order of the uniform estimate is $0(n^{-1/3+\varepsilon})$. Similar results have been shown by other authors, but under different assumptions on T_n.

1. Introduction

The aim of this study is to show that the asymptotic performance order for the uniform distance between the distribution function of randomly indexed kernel estimators of a probability density function (when properly normed) and standard normal distribution can be extended to include a wider family of stopping rules than heretofore considered. This problem has a number of applications in sequential analysis and, thus, has attracted the attention of several authors. Carroll [2] introduced Wolverton-Wagner-type [8] random kernel estimators to derive the conditions under which they fulfill the central limit theorem. Later, Isogai [3] and Basu and Sahoo [1], by extrapolating the condition on the stopping times proposed by Carroll, provided an exact approximation order of the uniform distance. Our attention here focuses on obtaining the rate of convergence in the central limit theorem for more general, yet realistic, stopping rules.

Throughout this paper, we shall let $\{U_i;\ i\in N\}$ be a sequence of real-valued independently distributed random variables (i. d. r. v.'s) not necessarily identically distributed on $L_3(\Omega, F, P, \boldsymbol{R})$, with $E|U|^3<\infty$. Write $S_0=0$ and $S_n=\sum_{j=1}^n U_j$. We assume that $EU_i=0$, $EU_i^2=\sigma_i^2$, $E|U_i|^3=\beta_i^3$, $s_n^2=\sum_{j=1}^n EU_j^2=\sum_{j=1}^n\sigma_j^2$, $B_n^3=\sum_{j=1}^n E|U_j|^3=\sum_{j=1}^n\beta_j^3$, and $L_n=B_n^3/s_n^3$. Let $\{T_n;\ n\in N\}$ denote a sequence of integer-valued r. v. s defined on the same probability space $\{\Omega, F, P\}$, not neces-

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sarily independent of the sequence $\{U_i;\ i\in \boldsymbol{N}\}$, and T is a positive r.v. also defined on $\{\Omega, F, P\}$ and not independent of $\{U_i;\ i\in \boldsymbol{N}\}$. The symbol "$c$" denotes a generic positive constant not necessarily the same at each appearance, while c_i, $i=1, 2, \cdots$, denote particular versions of c. Define $a_n=0(b_n)$ $(b_n\geqq 0)$, if $\exists c>0$, such that $|a_n|\leqq cb_n$ for all n. For $x\in \boldsymbol{R}$, we let $[x]=1$, if $x\leqq 1$ and $[x]=\max\{n\in \boldsymbol{N}: n\leqq x\}$, if $x>1$. $I(E)$ represents the indicator function of the event E, and $\lg_+ x=\max\{1, \log x\}$.

Given a sequence $\{X_i;\ i\in \boldsymbol{N}\}$ of independently and identically distributed r.v.s with density function $f(t)$, define

$$f_{T_n}(t)=\frac{1}{T_n}\sum_{j=1}^{T_n} h_j^{-1}K((t-X_j)/h_j) \tag{1.1}$$

as a random estimator for $f(t)$. Here, $K(\cdot)$ represents a kernel and $\{h_n;\ n\in \boldsymbol{N}\}$, a sequence of constants.

We shall deal here with univariate estimators; the extension to a multivariate case is straightforward. Rewriting the estimator given in (1.1) using the notations discussed above, it is clear that

$$T_n(f_{T_n}(t)-f(t))=\sum_{j=1}^{T_n} U_j+Z_{T_n}=S_{T_n}+Z_{T_n} \quad \text{say,} \tag{1.2}$$

where

$$U_j=h_j^{-1}K((t-X_j)/h_j)-Eh_j^{-1}K((t-X_j)/h_j)$$

$$Z_{T_n}=T_n(\hat{f}_{T_n}(t)-f(t)),$$

$$\hat{f}_{T_n}(t)=(1/T_n)\sum_{j=1}^{T_n} Eh_j^{-1}K((t-X_j)/h_j).$$

The sequence $\{U_j;\ j\in \boldsymbol{N}\}$, as defined in (1.2), consists of only i.d.r.v.'s which are not identical.

One of the objectives of this study is to introduce a certain class of stopping rules such that under some sufficient conditions, the uniform distance between the distribution of the random kernel estimator of the density (when properly normed) and standard normal distribution becomes the best possible.

When T is assumed to be constant, it is traditionally used to impose a similar condition to

$$P(|s_{T_n}^2 s_{[nT]}^{-2}-1|>\varepsilon_n)=0(\varepsilon_n^{1/2}), \qquad \text{where } n^{-1}\leqq \varepsilon_n\to 0. \tag{1.3}$$

The reason we use $s_{T_n}^2$ and $s_{[nT]}^2$ here instead of T_n and $[nT]$ respectively, is because U_is are just independently distributed not identical. On the other hand, when T is non-constant, an additional assumption on it is required which characterizes its dependence on the process $\{U_i;\ i\in \boldsymbol{N}\}$. Letting $\sigma(T)$ be the σ-field generated by T, define

$$\rho(\sigma(T), F_n)=\sup_{A\in\sigma(T)}\inf_{B\in F_n} P(A\Delta B)$$

and

$$\Delta(T_n)=\sup_{x\in \boldsymbol{R}}|P(S_{T_n}+Z_{T_n}\leqq x s_{T_n})-\Phi(x)|,$$

where Φ is the cumulative standard normal distribution function.

If, for some $0<\alpha\leqq 1$ and β,

(1.4) $$\rho(\sigma(T),\ F_n)=0(s_n^{-\alpha}(\lg_+ s_n)^{\beta}).$$

Then, as in Landers and Rogge [6], we shall show that

(1.5) $$\Delta(T_n)=0(\varepsilon_n^{1/2})+0(\delta_n),$$

where δ_n is defined in the following section.

It is worth mentioning that condition (1.4) is legitimate in the form presented. Specifically, if $EU_i=0$, $E|U_i|=\alpha_i$ and $EU_i^2=\sigma_i^2<\infty$ and if the stopping rule T satisfies $E\sum_{i=1}^{T}\alpha_i<\infty$ and $Es_T<\infty$, for $s_T^2=\sum_{i=1}^{T}\sigma_i^2$, then $\rho(\sigma(T),\ F_n)\leqq \sup_{B\in\sigma(T)}P(s_T^2\in B\Delta(s_T^2\cap\{s_1^2,\ \cdots,\ s_n^2\}))=P(s_T^2>s_n^2)\leqq s_n^{-1}Es_T=0(s_n^{-1})$. Also, some of the remarks pointed out by Landers and Rogge [6] for justifying the use of Hausdorff's metric $\rho(\cdot,\ \cdot)$, and the optimality of the rate in the uniform distance for the i.i.d. case, can be naturally extended to the i.d. case, which this paper deals with.

The remainder of the paper is organized as follows. In section 2, we extend the results of [6] to the sequences of i.d.r.v.s (not identical). The method presented here has some similarities to that of [6]; however, the details in the proofs are considerably different. Section 3 contains the proofs of the results stated in Section 2, and the proofs of certain auxiliary lemmas are delayed to Section 5. The main results on the kernel type estimators are demonstrated in Section 4.

2. The Order of Approximation of the Distribution of S_{T_n}.

In this section, we first exhibit the sufficient conditions for estimating the uniform distance between the distribution of the randomly indexed sum of i.r.v.s and $\Phi(x)$.

Let $\{U_i;\ i\in N\}$ be a sequence of i.r.v.'s and let $S_n=\sum_{j=1}^{n}U_j$. The list of conditions are:

C1: $EU_j=0,\ EU_j^2=\sigma_j^2,\ E|U_j|^3=\beta_j^3<\infty,\ s_n^2=\sum_{j=1}^{n}\sigma_j^2,\ B_n^3=\sum_{j=1}^{n}\beta_j^3$ and $L_n=B_n^3/s_n^3$.

C2: $L_n s_n$ tends to infinity as n tends to infinity.

Let the sequence of constants $\{\varepsilon_i;\ i\in N\}$ be such that $\varepsilon_n^{1/2}\geqq L_n$, and let $T_n:\Omega\to N$; $n\in N$, and $T:\Omega\to[c,\ \infty)$ be F-measurable with $c>0$. We then assume that

C3: $P(|s_{T_n}^2 s_{[nT]}^{-2}-1|>\varepsilon_n)=0(\varepsilon_n^{1/2})$, where $s_{T_n}^2=\sum_{j=1}^{T_n}\sigma_j^2$,

C4: $\rho(\sigma(T),\ F_n)=0(s_n^{-\alpha}(\lg_+ s_n)^{\beta})$ for $\alpha\in(0,\ 1]$, $\beta\in \boldsymbol{R}$,

$$C5: s_n=0(n^{\gamma}), \quad L_n=0(n^{-\delta}) \quad \text{for } 0<\delta\leqq\gamma<1$$

with $\delta\leqq 1/2$ and $\gamma(1-\alpha)<\delta$, where α is as defined in C4.

Our main result is now summarized in the following theorem.

Theorem 1. *If condition* C1-C5 *are satisfied, then*

(2.1) $$\Delta(T_n, T)=\sup_{x\in R}|P(S_{T_n}\leqq x s_{[nT]})-\Phi(x)|=0(\varepsilon_n^{1/2})+0(\delta_n), \quad \textit{and}$$

(2.2) $$\Delta(T_n)=\sup_{x\in R}|P(S_{T_n}\leqq x s_{T_n})-\Phi(x)|=0(\varepsilon_n^{1/2})+0(\delta_n),$$

where

$$\delta_n(\alpha, \beta, \gamma, \delta)=\delta_n=\begin{cases} L_n & \alpha=1,\ \beta<-3/2, \\ L_n \lg_+\lg_+ s_n & \alpha=1,\ \beta=-3/2, \\ L_n(\lg_+ s_n)^{\beta+3/2} & \alpha=1,\ \beta>-3/2, \\ L_n^{1-\gamma(1-\alpha)/\delta}(\lg_+ s_n)^{\beta+\alpha+1/2-\gamma(1-\alpha)} & \alpha<1,\ \beta\in \boldsymbol{R}. \end{cases}$$

The following remarks are in order prior to moving on to the proof of the Theorem.

(i) If we replace C3 by the strongest condition, i. e.,

$$P(|s_{T_n}^2 s_{[nT]}^{-2}-1|>\varepsilon_n)=0,$$

then arguing essentially the same way as [6], we can conclude that a better approximation order than $0(\delta_n)$ for (2.1) or (2.2) cannot be obtained.

(ii) If we replace C3 with a weaker condition, i. e.,

$$P(|s_{T_n}^2 s_{[nT]}^{-2}-1|>\varepsilon_n)=0(\varepsilon_n^{1/2}\alpha_n),$$

with α_n tending to infinity, then, in general, we can no longer obtain approximation order $0(\varepsilon_n^{1/2})+0(\delta_n)$ for (2.1) and (2.2) as stated in the Theorem.

(iii) If T is a constant, then $\rho(\sigma(T), F_n)=0$. We cannot obtain a better approximation than $0(\varepsilon_n^{1/2})$ for (2.1) and (2.2) under assumption C3. If T is a constant, our result falls into Isogai's approach.

(iv) If $s_n L_n=0(1)$, then the results still hold. This corresponds to the case where the process is "almost identically distributed". In the same class, of course, falls the Landers and Rogge case (i. i. d.).

3. Proofs

This segment of our work has considerable relationship to Landers and Rogge [6], where similar results are proved for the i. i. d. case. For reasons of convenience, we operate with the same notations as in their work. As mentioned earlier, the proofs of some useful auxiliary lemmas are presented in Section 5.

Proof of (2.2). It can be seen that

$$\Delta(T_n)=\sup_{x\in R}|P(S_{T_n}\leqq xs_{T_n})-\Phi(x)|$$
$$=\sup_{x\in R}|P(\xi_n S_{T_n}\leqq xs_{[nT]})-\Phi(x)|,$$

where $\xi_n=s_{[nT]}s_{T_n}^{-1}$. Consequently, from (2.1), (2.2) may be proved by using Lemma 2 and by showing that $P(|s_{[nT]}s_{T_n}^{-1}-1|>(2\varepsilon_n)^{1/2})=0(\varepsilon_n^{1/2})$. It is easy to check that

$$P(|\xi_n-1|>(2\varepsilon_n)^{1/2})\leqq P(|s_{[nT]}^2 s_{T_n}^{-2}-1|>2\varepsilon_n)$$
$$\leqq P(|s_{T_n}^2 s_{[nT]}^{-2}-1|>\varepsilon_n)=0(\varepsilon_n^{1/2}).$$

The proof of (2.2) is now complete.

Proof of (2.1). To show (2.1), it is sufficient to obtain that

(3.1) $$\mathrm{I}=\sup_{x\in R}|P(S_{[nT]}\leqq xs_{[nT]})-\Phi(x)|=0(\delta_n), \quad \text{and}$$

(3.2) $$\mathrm{II}=\sup_{x\in R}(P(\exists\nu\in I_n(\omega):S_\nu(\omega)\leqq xs_{[nT]})-P(\forall\nu\in I_n(\omega):S_\nu(\omega)\leqq xs_{[nT]}))$$
$$=0(\varepsilon_n^{1/2})+0(\delta_n),$$

where $I_n(\omega)=\{\nu\in N: (1-\varepsilon_n)s_{[nT]}^2\leqq s_\nu^2\leqq(1+\varepsilon_n)s_{[nT]}^2\}$. For convenience, we set $R_n(x)$ to be the event $\{\exists\nu\in I_n(\omega): S_\nu(\omega)\leqq xs_{[nT]}\}$ and $Q_n(x)$ to be the event $\{\forall\nu\in I_n(\omega): S_\nu(\omega)\leqq xs_{[nT]}\}$.

From C3, it follows that $P(T_n\notin I_n(\omega))=0(\varepsilon_n^{1/2})$ and $[nT]\in I_n(\omega)$. It can be shown that

(3.3) $$P(Q_n(x))+0(\varepsilon_n^{1/2})\leqq P(S_{T_n}\leqq xs_{[nT]})\leqq P(R_n(x))+0(\varepsilon_n^{1/2}), \quad \text{and}$$

(3.4) $$P(Q_n(x))\leqq P(S_{[nT]}\leqq xs_{[nT]})\leqq P(R_n(x)).$$

It follows from (3.3) and (3.4) that

(3.5) $$\Delta(T_n, T)\leqq\sup_{x\in R}|P(S_{[nT]}\leqq xs_{[nT]})-\Phi(x)|$$
$$+\sup_{x\in R}|P(S_{T_n}\leqq xs_{[nT]})-P(S_{[nT]}\leqq xs_{[nT]})|$$
$$\leqq\mathrm{I}+\mathrm{II}+0(\varepsilon_n^{1/2}).$$

The proof is completed, provided (3.1) and (3.2) are true.

Proof of (3.1). Let $N_1=\{2^i; i\in N\}$ and $N_n=\{\nu\in N_1; \nu\leqq[n/(c_1\log n)]\}$, where

$c_1>0$ is a fixed constant. Set $j(n)=\max N_n$, and assume that $s_n^2 \geqq 3$, and let c_1 be such that $L_\nu^{-2} \leqq L_n^{-2}/2$, for any $\nu \in N_n$ and $n \in N$.

For any event $B_m \in F$; $m \in N$, we define $B_m(\nu)=\{P(B_m | F_\nu)>1/2\} \in F_\nu$. Let $m \geqq 2$. We partition the event B_m as follows:

$$I(B_m)=I(B_m)-I(B_m(j(m)))+\Sigma_{\nu \in N_m}(I(B_m(\nu))-I(B_m(\nu/2)))+I(B_m(1)). \tag{3.6}$$

We define

$$D_\nu=\sup_{x \in R} | E\{I(S_m \leqq x s_m)-\Phi(x)\}\{I(B_m(\nu))-I(B_m(\nu/2))\} | \tag{3.7}$$

for $\nu=1, 2, \cdots, j(m)$. With the above definitions in mind, it can be seen that

$$\begin{aligned} \Delta_m &= \sup_{x \in R} | P(S_m \leqq x s_m,\ B_m)-\Phi(x) P(B_m) | \\ &\leqq E | I(B_m)-I(B_m(j(m))) | +\Sigma_{\nu \in N_m \cup \{1\}} D_\nu \\ &\leqq d(B_m,\ F_{j(m)})+\Sigma_{\nu \in N_m \cup \{1\}} D_\nu . \end{aligned} \tag{3.8}$$

The last inequality in (3.8) follows from Lemma 1. Moreover, via Lemma 4, it turns out that

$$\begin{aligned} D_\nu &= \sup_{x \in R} \left| \int (P(S_m \leqq x s_m | F_\nu)-\Phi(x))(I(B_m(\nu))-I(B_m(\nu/2)))dP \right| \\ &\leqq c_2 L_m \left\{ s_\nu P(B_m(\nu) \Delta B_m(\nu/2))+(s_\nu L_\nu)^{-1} \int_{B_m(\nu) \Delta B_m(\nu/2)} |S_\nu| \, dP \right\}, \quad \text{and} \end{aligned} \tag{3.9}$$

$$D_1 \leqq c_3 L_m \left\{ P(B_m(1))+\int_{B_m(1)} |U_1| \, dP \right\}. \tag{3.10}$$

In conjunction with (3.9) and (3.10), (3.8) yields

$$\begin{aligned} \Delta_m &\leqq d(B_m,\ F_{j(m)})+c_3 L_m \left\{ P(B_m(1))+\int_{B_m(1)} |U_1| \, dP \right\} \\ &\quad + c_2 L_m \Sigma_{\nu \in N_m} \left\{ s_\nu P(B_m(\nu) \Delta B_m(\nu/2))+(s_\nu L_\nu)^{-1} \int_{B_m(\nu) \Delta B_m(\nu/2)} |S_\nu| \, dP \right\} \\ &\leqq d(B_m,\ F_{j(m)})+c_3 L_m \left\{ P(B_m(1))+\int_{B_m(1)} |U_1| \, dP \right\} \\ &\quad + 2 c_2 L_m \Sigma_{\nu \in N_m} s_\nu d(B_m,\ F_{\nu/2})+c_2 L_m \Sigma_{\nu \in N_m} (s_\nu L_\nu)^{-1} \int_{B_m(\nu) \Delta B_m(\nu/2)} |S_\nu| \, dP. \end{aligned} \tag{3.11}$$

The last inequality follows from the fact that

$$P(B_m(\nu) \Delta B_m(\nu/2)) \leqq 2 d(B_m,\ F_{\nu/2}).$$

Next, we define $B_m=\{[nT]=m\} \in \sigma(T)$. For our purposes, we shall assume that $nc \geqq 3$, for $n \in N$, and that

(3.12)
$$\mathrm{I}=\sup_{x\in R}|P(S_{[nT]}\leqq xs_{[nT]})-\Phi(x)|$$
$$=\sup_{x\in R}|\textstyle\sum_{m\geq nc}(P(S_m\leqq xs_m,\ B_m)-\Phi(x)P(B_m))|$$
$$\leqq\textstyle\sum_{m\geq nc}\sup_{x\in R}|P(S_m\leqq xs_m,\ B_m)-\Phi(x)P(B_m))|.$$

Invoking (3.11), it is seen as in [6], that for any event $G_\nu=\{\omega\in\Omega:|S_\nu(\omega)|>2s_\nu(\lg_+L_\nu^{-1})^{1/2}\}$,

(3.13)
$$\mathrm{I}\leqq\textstyle\sum_{j=1}^4 R_j(n),$$

where
$$R_1(n)=\textstyle\sum_{m\geq nc}d(B_m,\ F_{j(m)}),$$
$$R_2(n)=c_4\textstyle\sum_{m\geq nc}L_m\left\{P(B_m(1))+\int_{B_m(1)}|U_1|\,dP\right\},$$
$$R_3(n)=c_5\sum_{m\geq nc,\,\nu\in N_m}L_m(4s_\nu^2\lg_+L_\nu^{-1})^{1/2}d(B_m,\ F_{\nu/2}),\quad\text{and}$$
$$R_4(n)=c_6\sum_{m\geq nc,\,\nu\in N_m}L_m(L_\nu s_\nu)^{-1}\int_{G_\nu\cap(B_m(\nu)\Delta B_m(\nu/2))}|S_\nu|\,dP.$$

The remaining part of the proof of (3.1) is devoted to showing that $R_i(n)=0(\delta_n)$, for $i=1$, 2, 3 and 4. It follows from Lemma 1 that

(3.14)
$$R_1(n)\leqq\textstyle\sum_{m\geq nc}d(B_m,\ F_{j([nc])})\leqq 4\rho(\sigma(T),\ F_{j([nc])}),$$

which, in conjunction with C4 and C5, yields

(3.15)
$$\rho(\sigma(T),\ F_{j([nc])})\leqq c_7(s_n/\log_+s_n)^{-\alpha}(\lg_+(s_n/\lg_+s_n))^{\beta}$$
$$\leqq c_8 s_n^{-\alpha}(\lg_+s_n)^{\alpha+\beta}$$
$$\leqq c_9 L_n^{1-\gamma(1-\alpha)/\delta}(\lg_+s_n)^{\beta+\alpha+1/2-\gamma(1-\alpha)}$$
$$\leqq c_{10}\delta_n\,.$$

Next, we proceed with $R_2(n)$. The events $B_m(1)$, for $m\geqq nc$, are mutually exclusive. It is deduced that

(3.16)
$$R_2(n)=c_4\textstyle\sum_{m\geq nc}L_m\left\{P(B_m(1))+\int_{B_m(1)}|U_1|\,dP\right\}\leqq c_{11}L_n\leqq c_{11}\delta_n\,.$$

The last statement follows because $E|U_1|<\infty$, $\sum_{m\geq nc}P(B_m(1))\leqq 1$ and $L_{[nc]}=0(L_n)$. To establish $R_3(n)=0(\delta_n)$, it can be seen that

(3.17)
$$R_3(n)=c_5\sum_{m\geq nc,\,\nu\in N_m}L_m(4s_\nu^2\lg_+L_\nu^{-1})^{1/2}d(B_m,\ F_{\nu/2})$$
$$=c_5\sum_{i\in N_1}\ \sum_{inc/2\leq m<inc,\,\nu\in N_m}L_m(4s_\nu^2\lg_+L_\nu^{-1})^{1/2}d(B_m,\ F_{\nu/2})$$
$$=c_{12}\sum_{i\in N_1}L_{[inc/2]}\sum_{inc/2\leq m<inc,\,\nu\in N_m}(4s_\nu^2\lg_+L_\nu^{-1})^{1/2}d(B_m,\ F_{\nu/2}).$$

As we have pointed out above, the events $B_m \in \sigma(T)$, for $m \geqq nc$ and are disjoint. According to Lemma 1 and C2, it follows that $L_\nu^{-1} \leqq c_{13} s_\nu$, and

$$(3.18) \qquad \begin{aligned} R_3(n) &\leqq 4c_{12} \sum_{i \in N_1} L_{[inc/2]} \sum_{\nu \in N_{[inc]}} (4s_\nu^2 \lg_+ L_\nu^{-1})^{1/2} \rho(\sigma(T), F_{\nu/2}) \\ &\leqq c_{14} \sum_{i \in N_1} L_{[inc/2]} \sum_{\nu \in N_{[inc]}} (4s_\nu^2 \lg_+ L_\nu^{-1})^{1/2} s_\nu^{-\alpha} (\lg_+ s_\nu)^\beta \\ &\leqq c_{15} \sum_{i \in N_1} L_{[inc/2]} \sum_{\nu \in N_{[inc]}} s_\nu^{1-\alpha} (\lg_+ s_\nu)^{\beta+1/2} . \end{aligned}$$

In view of C5 and Lemma 5, (3.18) is deduced to

$$(3.19) \qquad \begin{aligned} R_3(n) &\leqq c_{16} \sum_{i \in N_1} (in)^{-\delta} \sum_{\nu \in N_{[inc]}} \nu^{\gamma(1-\alpha)} (\log \nu)^{\beta+1/2} \\ &\leqq \begin{cases} c_{17} n^{-\delta+\gamma(1-\alpha)} (\log n)^{\beta+1/2-\gamma(1-\alpha)} & \alpha<1, \ \beta \in \boldsymbol{R}, \\ c_{17} n^{-\delta} (\log n)^{3/2+\beta} & \alpha=1, \ \beta>-3/2, \\ c_{17} n^{-\delta} \log\log n & \alpha=1, \ \beta=-3/2, \\ c_{17} n^{-\delta} & \alpha=1, \ \beta<-3/2, \end{cases} \\ &\leqq c_{17} \delta_n . \end{aligned}$$

It remains to be shown that $R_4(n) = 0(\delta_n)$.

$$(3.20) \qquad \begin{aligned} R_4(n) &\leqq c_6 \sum_{m \geq nc, \nu \in N_1} L_m (L_\nu s_\nu)^{-1} \int_{G_\nu \cap (B_{m(\nu)} \Delta B_{m(\nu/2)})} |S_\nu| \, dP \\ &\leqq c_{18} L_n \sum_{m \geq nc, \nu \in N_1} (L_\nu s_\nu)^{-1} \Big(\int_{G_\nu \cap B_{m(\nu)}} + \int_{G_\nu \cap B_{m(\nu/2)}} \Big) \\ &\leqq c_{19} L_n \sum_{\nu \in N_1} L_\nu^{-1} \int_{G_\nu} |S_\nu^*| \, dP, \qquad \text{where } S_n^* = S_n / s_n . \end{aligned}$$

The following arguments are similar to those in [4]. Since $E|X| \leqq \sum_{k=0}^\infty P(|X| > k)$, then by setting

$$Y = (S_\nu^*/2)(\lg_+ L_\nu^{-1})^{-1/2} I(|S_\nu^*| > 2(\lg_+ L_\nu^{-1})^{1/2}) ,$$

it follows from Lemma 6 and Markov's inequality that for $\nu > 1$

$$\begin{aligned} L_\nu^{-1} \int_{G_\nu} |S_\nu^*| \, dP &\leqq 2(\lg_+ L_\nu^{-1})^{1/2} L_\nu^{-1} \sum_{k=0}^\infty P(|S_\nu^*| > 2k(\lg_+ L_\nu^{-1})^{1/2}) \\ &\leqq 2(\lg_+ L_\nu^{-1})^{1/2} L_\nu^{-1} \sum_{k=0}^\infty \Big\{ c_{20} L_\nu k^{-6} (\lg_+ L_\nu^{-1})^{-3} + \sum_{j=1}^\nu P\Big(|U_j| > \frac{k}{3} s_\nu (\lg_+ L_\nu^{-1}) \Big)^{1/2} \Big\} . \end{aligned}$$

Since $L_\nu = 0(\nu^{-\delta})$ and $i \log 2 \leqq \log \nu(i)$ for $\nu \in N_1$, it results from Lemma 6 in [4] that

$$(3.21) \qquad R_4(n) \leqq c_{22} L_n = 0(\delta_n) .$$

By inserting (3.15), (3.16), (3.19) and (3.21) into (3.13), the proof of (3.1) is completed.

Proof of (3.2). We set $I_n(m)=\{\nu\in N: s_m^2(1-\varepsilon_n)\leqq s_\nu^2\leqq s_m^2(1+\varepsilon_n)\}$. It is easy to see that

$$(3.22)\qquad P(Q_n(x))-P(R_n(x))=\sum_{m\geqq nc}(P(Q_n(x)\cap B_m)-P(R_n(x)\cap B_m))$$

$$=\sum_{m\geqq nc}(P(B_m,\ \exists\nu\in I_n(m): S_\nu\leqq xs_m)-P(B_m,\ \forall\nu\in I_n(m): S_\nu\leqq xs_m)).$$

Let $A_m=\{P(B_m\mid F_{j([nc])})>1/2\}\in F_{j([nc])}$. Clearly, $\{A_m,\ m\geqq nc\}$ consists of mutually exclusive events. By Lemma 1, $P(B_m\Delta A_m)=d(B_m,\ F_{j([nc])})$. Thus,

$$(3.23)\qquad P(Q_n(x))-P(R_n(x))\leqq 2\sum_{m\geqq nc}d(B_m,\ F_{j([nc])})$$

$$+\sum_{m\geqq nc}(P(A_m,\ \exists\nu\in I_n(m): S_\nu\leqq xs_m)-P(A_m,\ \forall\nu\in I_n(m): S_\nu\leqq xs_m)).$$

From Lemma 1 and (3.15)

$$(3.24)\qquad \sum_{m\geqq nc}d(B_m,\ F_{j([nc])})\leqq 4\rho(\sigma(T),\ F_{j([nc])})\leqq 4c_{10}\delta_n\,.$$

From Lemma 7,

$$(3.25)\qquad P(A_m,\ \exists\nu\in I_n(m): S_\nu\leqq xs_m)-P(A_m,\ \forall\nu\in I_n(m): S_\nu\leqq xs_m)$$

$$=\int_{A_m}P(\exists\nu,\ l\in I_n(m): S_\nu\leqq xs_m<S_l\mid F_{j([nc])})dP$$

$$\leqq c_{23}P(A_m)(\hat{L}_{r(n)}+\{2\varepsilon_n s_m^2/(s_m^2(1-\varepsilon_n)-s_{j([nc])}^2)\}^{1/2}\leqq c_{24}P(A_m)\varepsilon_n^{1/2},$$

where $r(n)=\min\{\nu\in N: s_m^2(1-\varepsilon_n)\leqq s_\nu^2\leqq s_m^2(1+\varepsilon_n)\}=\min I_n(m)$. The last inequality in (3.25) follows, since $L_n\leqq\varepsilon_n^{1/2}$. In view of (3.24) and (3.25), it is observed that

$$(3.26)\qquad \mathrm{II}\leqq 8c_{10}\delta_n+c_{25}\varepsilon_n^{1/2}\sum_{m\geqq nc}P(A_m)=0(\delta_n)+0(\varepsilon_n^{1/2})\,.$$

The proof of Theorem 1 is now complete.

4. An approximation Order of Convergence of the Distribution of $f_{T_n}(t)$

In this section, we exploit the method developed above to obtain the rate of convergence for the uniform distance between the distribution of our random estimator for the p.d.f. and the standard normal distribution.

We first state the conditions for the kernel function, the sequence $\{h_n: n\in N\}$, and the probability density function.

Assume $K(\cdot)$ to be a measurable function with

K_1 : $\sup_{t\in R}|K(t)|<\infty$,

K_2 : $\int|K(t)|\,dt<\infty$,

K_3 : $\int K(t)dt=1$,

K_4 : $\lim_{|t|\to\infty}|tK(t)|=0$,

K_5 : $\int tK(t)dt=0$.

For the sequence $\{h_n;\ n\in N\}$, we assume that it is a monotone decreasing sequence and

H_1 : $\lim_{n\to\infty}h_n=0$,

H_2 : $\lim_{n\to\infty}nh_n=\infty$,

H_3 : $\lim_{n\to\infty}(h_n^r/n)\sum_{j=1}^n h_j^{-r}=\gamma_r$ for $r=1, 2$.

It is traditional in such work to assume that

$$H_4:\ h_n=n^{-\lambda},\qquad \lambda\in(1/3,\ 1)\,.$$

As far as the probability density is concerned, we shall assume that

F_1 : $f(\cdot)$ is uniform continuous, integrable and bounded (in order to be a density function),

F_2 : $f''(t)$ exists for all $t\in R$, and $\sup_{t\in R}|f''(t)|<\infty$.

Adapting the notations presented in Sections 1 and 2, we shall present a result similar to that given by Yamato [9] and Basu and Sahoo [1]. The purpose of this result is two-fold: First, it provides the rate of convergence for the uniform distance between the distribution of Wolverton-Wagner's estimator (when properly normed) and the standard normal distribution (for fixed size sample), and then it states the exact approximation order of L_n, upon which the whole analysis depends.

Theorem 2. *If* K_1-K_4, H_1-H_3 *and* F_1 *hold, then for fixed* $t\in \boldsymbol{R}$,

$$M_n=\sup_{x\in R}\left|P\Big(\frac{f_n(t)-Ef_n(t)}{(\mathrm{Var}\,(f_n(t)))^{1/2}}\leq x\Big)-\Phi(x)\right|=0((nh_n)^{-1/2}),$$

where $f_n(t)$ *is the fixed-size kernel density estimator.*

In addition, if H_4 *holds, then* $M_n=0(n^{-(1-\lambda)/2})$.

Proof. Since $(f_n(t)-Ef_n(t))/(\mathrm{Var}\,(f_n(t)))^{1/2}=S_n/s_n$, then by Petrov ([7], p.

115), it follows that

(4.1) $$M_n \leqq c L_n .$$

If we show that $L_n = 0((nh_n)^{-1/2})$, the proof of Theorem 2 is completed. By C_r-inequality,

(4.2) $$B_n^3 = \Sigma_{j=1}^n E|U_j|^3 \leqq 4\Sigma_{j=1}^n \{E|Y_j|^3 + |EY_j|^3\}, \quad Y_j = \frac{1}{h_j} K((t-X_j)/h_j).$$

As in [9, (5.8) and (5.9)],

(4.3) $$E|Y_j|^3 \leqq \|f\| \int |K(u)|^3 du / h_j^2, \quad \|f\| = \sup_{t\in \boldsymbol{R}} f(t), \quad \text{and}$$

(4.5) $$|EY_j| \leqq \|f\| \int |K(u)| du .$$

Consequently, since H3 holds, it is seen that for large n,

(4.6) $$B_n^3 h_n^2 / n \leqq c_2 \gamma_2 \|f\| \int |K(u)|^3 du .$$

We shall now examine the behavior of $\mathrm{Var}(S_n)$. It is clear that

(4.7) $$\mathrm{Var}(S_n) = \Sigma_{j=1}^n \left\{ h_j^{-2} \int K^2\left(\frac{u}{h_j}\right) f(t-u) du - \left(h_j^{-1} \int K\left(\frac{u}{h_j}\right) f(t-u) du \right)^2 \right\}.$$

Arguments similar to those of [9, Section 3], can be used to show that if $f(t)$ is uniform, continuous, and bounded, and if K_1-K_4 and H_1-H_3 hold, then $\mathrm{Var}(S_n) \asymp (n/h_n)$, i.e., $(h_n/n)\mathrm{Var}(S_n)$ is bounded away from zero and infinity.

We conclude, as does Yamato [9, (5.11)], that

(4.8) $$(nh_n)^{1/2} L_n = (B_n^3 h_n^2 / n)(s_n^2 h_n / n)^{-3/2} = 0(1) .$$

This completes the proof of Theorem 2.

The rest of this section is consigned to the random estimator $f_{T_n}(t)$. Calling upon Theorem 1, the following theorem may also be proved.

Theorem 3. *If conditions* C_1, C_3, C_4, K_1-K_4, H_1-H_4 *and* F_1 *hold, then for fixed* $t \in \boldsymbol{R}$,

$$\sup_{x\in \boldsymbol{R}} |P(T_n(f_{T_n}(t) - \hat{f}_{T_n}(t))/s_{[nT]} \leqq x) - \Phi(x)| = 0(\varepsilon_n^{1/2}) + 0(\delta_n) .$$

Comments. (i) Since $s_n^2 \asymp n^{1+\lambda}$ and $L_n = 0(n^{-(1-\lambda)/2})$, it follows that

$P(|s_{T_n}^2 s_{[nT]}^{-2} - 1| > \varepsilon_n) = 0(\varepsilon_n^{1/2})$ is equivalent to $P(|T_n^{1+\lambda}[nT]^{-1-\lambda} - 1| > \varepsilon_n) = 0(\varepsilon_n^{1/2})$.

(ii) We note that C2 and C5 are always valid because of H_4 and (4.8).

As an extension, the next Theorem asserts that the conclusions of Theorem

3 still hold even if $\hat{f}_{T_n}(t)$ is replaced by $f(t)$.

Theorem 4. *If* C1, C3, C4, K1-K5, H1-H4 *and* F1-F2 *hold, then for fixed* $t\in \boldsymbol{R}$,

$$\sup_{x\in R}|P(T_n(f_{T_n}(t)-f(t))s_{[nT]}^{-1}\leqq x)-\Phi(x)|=0(\varepsilon_n^{1/2})+0(\delta_n).$$

Proof. As pointed out in (1.2), $T_n(f_{T_n}(t)-f(t))=S_{T_n}+Z_{T_n}$. The asymptotic behavior of S_{T_n} is already known in Theorem 3. Hence, it remains to find out how Z_{T_n} acquits. Using K_1-K_3 and K_5, H_4 and F_2, we see that

$$\begin{aligned}(4.9)\qquad |Z_{T_n}| &=\left|\textstyle\sum_{j=1}^{T_n}\int K(u)(f(t-h_ju)-f(t))du\right|\\ &=\left|\textstyle\sum_{j=1}^{T_n}\int K(u)\{h_juf'(t)+(h_j^2/2)u^2f''(v)\}du\right|\\ &=c_1\textstyle\sum_{j=1}^{T_n}h_j^2=c_2T_n^{1-2\lambda}\qquad \text{a.e.,}\end{aligned}$$

where v belongs to (t, u) or (u, t). Since $\varepsilon_n^{1/2}\geqq L_n=cn^{-(1-\lambda)/2}$ ($\lambda>1/3$), it follows that

$$\begin{aligned}(4.10)\qquad P(T_n(\hat{f}_{T_n}(t)-f(t))s_{[nT]}^{-1}>\varepsilon_n^{1/2}) &\leqq P(T_n^{1-2\lambda}[nT]^{(\lambda-1)/2}>c_3\varepsilon_n^{1/2})\\ &\leqq P(T_n^{1-2\lambda}[nT]^{2\lambda-1}>c_4n^{-1+3\lambda})\\ &\leqq(1/2)P(|(T_n^{1+\lambda}[nT]^{-1-\lambda})-1|>c_5n^{(-1+3\lambda)(1+\lambda)/(1-\lambda)}-1)\\ &\leqq(1/2)P(|s_{T_n}^2s_{[nT]}^{-2}-1|>\varepsilon_n)=0(\varepsilon_n^{1/2}).\end{aligned}$$

By Lemma 3 and by choosing $\beta_n=c_6\varepsilon_n^{1/2}+c_7\delta_n$ and $\alpha_n=c_8\varepsilon_n^{1/2}$, the result is seen immediately.

The next Corollary is a consequence of Theorem 4.

Corollary. *If* C_1, C_3, C_4, K_1-K_5, H_1-H_4 *and* F_1-F_2 *hold, then for fixed* $t\in \boldsymbol{R}$,

$$\sup_{x\in R}|P(T_n(f_{T_n}(t)-f(t))s_{T_n}^{-1}\leqq x)-\Phi(x)|=0(\varepsilon_n^{1/2})+0(\delta_n).$$

5. Auxiliary Results

In this section, we collect all the Lemmas which were used in the proofs of Theorems 1-4.

Lemma 1 ([5]). *Let* $F_{(1)}$ *and* $F_{(2)}$ *be subfields of* F, *then*

(i) *if* $\{B_n;\ n\in N\}$ *is a sequence of disjoint events of* $F_{(1)}$, *the following result is true*

$$\textstyle\sum_{n\in N}d(B_n, F_{(2)})\leqq 4\rho(F_{(1)}, F_{(2)}),\qquad \textit{and}$$

(ii) *if* $A \in F$, *and if* $B=\{P(A|F_{(1)})>1/2\}$, *then we have that*

$$P(A \Delta B)=d(A, F_{(1)}).$$

The proofs of the following two lemmas are simple and, therefore, are omitted.

Lemma 2. *Let* $\{\xi_n ; n \in N\}$ *and* $\{Y_n ; n \in N\}$ *be sequences of r.v.s and let* $W_n=\xi_n Y_n$. *Suppose that* $\{\alpha_n ; n \in N\}$ *and* $\{\beta_n ; n \in N\}$ *are two sequences of positive constants which tend to zero as n tends to infinity. If*

$$\sup_{x \in R}|P(Y_n \leqq x)-\Phi(x)|=0(\beta_n) \quad \text{and} \quad P(|\xi_n-1|>\alpha_n)=0(\beta_n),$$

then $\sup_{x \in R}|P(W_n \leqq x)-\Phi(x)|=0(\beta_n)+0(\alpha_n)$.

Lemma 3. *Let* $\{Z_n ; n \in N\}$ *and* $\{Y_n ; n \in N\}$ *be sequences of r.v.s and let* $W_n=Z_n+Y_n$, *for* $n=1, 2, \cdots$. *Suppose that for any sequence* $\{\beta_n ; n \in N\}$ *of positive constants, we have that*

$$\sup_{x \in R}|P(Z_n \leqq x)-\Phi(x)|=0(\beta_n).$$

Then, for any sequence of positive constants $\{\alpha_n ; n \in N\}$, *it follows that*

$$\sup_{x \in R}|P(W_n \leqq x)-\Phi(x)|=0(\alpha_n)+0(\beta_n)+P(|Y_n|>\alpha_n).$$

Lemma 4. *Let* $\{U_i ; i \in N\}$ *be a sequence of i.r.v.s such that* $EU_i=0$, $EU_i^2=\sigma_i^2$ *and* $E|U_i|^3<\infty$. *Let* $\{S_n ; n \in N\}$ *be a sequence of partial sums with* $s_n^2=\sum_{j=1}^n \sigma_j^2$, $B_n^3=\sum_{j=1}^n \beta_j^3$ *and* $L_n=B_n^3/s_n^3$. *Define* $F_k=\sigma(U_1, U_2, \cdots, U_k)$, *then for any* $2L_k^{-1} \leqq L_n^{-1}$ *we have P-a.e.*

$$\sup_{x \in R}|P(S_n \leqq x s_n | F_k)-\Phi(x)|=c L_n\{s_k+(s_k L_k)^{-1}|S_k|\}.$$

Proof. We note that

$$\begin{aligned}
\Delta_n(F_k) &= \sup_{x \in R}|P(S_n \leqq x s_n | F_k)-\Phi(x)| \\
&= \sup_{x \in R}|P(S_k+(S_n-S_k) \leqq x s_n | F_k)-\Phi(x)| \\
&= \sup_{x \in R}\left|P\left(\frac{S_n-S_k}{(s_n^2-s_k^2)^{1/2}} \leqq x \frac{s_n}{(s_n^2-s_k^2)^{1/2}}-\frac{S_k}{(s_n^2-s_k^2)^{1/2}} \middle| F_k\right)-\Phi(x)\right| \\
&\leqq \sup_{x \in R}|P(S_n-S_k \leqq x(s_n^2-s_k^2)^{1/2})-\Phi(x)| \\
&\quad +\sup_{x \in R}\left|\Phi\left(x \frac{s_n}{(s_n^2-s_k^2)^{1/2}}-\frac{S_k}{(s_n^2-s_k^2)^{1/2}}\right)-\Phi(x)\right| \\
&= T_1+T_2, \quad \text{say.}
\end{aligned}$$

It is clear ([7], p. 114) that

$$T_2 \leqq (s_n(s_n^2-s_k^2)^{-1/2}-1)/\sqrt{2\pi e}+|S_k|(s_n^2-s_k^2)^{-1/2}/\sqrt{2\pi}$$
$$\leqq c_1\{s_k^2/(s_n^2-s_k^2)+|S_k|/(s_n^2-s_n^{-2})^{1/2}\},$$

since $2s_k^3/B_k^3 \leqq s_n^3/B_n^3$ implies that $2B_n^3/B_k^3 \leqq s_n^3/s_k^3$. But $B_n^3/B_k^3 \geqq 1$, hence $2^{2/3}s_k^2 \leqq s_n^2$. And since $s_k^2/s_n^2<1$, it follows that $s_k^2/s_n^2<s_k/s_n$. Under these considerations, we obtain that $T_2 \leqq c_2 s_n^{-1}\{s_k+|S_k|\}$. Since s_n is a monotone increasing with respect to n, and $s_n L_n$ tends to infinity, T_2 can be bounded by $T_2 \leqq c_3 L_n\{s_k+(s_k L_k)^{-1}|S_k|\}$. From [7, p. 115], and since $2^{2/3}s_k^2 \leqq s_n^2$, it follows that $T_1 \leqq c_4(B_n^3-B_k^3)(s_n^2-s_k^2)^{-3/2} \leqq c_5 L_n$. This completes the proof of the lemma.

Lemma 5 ([6]). *Let $N_1=\{2^\nu;\ \nu\in \boldsymbol{N}\}$ and $N_n=\{\nu\in N_1:\nu\leqq[n/\log n]\}$. Then*

$$\sum_{\nu\in N_n}\nu^\varepsilon(\log \nu)^\delta=\begin{cases} 0(1) & \text{if } \varepsilon=0,\ \delta<-1, \\ 0(\log\log n) & \text{if } \varepsilon=0,\ \delta=-1, \\ 0((\log n)^{\delta+1}) & \text{if } \varepsilon=0,\ \delta>-1, \\ 0(n^\varepsilon(\log n)^{\delta-\varepsilon}) & \text{if } \varepsilon>0,\ \delta\in \boldsymbol{R}. \end{cases}$$

Lemma 6. *Let $\{U_i;\ i\in \boldsymbol{N}\}$ be a sequence of i.r.v.'s with $EU_i=0$, $EU_i^2=\sigma_i^2$ and $E|U_i|^3=\beta_i^3<\infty$. Let $\{S_n;\ n\in \boldsymbol{N}\}$ be a sequence of partial sums with $s_n^2=\sum_{j=1}^n\sigma_j^2$, $B_n^3=\sum_{j=1}^n\beta_j^3$ and $L_n=B_n^3/s_n^3$. Then, for all $t>0$ with $t^2\geqq 4\log_+ L_n^{-1}$ we have*

$$P(|S_n|>s_n t)\leqq c_1 L_n t^{-6}+R_n\,, \tag{5.1}$$

where $R_n=\sum_{k=1}^n P(|U_k|>c_2 t s_n)$, $c_2=1/6$.

Proof. The idea here is to show that $P(S_n\geqq s_n t)$ is bounded above by the R. H. S. of (5.1). Next, replacing the U_k by $-U_k$ and repeating the proof of the latter case, we establish that $P(S_n<-s_n t)$ is also bounded above by the same expression as in (5.1). This confirms that (5.1) is true. We proceed by showing that

$$P(S_n\geqq s_n t)\leqq c_1 L_n t^{-6}+R_n\,.$$

We define $U_k^*=U_k I(|U_k|\leqq c_2 s_n t)$, $c_2=1/6$, $k=1, 2, \cdots, n$, and $S_n^*=\sum_{j=1}^n U_j^*$. It is easy to check that

$$P(S_n\geqq t s_n)\leqq P(S_n^*\geqq t s_n)+\sum_{j=1}^n P(|U_j|>c_2 s_n t)\,. \tag{5.2}$$

Set $h=(s_n t)^{-1}\{2\lg_+ L_n^{-1}+12\lg_+ t\}$. Via Markov's inequality and independence,

$$P(S_n^*>t s_n)\leqq \exp(-h s_n t)\,\mathrm{E}\exp(hS_n^*) \tag{5.3}$$
$$\leqq L_n^{-2}t^{-12}\prod_{j=1}^n \mathrm{E}\exp(hU_j^*)\,.$$

Since $hU_j^* \leqq (2\lg_+ L_n^{-1} + 12\lg_+ t)c_2 = d_n$, a.e., it follows that

(5.4) $$\exp(hU_j^*) = 1 + hU_j^* + (hU_j^*)^2/2 + ((hU_j^*)^3/6)(1 + hU_j^*/3 + \cdots)$$
$$\leqq 1 + hU_j^* + (hU_j^*)^2/2 + (h^3/6)|U_j^*|^3 \exp(d_n).$$

Taking the expectations from both sides of (5.4), it can be seen that

$$E\exp(hU_j^*) \leqq \exp(hEU_j^* + (h^2/2)EU_j^{*2} + (h^3/6)E|U_j^*|^3 \exp(d_n)).$$

Now, since $EU_j = 0$, it implies that

(5.5) $$|hEU_j^*|) \leqq hE|U_j^*| I(|U_j| > c_2 s_n t) \leqq h\sigma_j^2/(c_2 s_n t) \leqq c_4 \sigma_j^2/s_n^2,$$

and

(5.6) $$h^3 \exp(d_n) E|U_j^*|^3 \leqq L_n^{-1/3} t^2 (s_n t)^{-3} \{2\lg_+ L_n^{-1} + \lg_+ t\}^3 \beta_j^3.$$

In conjunction with (5.5) and (5.6), (5.3) becomes

$$P(S_n^* > ts_n) \leqq L_n^{-2} t^{12} \exp(c_4 + (1/2)h^2 s_n^2 + L_n^{2/3} t \{s\lg_+ L_n^{-1} + \lg_+ t\}^3).$$

But for sufficiently large n, $L_n^{2/3} t \{s\lg_+ L_n^{-1} + \lg_+ t\}^3$ is bounded above by a constant c_5, say, and $(1/2)h^2 s_n^2$ is bounded by $\lg_+ L_n^{-1} + 6\lg_+ t$. This confirms that $P(S_n^* > ts_n) \leqq c_1 L_n t^{-6}$, which completes the proof of the lemma, by noticing that

$$R_n = \sum_{j=1}^n P(|U_j| > c_2 s_n t) \leqq \sum_{j=1}^n E|U_j|^3/(c_2 s_n t)^3 \leqq c_8 L_n t^{-3}.$$

Lemma 7. *Let $\{U_i;\ i \in N\}$ be a sequence of i.r.v.'s with $EU_i = 0$, $EU_i^2 = \sigma_i^2$ and $E|U_i|^3 = \beta_i^3 < \infty$. Let $\{S_n;\ n \in N\}$ be a sequence of partial sums with $s_n^2 = \sum_{j=1}^m \sigma_j^2$, $B_n^3 = \sum_{j=1}^n \beta_j^3$ and $L_n = B_n^3/s_n^3$. Then for $I([p, q]) = \{\nu \in N;\ s_p^2 \leqq s_\nu^2 \leqq s_q^2\}$ $(p \leqq q)$, we have that*

(i) $\sup_{x \in R} P(\exists \mu, \nu \in I([p, q]): S_\nu \leqq x < S_\mu) \leqq c_1 L_p + c_2 (s_q^2 - s_p^2)^{1/2} s_p^{-1}$

and for $F_k \equiv \sigma(U_1, \cdots, U_k)$ $(k < p \leqq q)$, we have that

(ii) $\sup_{x \in R} P(\exists \mu, \nu \in I([p, q]): S_\nu \leqq x < S_\mu | F_k) \leqq c_1 \hat{L}_p + c_2 (s_q^2 - s_p^2)^{1/2} (s_p^2 - s_k^2)^{-1/2}$,

where $\hat{L}_p = (B_p^3 - B_k^3)(s_p^2 - s_k^2)^{-3/2}$.

Proof. We shall first show (i). It is not hard to see that

$$P(\exists \mu, \nu \in I([p, q]): S_\nu \leqq x < S_\mu)$$
$$= P(\exists \nu \in I([p, q]): S_\nu \leqq x < S_p) + P(\exists \mu \in I([p, q]): S_p \leqq x < S_\mu).$$

Since we can replace X_i by $-X_i$, it suffices to show that

$$P(S_p \leqq x < \max_{p < j \leq q} S_j) \leqq c_1 L_p + c_2 (s_q^2 - s_p^2)^{1/2} s_p^{-1}.$$

Set $H = \max_{p < j \leq q}(S_j - S_p)$, then using [7, p. 115] it follows that

$$P(S_p \leqq x < \max_{p<j\leqq q} S_j) \leqq P(x-H \leqq S_p \leqq x)$$

$$= \int P(x-h \leqq S_p \leqq x) dP(H \leqq h)$$

$$\leqq c_1 L_p + \int |\Phi(x/s) - \Phi((x-h)/s_p)| dP(H \leqq h)$$

$$\leqq c_1 L_p + c_2 E|H|/s_p$$

$$\leqq c_1 L_p + c_2 (EH^2)^{1/2}/s_p$$

$$\leqq c_1 L_p + c_3 (s_q^2 - s_p^2)^{1/2}/s_p \,.$$

The last statement follows from the Marcinkiewicz-Zygmund Inequality for independent variables.

To show (ii), we have that P-a.e.

$$P(S_p \leqq x < \max_{p<j\leqq q} S_j | F_k) \leqq P(x-H-S_k \leqq S_p - S_k \leqq x - S_k | F_k)$$

$$= \int P(x-h-S_k \leqq S_p - S_k \leqq x - S_k | F_k) dP|(H \leqq h)$$

$$\leqq c_1 \hat{L}_p + \int |\Phi((x-S_k)(s_p^2 - s_k^2)^{-1/2}) - \Phi((x-S_k-h)(s_p^2-s_k^2)^{-1/2})| dP(H \leqq h)$$

$$\leqq c_1 \hat{L}_p + c_2 (s_q^2 - s_p^2)^{1/2} (s_p^2 - s_k^2)^{-1/2} \,.$$

This completes the proof of Lemma 7.

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