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LOCAL HERMITIAN GEOMETRY OF COMPLEX SURFACES IN P^3: TOTALLY PARABOLIC SURFACES[(*)]

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Abstract. A study of Hermitian geometry of holomorphically immersed complex surfaces in P^3 is given. A normalization of the complex second fundamental form is given and is used to extract the global invariants, complex principal curvatures. We give formulae expressing the Chern forms in terms of the Kaehler form and the complex principal curvatures. For compact surfaces in P^3 we obtain a pair of Gauss-Bonnet type formulae. As an application of our Gauss-Bonnet formulae we prove that an algebraic surface in P^3 with an immersive osculating map is a quadric. The notion of a parabolic surface in P^3 is introduced. Compact parabolic surfaces do not exist since they would have to satisfy the critical Chern number equality, $(c_1)^2=3c_2$. An infinite family of local parabolic surfaces are constructed using the method of prolongation. In the process we also give an infinite family of non-left-invariant integrable distributions on $U(4)$.

Introduction.

In this paper we present a study of Hermitian geometry of holomorphically immersed complex surfaces in P^3 with the Fubini-Study metric using the method of moving frames.

Let M be a complex surface and consider a holomorphic immersion

$$f: M \longrightarrow P^3.$$

We construct what we call the *complex second fundamental form of* f which is a smooth section of the second symmetric power of the holomorphic cotangent bundle of M. Normalizing the complex second fundamental form we uncover two nonnegative functions κ_1, κ_2 on M which we call *complex-principal curvatures.* The two global invariants κ_1 and κ_2 in turn enable us to give a satisfactory description of complex surfaces in P^3, and the resulting picture is not unlike that of real surfaces in the Euclidean 3-space.

Our normalization of the complex second fundamental form yields formulae

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(2.8–2.12) expressing the Chern forms in terms of the Kähler form and the complex prinicipal curvatures. These formulae may have applications in value distribution theory. For a compact surface in $\boldsymbol{P}^3$ we obtain a pair of Gauss-Bonnet formulae (**2**, Theorem 2) expressing the Chern numbers in terms of the degree (or equivalently, the induced volume) and the integral of the sum and product of complex principal curvatures. Generalized Gauss-Bonnet formulae for algebraic hypersurfaces in $\boldsymbol{P}^n$ are also given in **2**. Combining our results with the well-known calculation of the Chern numbers of projective algebraic hypersurfaces we then give formulae expressing the integrals of elementary symmetric polynomials in the complex principal curvatures in terms of the degree. As an application of our Gauss-Bonnet formulae we prove that (**2**, Theorem 6) an algebraic surface in $\boldsymbol{P}^3$ with an immersive osculating map must be a quadric.

We call a complex surface in $\boldsymbol{P}^3$ (totally) complex-umbilic if the two principal curvatures coincide everywhere. We prove (**3**, Theorem) that a complex surface in $\boldsymbol{P}^3$ is complex-umbilic if and only if the induced metric on it is Kähler-Einstein. This theorem combined with an earlier theorem of Chern [C] yields a complete local classification of complex-umbilic surfaces: A complex umbilic surace in $\boldsymbol{P}^3$ is either a piece of a $\boldsymbol{P}^2$, or congruenct to a piece of the normalized quadric Q_2.

We call a complex surface in $\boldsymbol{P}^3$ a (totally) parabolic surface if one of the complex-principal curvatures vanishes everywhere. Unfortunately there are no compact parabolic surfaces in $\boldsymbol{P}^3$: As a consequence of our Gauss-Bonnet formulae a compact parabolic surface satisfies the critical Chern number equality, $c_1^2=3c_2$. On the other hand a surface satisfying the critical Chern number equality can be uniformized by the complex 2-ball (cf. [H]), and the Lefschetz theorem on hyperplane sections implies that a surface in $\boldsymbol{P}^3$ is simply connected. Indeed there are not even immersed compact parabolic surfaces: A theorem of Fulton-Hansen [FH] implies that a holomorphic immersion from a connected compact M^2 into $\boldsymbol{P}^3$ has to be an embedding. However, there are an abundant supply of local parabolic surfaces in $\boldsymbol{P}^3$: We prove that (**5**, Theorem 4) given any nonconstant function greater than 1, there is a local parabolic surface whose nonzero principal curvature is the given function. In fact, we construct these surfaces up to integration involving ordinary differential equations. It is worth remarking that in the course of proving Theorem 4 we discover a family of non-left-invariant completely integrable distributions on $U(4)$. The resulting foliations of $U(4)$ seem to possess some interesting properties.

We now explain the organization of our paper.

§1 is essentially expository, and contains a moving frame theoretic sketch of complex submanifolds of $\boldsymbol{P}^n$. We explain the construction of the Frenet bundle along a complex submanifold in $\boldsymbol{P}^n$ as the *first order reduction* of the unitary frame bundle along the submanifold. The so called *infinitesimal Plücker formulae* (1.6) are seen to be an immediate consequence of this first order reduction process.

In § 2 the results of § 1 are applied to complex hypersurfaces in $\boldsymbol{P}^n$. The complex second fundamental form is defined, and normalized. The normalization of the complex second fundamental form is seen as a second order reduction of the unitary frame bundle. This reduction at once leads to the complex-principal curvatures. Chern's curvature theoretic formulation of characteristic classes combined with the second order reduction produces generalized Gauss-Bonnet formulae for hypersurfaces.

In § 3 the notion of a constant isotropy type surface in $\boldsymbol{P}^3$ is defined, and we find that there are three constant isotropy type surfaces: complex-umbilic, parabolic, and generic surfaces. We take care of the complex-umbilic case in this section.

In § 4 we prolong the exterior differential system describing the Frenet bundle along a parabolic surface several times to arrive at a set of local structure equations (4.58-4.71). § 5 contains the main results on parabolic surfaces.

1. Moving frames.

This section is largely expository and serves to set up the subsequent notation.

Let $G(n, k)$ denote the Grassmann manifold of k-planes in $\boldsymbol{C}^n$. A k-plane $\Lambda \in G(n, k)$ can be represented by a decomposible k-vector $e_1 \wedge \cdots \wedge e_k$, where $(e_1, \cdots, e_k)$ is a unitary k-frame in $\boldsymbol{C}^n$. We will write

$$\Lambda = [e_1 \wedge \cdots \wedge e_k].$$

Let (ε_α) denote the canonical basis of $\boldsymbol{C}^n$ and put

$$\Lambda_0 = [\varepsilon_1 \wedge \cdots \wedge \varepsilon_k].$$

Then the holomorphic projection

$$U(n) \longrightarrow G(n, k), \qquad g \longmapsto g(\Lambda_0)$$

gives an explicit identification

$$G(n, k) = U(n)/U(k) \times U(n-k),$$

where

$$U(k) \times U(n-k) = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A \in U(k),\ B \in U(n-k) \right\}$$

is the isotropy subgroup at $\Lambda_0 \in G(n, k)$.

Let $\mathfrak{m}$ denote the orthogonal complement (relative to the Killing form) to

$\mathfrak{h}$ in $\mathfrak{u}(n)$, where $\mathfrak{h}$ denotes the Lie algebra of $U(k)\times U(n-k)$, so that

$$\mathfrak{u}(n)=\mathfrak{h}\oplus\mathfrak{m}.$$

Note that $\mathfrak{m}$ is identified with the tangent space of $G(n, k)$ at Λ_0. Let $\Omega=(\Omega^a_b)$ denote the $\mathfrak{u}(n)$-valued Maurer-Cartan form of $U(n)$. Take a local section s of $U(n)\to G(n, k)$ and put $\omega=s^*\Omega$. The usual metric, invariant and Kähler, on $G(n, k)$ is given by

$$\sum\omega^\alpha_i\otimes\bar{\omega}^\alpha_i, \quad \text{where} \quad 1\leqq i\leqq k\,;\ k+1\leqq\alpha\leqq n.$$

The forms (Ω^α_i) give the $\mathfrak{m}$-component of Ω; (ω^α_i) are all of type (1, 0).

Consider a holomorphic immersion

$$F:\ N\longrightarrow G(n, k),$$

where N is a complex manifold. Take a local section $e=(e_1, \cdots, e_n)$ of the pullback bundle $F^{-1}U(n)\to N$ and put $\omega=e^*\Omega$. Then the Kähler form of the induced metric on N is

$$\Phi=\frac{i}{2}\sum_{\alpha,i}\omega^\alpha_i\wedge\bar{\omega}^\alpha_i, \quad \text{where} \quad 1\leqq i\leqq k\,;\ k+1\leqq\alpha\leqq n. \tag{1.1}$$

Let $S\to G(n, k)$ denote the universal bundle. Observe that (e_i) (the first k vectors of e) form a local unitary frame for the bundle $F^{-1}S\to N$. The corresponding curvature forms are

$$\chi^j_i=d\omega^j_i+\omega^j_h\wedge\omega^h_i=-\omega^j_\alpha\wedge\omega^\alpha_i, \tag{1.2}$$

where $1\leqq h, i, j\leqq k$. Let $\tau_1(F^{-1}S)$ denote the first Chern form of $F^{-1}S$. From (1.1) and (1.2) we then obtain

$$\tau_1(F^{-1}S)=\frac{i}{2\pi}\operatorname{trace}(\chi)=-\frac{1}{\pi}\Phi. \tag{1.3}$$

In the remainder of this section we give a brief exposition of the local theory of complex submanifolds in $\boldsymbol{P}^n$ via the method of moving frames.

Consider a holomorphic map $f: N\to\boldsymbol{P}^n$, where N is an m-dimensional complex manifold. There is a local holomorphic lifting

$$\hat{f}: U\subset N\longrightarrow \boldsymbol{C}^{n+1}\setminus\{0\}, \qquad z=(z_i)\longmapsto {}^t(\hat{f}^0(z), \cdots, \hat{f}^n(z)),$$

where (z_i) are local holomorphic coordinates.

The *first order osculating space* of f at $z\in M$ is defined to be

$$T^1_z=\operatorname{span}\{\hat{f}, \partial\hat{f}/\partial z_i\}_z\subset\boldsymbol{C}^{n+1}.$$

Near a general point $\dim T^1=m+1$. As our discussion is of local nature we assume that $\dim T^1\equiv m+1$. Note that this amounts to assuming that f is an immersion.

The *first osculating map* is the holomorphic map

$$f^{(1)}: N \longrightarrow G(n+1, m+1), \qquad z \longmapsto T_z^1.$$

The *second order osculating space* of f at z is

$$T_z^2 = \text{span}\{\hat{f}, \partial\hat{f}/\partial z_i, \partial^2\hat{f}/\partial z_i \partial z_j\}_z \subset \boldsymbol{C}^{n+1}.$$

Observe that

$$\dim T^2 \leqq m+1+\binom{m+1}{2}.$$

Invoking the localism again we assume that the dimension of T^2 is constant.

The *second osculating map* is the holomorphic map

$$f^{(2)}: N \longrightarrow G(n+1, t+1), \qquad z \longmapsto T_z^2,$$

where $t+1=\dim T^2$.

The *i-th order osculating space* of f at a point $z \in M$, denoted by T_z^i, is defined to be the subspace of $\boldsymbol{C}^{n+1}$ spanned by $\hat{f}$ and its partials of order up to i at the point z. The *i-th osculating map,* denoted by $f^{(i)}$, is the map taking z to T_z^i. Near a general point of N the dimensions of osculating spaces are constant. Assume that the osculating dimensions of f are constant everywhere on N, and put

$$\dim T^i = 1+t_i.$$

The *osculating order* of f, denoted by $\mathfrak{o}(f)$, is defined to be the smallest integer such that

$$t_{\mathfrak{o}(f)} = t_{\mathfrak{o}(f)+1}, \quad \text{but} \quad t_{\mathfrak{o}(f)} \neq t_{\mathfrak{o}(f)-1}.$$

We assume that f is *linearly full* i.e.,

$$T^{\mathfrak{o}(f)} = \boldsymbol{C}^{n+1}.$$

We call the strictly increasing monotone sequence $(t_1, \cdots, t_{\mathfrak{o}(f)})$ the osculating sequence of f. Observe that we always have

$$t_1 = m, \ t_{\mathfrak{o}(f)} = n, \ 0 < t_i - t_{i-1} \leqq \binom{m+i-1}{i}.$$

We also put

$$T^0 = f^{-1}S,$$

where $S \rightarrow \boldsymbol{P}^n$ is the universal bundle.

For each i, $1 \leqq i \leqq \mathfrak{o}(f)$, we define

$$\Delta^i \longrightarrow N$$

to be the subbundle of the trivial bundle $\boldsymbol{C}^{n+1} \times N$ whose fibre at $z \in N$ is given by

$$\Delta_z^i = \text{the orthogonal complement to } T_z^{i-1} \text{ in } T_z^i,$$

i.e.,

$$\Delta_z^i = [e_{t_{i-1}+1} \wedge \cdots \wedge e_{t_i}]_z .$$

We thus have the Whitney sum decomposition

$$\boldsymbol{C}^{n+1} \times N = T^0 \oplus \Delta^1 \oplus \cdots \oplus \Delta^{o(f)},$$

and

$$\dim \Delta^i = s_i = t_i - t_{i-1} .$$

The *Frenet bundle,* denoted by $\mathcal{F} \to N$, is a $U(1) \times U(s_1) \times \cdots \times U(s_{o(f)})$-reduction of the $U(1) \times U(n)$-principal bundle $f^{-1}U(n+1) \to N$. A local section

$$e = (e_0, \cdots, e_n)$$

of the Frenet bundle, called a Frenet frame along f, is characterized by the conditions

$$[e_0] = f,\ [e_0 \wedge \cdots \wedge e_{t_i}] = f^{(i)} \qquad \text{for every } i. \tag{1.4}$$

Let $\Omega = (\Omega_b^a)$, $0 \leqq a, b \leqq n$, denote the $\mathfrak{u}(n+1)$-valued Maurer-Cartan form of $U(n+1)$. Put $\omega = e^* \Omega$, where e is a Frenet frame along f. We then have

$$de_a = \omega_a^b \otimes e_b ,$$

and the conditions in (1.4) are reflected by the following set of exterior equations:

$$\omega_a^b = 0 \qquad \text{for} \quad a \leqq t_i,\ b \geqq t_{i+1} . \tag{1.5}$$

Let Φ_i denote the Kähler form of the i-th osculating (possibly singular) metric given by the pullback of the standard metric on $G(n+1, t_i+1)$. Also let ∇^i denote the induced connection on the bundle

$$\Delta^{i*} \otimes \Delta^{i+1} \longrightarrow N.$$

The following set of formulae routinely follow from the standard Chern form computation together with the relations given in (1.5), and were first written down by Tai [T]:

$$\tau_1(\nabla^i) = \frac{1}{\pi}(-s_{i+1}\Phi_{i-1} + (s_i + s_{i+1})\Phi_i - s_i \Phi_{i+1}), \tag{1.6}$$

where τ_1 denotes the first Chern form.

2. The complex second fundamental form and the Chern forms.

We consider a holomorphic immersion

$$f:\ M \longrightarrow \boldsymbol{P}^{m+1},$$

where M is an m-dimensional complex manifold.

Remark. If M is compact and connected, then a theorem of Fulton-Hansen [FH] implies that f has to be an embedding.

The (first) osculating map of f is the holomorphic map

$$f^{(1)}:\ M \longrightarrow G(m+2, m+1)=\boldsymbol{P}^{m+1*}, \qquad z \longmapsto T_z^1.$$

Throughout this section we will adhere to the following index convention:

$$1 \leqq i, j, k, \cdots \leqq m;\ 0 \leqq a, b, c, \cdots \leqq m+1.$$

The Frenet bundle, $\mathcal{F} \to M$, is a $U(1)\times U(m)\times U(1)$-reduction of the pullback bundle $f^{-1}U(m+2)\to M$; if $e=(e_a)$ is a Frenet frame, then

$$[e_0]=f, \qquad [e_0\wedge \cdots \wedge e_m]=f^{(1)}. \tag{2.1}$$

Choose a Frenet frame e, and put

$$\omega_b^a=e^*\Omega_b^a,$$

where $\Omega=(\Omega_b^a)$ is the skew-Hermitian Maurer-Cartan form of $U(m+2)$. The forms (ω_b^a) satisfy

$$de_a=\omega_a^b \otimes e_b.$$

The conditions in (2.1) imply that

$$\omega_0^{m+1}=\omega_{m+1}^0=0. \tag{2.2}$$

Let ds^2 denote the normalized Fubini-Study metric on $\boldsymbol{P}^{m+1}$ so that its holomorphic sectional curvature equals 4. The holomorphic immersion f pulls back ds^2 to M giving it a Kähler metric. The forms (ω_0^i) form a local type (1, 0) unitary coframe on (M, f^*ds^2), and the corresponding Kähler form is given by

$$\Phi=\frac{i}{2}(\omega_0^1\wedge\bar{\omega}_0^1+\cdots+\omega_0^m\wedge\bar{\omega}_0^m).$$

Exterior differentiation of both sides of the equation $\omega_0^{m+1}=0$ leads to

$$\omega_1^{m+1}\wedge\omega_0^1+\cdots+\omega_m^{m+1}\wedge\omega_0^m=0.$$

The holomorphy of $f^{(1)}$ is reflected by the fact that the forms (ω_i^{m+1}) are all of type (1, 0). By Cartan's lemma

$$\omega_i^{m+1}=X_{ij}\omega_0^j$$

for some complex-valued local functions X_{ij} with $X_{ij}=X_{ji}$.

Definition. The type (2, 0) symmetric form

$$\mathrm{II}=\omega_i^{m+1}\cdot\omega_0^i=X_{ij}\omega_0^i\cdot\omega_0^j$$

is called the complex second fundamental form.

Theorem 1 (The first normal form). *Let $f: M\to \boldsymbol{P}^{m+1}$ be a holomorphic immersion. Then in a neighborhood of any point in M there exists a Frenet frame e such that*

(2.3) $$e^*\Omega_i^{m+1}=k_i e^*\Omega_0^i \text{ (no sum)},$$

where the k_i's are globally defined real-valued functions on M with

$$0\leqq k_m\leqq k_{m-1}\leqq \cdots \leqq k_1.$$

Proof. We want to see how the complex symmetric matric $X=(X_{ij})$ transforms under a change of Frenet frame. Let e, $\tilde{e}$ be two Frenet frames. Then on their common domain the two frames are related by

$$e=\tilde{e}\cdot g$$

for some $U(1)\times U(m)\times U(1)$-valued local function $g=(\exp(it), A, \exp(is))$. Define tilded quantities using $\tilde{e}$: $\tilde{\omega}=\tilde{e}^*\Omega$, $\tilde{X}_{ij}\tilde{\omega}_0^j=\tilde{\omega}_i^{m+1}$. From the formula

$$\omega=\mathrm{Ad}(g^{-1})\tilde{\omega}$$

we compute that

(2.4) $$X=\exp(-i(s+t))\,{}^tA\tilde{X}A.$$

It now follows from a result of Chern (see [C] p. 28) that we can make X diagonal. The rest follows routinely. □

Given a holomorphic immersion $f: M\to \boldsymbol{P}^{m+1}$ we will call the global functions

$$\kappa_i=(k_i)^2:\ M\longrightarrow \boldsymbol{R}$$

the *complex principal curvatures*. We also let σ_i denote the i-th elementary symmetric polynomial of (κ_i). For example,

(2.5) $$\sigma_1=\kappa_1+\cdots+\kappa_m,\qquad \sigma_m=\kappa_1\cdots\kappa_m.$$

Let ∇ denote the canonical connection (i.e., metric and type (1, 0)) on the holomorphic tangent bundle $TM\to M$ coming from the Kähler metric f^*ds^2. In the following we will compute the curvature matrix, χ, of ∇ using the unitary coframe (ω_0^i). We have

$$d\omega_0^i=-\theta_j^i\wedge\omega_0^j,$$

where $\theta=(\theta_j^i)$ is the connection matrix. The curvature forms (χ_j^i) are given by

$$\chi_j^i=d\theta_j^i+\theta_k^i\wedge\theta_j^k.$$

Using the Maurer-Cartan structure equations of $U(m+2)$ we obtain

$$d\omega_0^i=-\omega_0^i\wedge\omega_0^0-\omega_j^i\wedge\omega_0^j=-(\omega_j^i-\delta_j^i\omega_0^0)\wedge\omega_0^j.$$

Thus

$$\theta_j^i=\omega_j^i-\delta_j^i\omega_0^0.$$

We easily have

$$\theta_k^i\wedge\theta_j^k=\omega_k^i\wedge\omega_j^k.$$

Again using the Maurer-Cartan structure equations of $U(m+2)$ we obtain

$$d\theta_j^i=d(\omega_j^i-\delta_j^i\omega_0^0)=-\omega_0^i\wedge\omega_j^0-\omega_k^i\wedge\omega_j^k-\omega_{m+1}^i\wedge\omega_j^{m+1}+\delta_j^i\omega_k^0\wedge\omega_0^k.$$

It follows that

$$\chi_j^i=\omega_0^i\wedge\bar{\omega}_0^i+\bar{\omega}_i^{m+1}\wedge\omega_j^{m+1}-\delta_j^i\sum\bar{\omega}_0^k\wedge\omega_0^k.$$

Using the first normal form (2.3) we can rewrite the above as

$$\chi_j^i=\omega_0^i\wedge\bar{\omega}_0^j+\delta_j^i\sum\omega_0^k\wedge\bar{\omega}_0^k+k_ik_j\bar{\omega}^i\wedge\omega^j. \tag{2.6}$$

We have, in particular,

$$\text{trace}\,\chi=-2i(m+1)\Phi-\sum\kappa_i\omega_0^i\wedge\bar{\omega}_0^i, \tag{2.7}$$

where Φ is the Kähler form of (M, f^*ds^2).

Note that if we let (h_i) denote the holomorphic sectional curvatures relative to the unitary coframe (ω_0^i), then

$$h_i=2(2-\kappa_i).$$

The i-th Chern form of ∇, denoted by $\tau_i(M, \nabla)$, is given by

$$\tau_i(M, \nabla)=P^i\left(\frac{i}{2\pi}\cdot\chi\right),$$

where P^i denotes the i-th elementary symmetric polynomial in the eigenvalues of the matrix $(i/2\pi)\cdot\chi$. For example,

$$\tau_1(M, \nabla)=\frac{i}{2\pi}\text{trace}\,\chi,$$

$$\tau_2(M, \nabla)=\left(\frac{i}{2\pi}\right)^2\left(\sum_{j<k}\chi_j^j\chi_k^k-\chi_k^j\chi_j^k\right),$$

$$\tau_m(M, \nabla)=\left(\frac{i}{2\pi}\right)^m\det\chi.$$

From (2.7) we obtain

$$\tau_1(M, \nabla)=\frac{1}{\pi}(m+1)\Phi-\frac{i}{2\pi}\sum\kappa_i\omega_0^i\wedge\bar{\omega}_0^i. \tag{2.8}$$

Put

$$\Phi_1=\frac{i}{2}\sum\omega_i^{m+1}\wedge\bar{\omega}_i^{m+1}.$$

Observe that Φ_1 is the Kähler form of the (possibly singular) osculating metric $f^{(1)*}ds_1^2$, where ds_1^2 denotes the standard metric on $G(m+2, m+1)$. We can now rewrite (2.8) as

$$\tau_1(M, \nabla)=\frac{1}{\pi}((m+1)\Phi-\Phi_1). \tag{2.9}$$

For $m=2$ we calculate that

$$\tau_2(M^2, \nabla)=\frac{1}{\pi^2}(3-\sigma_1+\sigma_2)\Phi^2. \tag{2.10}$$

For $m=3$ we calculate that

$$\sigma_2(M^3, \nabla)=\frac{-1}{4\pi^2}\sum_{i<j}(12-3(\kappa_i+\kappa_j)+2\kappa_i\kappa_j)\Psi_{ij}, \tag{2.11}$$

where $\Psi_{ij}=\omega_0^i\wedge\bar{\omega}_0^i\wedge\omega_0^j\wedge\bar{\omega}_0^j$. Also

$$\tau_3(M^3, \nabla)=\frac{1}{3\pi^2}(12-3\sigma_1+2\sigma_2-3\sigma_3)\Phi^3. \tag{2.12}$$

Theorem 2 (Generalized Gauss-Bonnet). *Let $f: M^2\to \boldsymbol{P}^3$ be a smooth algebraic surface given by a holomorphic embedding, and also let c_1^2, c_2 denote the Chern numbers of M. Then*

$$c_1^2=9d+\frac{1}{\pi^2}\int_M(\sigma_2-3\sigma_1)\Phi^2, \tag{2.13}$$

$$c_2=3d+\frac{1}{\pi^2}\int_M(\sigma_2-\sigma_1)\Phi^2, \tag{2.14}$$

where d is the degree of the projective variety $f(M)\subset\boldsymbol{P}^3$.

Proof. Wirtinger's theorem states that

$$d=\frac{1}{\pi^2}\int_M\Phi^2.$$

The formula in (2.13) follows from (2.9) upon integration, and (2.14) follows from (2.10). □

Theorem 3 (Generalized Gauss-Bonnet). *Let $f: M^3\to\boldsymbol{P}^4$ be a smooth algebraic 3-fold given by a holomorphic embedding, and also let c_1^3, c_1c_2, c_3 be the Chern numbers of M. Then*

$$c_1^3=64d+\frac{1}{\pi^3}\int_M(-16\sigma_1+4\sigma_2-\sigma_3)\Phi^3, \tag{2.15}$$

(2.16) $$c_1c_2=24d+\frac{1}{\pi^3}\int_M\left(-6\sigma_1+\frac{7}{3}\sigma_2-\sigma_3\right)\Phi^3,$$

(2.17) $$c_3=4d+\frac{1}{\pi^3}\int_M\left(-\sigma_1+\frac{2}{3}\sigma_2-\sigma_3\right)\Phi^3,$$

where d is the degree of $f(M)\subset \boldsymbol{P}^4$.

Proof. Wirtinger's theorem becomes

$$d=\frac{1}{\pi^3}\int_M\Phi^3,$$

and the result follows from (2.9), (2.11), (2.12) upon integration. □

Given a compact embedded hypersurface $f: M\to \boldsymbol{P}^{m+1}$ we put

$$K_i=\frac{1}{\pi^m}\int_M\sigma_i\Phi^m,$$

which we call the i-th total curvature. We will express the total curvatures K_i in terms of the projective degree of M in the following. Realize $M=f(M)\subset \boldsymbol{P}^{m+1}$ as the zero locus of a transversal holomorphic section of the line bundle

$$H^{\otimes d}\longrightarrow \boldsymbol{P}^{m+1},$$

where $H=S^*\to \boldsymbol{P}^{m+1}$ is the hyperplane bundle. (In other words, M is a smooth divisor in the complete linear system $|dH|$ on $\boldsymbol{P}^{m+1}$.) From the direct sum

$$TM\oplus H^{\otimes d}|_M=T\boldsymbol{P}^n|_M$$

and the Whitney product formula we obtain

(2.18) $$c(T\boldsymbol{P}^{m+1})|_M=c(TM)\cdot c(H^{\otimes d})|_M.$$

Put

$$h=c_1(H)|_M.$$

By a version of the Wirtinger theorem we find that

(2.19) $$h^m=h^m([M])=\int_M h^m=d=\deg(M).$$

A quick proof of (2.19) can be given as follows: The Poincaré dual of a $\boldsymbol{P}^1$ in $\boldsymbol{P}^{m+1}$ is $[(1/\pi^m)\Phi^m]\in H^{2m}_{dR}(\boldsymbol{P}^{m+1})$, where Φ is the Kähler form of the Fubini-Study metric. (Our Fubini-Study metric is normalized so that the volume of $\boldsymbol{P}^{m+1}$ is π^{m+1}.) On the other hand, by (1.3) the first Chern class of the hyperplane bundle H is given by $[(1/\pi)\Phi]\in H^2_{dR}(\boldsymbol{P}^{m+1})$. So

$$h^m([M])=\int_M\frac{1}{\pi^m}\Phi^m|_M=\#(M,\boldsymbol{P}^1),$$

where # denotes the intersection pairing. But

$$\#(M, \boldsymbol{P}^1)=d=\deg(M).$$

Using (2.18) together with (2.19) we can compute the Chern numbers of a degree d smooth hypersuface $M\subset \boldsymbol{P}^{m+1}$. For example, for $m=2$ we obtain

$$c_1^2=d(4-d)^2, \tag{2.20}$$

$$c_2=d(d^2-4d+6). \tag{2.21}$$

For algebraic three-folds in $\boldsymbol{P}^4$ we obtain

$$c_1^3=d(5-d), \tag{2.22}$$

$$c_1c_2=d(5-d)(d^2-5d+10), \tag{2.23}$$

$$c_3=d(-d^3+5d^2-10d+10). \tag{2.24}$$

Combining (2.20, 2.21) with (2.13, 2.14) we obtain

Theorem 4. *Let M^2 be an algebraic surface in $\boldsymbol{P}^3$, and put*

$$K_i=\frac{1}{\pi^2}\int_M \sigma_i\Phi^2, \quad d=\deg(M).$$

Then

$$K_1=2d(d-1),$$

$$K_2=d(d-1)^2.$$

Combining (2.22–2.24) with (2.15–2.17) we obtain

Theorem 5. *Let M^3 be an algebraic three-fold in $\boldsymbol{P}^4$ of degree d, and also let K_i denote the i-th total curvature. Then*

$$K_1=3d(d-1),$$

$$K_2=3d(d-1)^2,$$

$$K_3=d(d-1)^3.$$

We close this section by giving an application of our Gauss-Bonnet formulae.

Theorem 6. *Let $f: M^2\to \boldsymbol{P}^3$ be a holomorphic embedding from a compact surface, and suppose that the osculating map*

$$f^{(1)}: M \longrightarrow G(4, 3)=\boldsymbol{P}^{3*}$$

is everywhere immersive. Then $f(M)$ is a quadric.

Proof. Let ∇^1 denote the type (1, 0) metric connection of M with the osculating metric, i.e., the Kähler metric $f^{(1)*}ds_1^2$. The curvature forms, $\chi'=(\chi'^i_j)$, of $(M, f^{(1)*}ds_1^2)$ with respect to (ω_1^3, ω_2^3) are computed to be

$$\chi'^1_1=-2\omega^3_1\wedge\omega^1_3-\omega^3_2\wedge\omega^2_3+\omega^1_0\wedge\omega^0_1,$$

$$\chi'^2_2=-2\omega^3_2\wedge\omega^2_3-\omega^3_1\wedge\omega^1_3+\omega^2_0\wedge\omega^0_2,$$

$$\chi'^1_2=-\bar{\chi}'^2_1=\omega^2_0\wedge\omega^0_1+\omega^2_3\wedge\omega^3_1.$$

Let $\tau_i(M, \nabla^1)$, $i=1, 2$, denote the i-th Chern form of $TM\to M$ with the connection ∇^1. Consulting the above equations we find that

$$\tau_1(M, \nabla^1)=\frac{1}{\pi}(3\Phi_1-\Phi),$$

$$\tau_2(M, \nabla^1)=\frac{1}{\pi^2}(1+3\sigma_2-\sigma_1)\Phi^2.$$

Integration over M yields

$$c_1^2=d+\frac{1}{\pi^2}\int_M(9\sigma_2-3\sigma_1)\Phi^2,$$

$$c_2=d+\frac{1}{\pi^2}\int_M(3\sigma_2-\sigma_1)\Phi^2.$$

Combining these formulae with our Gauss-Bonnet formulae we obtain

$$3c_2-c_1^2=2d.$$

On the other hand the formulae in (2.20, 2.21) imply that

$$3c_2-c_1^2=2(d-1)^2d.$$

Consequently the degree must equal 2. □

3. Constant isotropy type surfaces in P^3.

For the rest of this paper we deal exclusively with holomorphically immersed complex surfaces in P^3. We will focus on local Hermitian geometry of such surfaces.

Consider a holomorphic immersion

$$f: M=M^2 \longrightarrow P^3.$$

Take a Frenet frame $e: U\subset M\to\mathcal{F}$ and put $\omega=e^*\Omega$. We then have

$$\omega^3_0=0. \tag{3.1}$$

We know from 2 that we can choose e so that in addition to (3.1)

$$\omega^3_1=k_1\omega^1_0,\quad \omega^3_2=k_2\omega^2_0,\quad 0\leqq k_2\leqq k_1. \tag{3.2}$$

Moreover, the k_i's are globally defined functions on M.

Any Frenet frame achieving the normal form (3.2) will be called a *second* *order* Frenet frame.

Definition. i) A point $p \in M$ is called a complex-umbilic point of f if

$$k_1(p)=k_2(p);$$

ii) $p \in M$ is called a parabolic point if

$$k_2(p)=0 \quad \text{and} \quad k_1(p)>0;$$

iii) otherwise, $p \in M$ is called a generic point.

If every point is complex-umbilic, then the immersion is said to be *totally complex-umbilic*. We define the notions of *totally parabolic* and *totally generic* immersions analogously.

In order to carry out our analysis further it becomes necessary to assume that the immersion is of *constant isotropy type*, i. e., it is totally complex-umbilic, totally papabolic, or totally generic. (Near a generic point, points stay generic. So the total genericity assumption is a global assumption.) The upshot is that the totality of second order frames at a point depends on the type of the point. Immersions of constant isotropy type (at every order) are also called immersions with constant isotropy tower by some authors. See [G] or [J].

There are not many totally complex-umbilic surfaces as the next theorem shows.

Theorem. *Let $f: M \to \boldsymbol{P}^3$ be a holomorphically immersed surface. Then f is totally complex-umbilic if and only if the induced metric on M is Kähler-Einstein.*

Proof. The induced metric on M is given by

$$ds_M^2=\omega_0^1 \otimes \bar{\omega}_0^1+\omega_0^2 \otimes \bar{\omega}_0^2.$$

The curvature matrix, χ, of M with respect to (ω_0^1, ω_0^2) is given by

$$\chi_j^i=\omega_0^i \wedge \bar{\omega}_0^j+\delta_j^i \sum \omega_0^k \wedge \bar{\omega}_0^k+k_i k_j \bar{\omega}^i \wedge \omega^j.$$

Consequently,

$$\text{trace}\, \chi=(3-\kappa_1)\omega_0^1 \wedge \bar{\omega}_0^1+(3-\kappa_2)\omega_0^2 \wedge \bar{\omega}_0^2.$$

The Kähler form of ds_M^2 is given by

$$\Phi=\frac{i}{2}(\omega_0^1 \wedge \bar{\omega}_0^1+\omega_0^2 \wedge \bar{\omega}_0^2).$$

Now the metric ds_M^2 is Kahler-Einstein if and only if

$$\text{trace}\, \chi=\lambda \Phi,$$

for some complex-valued function λ (which, in fact, has to be a constant). The result follows easily. □

Remark. Consulting (2.7) we see that the preceding result holds true for hypersurfaces in any $\boldsymbol{P}^n$.

Chern [C] has shown that a Kähler-Einstein surface in $\boldsymbol{P}^3$ is locally either a piece of a $\boldsymbol{P}^2$ or congruent to a piece of the normalized quadric

$$Q_2=\{x^2+y^2+z^2+w^2=0:\ x,\ y,\ z,\ w \text{ homogeneous coordinates}\}\subset \boldsymbol{P}^3.$$

This result together with a bit of computation yields

Theorem. *Suppose $f: M\to \boldsymbol{P}^3$ is a totally complex-umbilic immersion. Then either $\kappa_1=\kappa_2\equiv 0$ (a piece of a $\boldsymbol{P}^2$), or $\kappa_1=\kappa_2\equiv 1$ (congruent to a piece of Q_2).*

Remark. We could have obtained the above theorem directly by exterior differentiating both sides of the equations in (3.2), and setting $k_1=k_2$.

Let $f: M\to \boldsymbol{P}^3$ be a totally generic immersion. If

$$e: U\subset M \longrightarrow \mathcal{F}$$

is any second order Frenet frame, then

$$e^*\Omega_i^3=X_{ij}e^*\Omega_0^j,$$

$$X=(X_{ij})=\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}: M\longrightarrow M(2,\boldsymbol{R}),\quad 0<k_2<k_1.$$

To compute the structure group of the second order Frenet bundle,

$$\mathcal{F}_2\longrightarrow M,$$

we recall the transformation rule (2.5):

$$\tilde{X} = e^{-i(s+t)}\,{}^tAXA. \tag{3.3}$$

By definition, the second isotropy group at $p\in M$ is

$$G_2(p)=\{(e^{is},A,e^{it})\in U(1)\times U(2)\times U(1):\ X(p)=\tilde{X}(p)\}.$$

Lemma. *For every $p\in M$, $G_2(p)$ is given by*

$$G_2=\left\{\left(e^{is},\exp\left(\frac{i}{2}(s+t)\right)\cdot\begin{bmatrix}\delta & 0\\ 0 & \varepsilon\end{bmatrix},e^{it}\right):\ s,t\in\boldsymbol{R},\ \delta,\ \varepsilon=\pm 1\right\} \tag{3.4}$$

$$\cong U(1)^3\times \boldsymbol{Z}_4.$$

Proof. We have

$$SU(2)=\left\{\begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix} : z,\ w\in \boldsymbol{C},\ |z|^2+|w|^2=1\right\}.$$

We also have a epimorphism

$$U(1)\times SU(2) \longrightarrow U(2), \qquad (e^{i\theta},\ A)\longmapsto e^{i\theta}A.$$

Thus we can write any $A\in U(2)$ as

$$A=e^{i\theta}\begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix}, \quad |z|^2+|w|^2=1. \tag{3.5}$$

With $A\in U(2)$ as in the above and $X=\tilde{X}=\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$ the equation in (3.3) gives the following set of conditions on $s,\ t,\ \theta,\ z,\ w$:

$$e^{i(s+t)}k_1=e^{2i\theta}(z^2k_1+\bar{w}^2k_2),$$

$$0=zwk_1-\bar{z}\bar{w}k_2,$$

$$e^{i(s+t)}k_2=e^{2i\theta}(w^2k_1+\bar{z}^2k_2).$$

The rest is straightforward. □

Summarizing, a second order frame along a totally generic immersion

$$f: M\longrightarrow \boldsymbol{P}^3$$

is a Frenet frame giving the first normal form

$$\omega_i^3=k_i\omega_0^i, \quad 0<k_2<k_1.$$

If e, and $\tilde{e}$ are any two second order frames, then on their common domain they are related by the formula

$$\tilde{e}=e\cdot g,$$

where g is a local G_2-valued smooth function. The second order Frenet bundle, $\mathcal{F}_2\to M$, is a G_2-principal bundle obtained from the $U(1)\times U(2)\times U(1)$-principal Frenet bundle by a reduction process.

Suppose now that $f: M\to \boldsymbol{P}^3$ is a totally parabolic immersion. The structure group of the second order Frenet bundle, again denoted by G_2, is the isotropy group of the action (3.3), where

$$X=\tilde{X}=\begin{bmatrix} k_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad k_1>0.$$

We compute that

$$G_2=\left\{\left(e^{is},\ \begin{bmatrix} \pm e^{i(s+t)/2} & 0 \\ 0 & e^{ir} \end{bmatrix},\ e^{it}\right) : s,\ t,\ r\in\boldsymbol{R}\right\}\cong U(1)^3\times \boldsymbol{Z}_2. \tag{3.6}$$

4. Totally parabolic surfaces: structure equations.

Consider a totally parabolic immersion

$$f: M \longrightarrow \boldsymbol{P}^3$$

from a complex two-manifold M. Let

$$e: U \subset M \longrightarrow \mathcal{F}_2 \subset U(4)$$

be any second order Frenet frame, and put

$$\omega = e^*\Omega.$$

We then have

$$\omega_0^3 = 0, \tag{4.1}$$

$$\omega_1^3 = k_1\omega_0^1, \quad \omega_2^3 = 0, \quad k_1 > 0. \tag{4.2}$$

Exterior differentiating both sides of the equations in (4.2), and writing modulo equations in (4.1, 4.2), we obtain

$$[d\log k_1 - (2\omega_1^1 - \omega_0^0 - \omega_3^3)] \wedge \omega_0^1 - \omega_2^1 \wedge \omega_0^2 = 0, \tag{4.3}$$

$$\omega_2^1 \wedge \omega_0^1 = 0. \tag{4.4}$$

Note that $d\log k_1$ is purely real and that $(2\omega_1^1 - \omega_0^0 - \omega_3^3)$ is purely imaginary. Put

$$\omega_0^1 = \alpha^1 + i\alpha^3, \quad \omega_0^2 = \alpha^2 + i\alpha^4, \tag{4.5}$$

$$\omega_2^1 = \zeta^1 + i\zeta^2, \tag{4.6}$$

$$i(2\omega_1^1 - \omega_0^0 - \omega_3^3) = \eta, \tag{4.7}$$

for some real 1-forms $\alpha^1, \alpha^2, \cdots, \eta$. Substitution of the equations in (4.5, 4.6, 4.7) into the relations in (4.3, 4.4) yields

(4.8a)
(4.8b)
(4.8c)
(4.8d)

$$\begin{pmatrix} \zeta^1 & 0 & -\zeta^2 & 0 \\ \zeta^2 & 0 & \zeta^1 & 0 \\ d\log k_1 & -\zeta^1 & -\eta & \zeta^2 \\ \eta & -\zeta^2 & d\log k_1 & -\zeta^1 \end{pmatrix} \wedge \begin{pmatrix} \alpha^1 \\ \alpha^2 \\ \alpha^3 \\ \alpha^4 \end{pmatrix} = 0.$$

The forms $\zeta^1, \zeta^2, \eta, d\log k_1$ are real 1-forms and they can be expressed as linear combinations of $\alpha^1, \cdots, \alpha^4$. So we introduce 16 real variables

$$(g_i;\ o_i;\ s_i;\ s_i'), \quad 1 \leqq i \leqq 4,$$

by setting

(4.9) $$d\log k_1 = g_i\alpha^i, \quad \eta = o_i\alpha^i, \quad \zeta^1 = s_i\alpha^i, \quad \zeta^2 = s_i'\alpha^i.$$

Assume that the primary invariant κ_1 is not a constant. (In fact, as we will show in **5**, κ_1 may not be a constant.)

Substituting (4.9) into (4.8), and expanding with respect to the basis

$$\{\alpha^1\wedge\alpha^2,\ \alpha^1\wedge\alpha^3,\ \alpha^1\wedge\alpha^4,\ \alpha^2\wedge\alpha^3,\ \alpha^2\wedge\alpha^4,\ \alpha^3\wedge\alpha^4\}$$

of $\Lambda^2(T^*M)$, we obtain the following system of linear equations in $(g_i;\ o_i;\ s_i;\ s_i')$:

(4.10a) $$s_2=0, \quad s_3+s_1'=0, \quad s_4=0, \quad s_2'=0, \quad s_4'=0,$$

(4.10b) $$s_1-s_3'=0,$$

(4.10c) $$g_2+s_1=0, \quad g_3+o_1=0, \quad g_4+s_3=0, \quad o_2-s_3=0, \quad o_4+s_3'=0,$$

(4.10d) $$o_2+s_1'=0, \quad g_1-o_3=0, \quad o_4+s_1=0, \quad g_2+s_3'=0, \quad s_2-s_4'=0.$$

The linear system (4.10) has exactly 4-dimensional solutions, and they are given by

$$g_i=\text{arbitrary},$$

$$o_1=-g_3, \quad o_2=-g_4, \quad o_3=g_1, \quad o_4=g_2,$$

$$s_1=-g_2, \quad s_2=0, \quad s_3=-g_4, \quad s_4=0,$$

$$s_1'=g_4, \quad s_2'=0, \quad s_3'=-g_2, \quad s_4'=0.$$

We thus obtain

(4.11) $$d\log k_1=g_i\alpha^i,$$

(4.12) $$\eta=-g_3\alpha^1-g_4\alpha^2+g_1\alpha^3+g_2\alpha^4,$$

(4.13) $$\zeta^1=-g_2\alpha^1-g_4\alpha^3,$$

(4.14) $$\zeta^2=g_4\alpha^1-g_2\alpha^3.$$

The equations in (4.13, 4.14) can be combined into a single equation:

(4.15) $$\omega_3^1=i(g_4+ig_2)\omega_0^1.$$

The form $d\log k_1$ is globally defined on M. On the other hand the forms (α^i) are subject to the G_2-isotropy action, and so are (g_i). We will compute a normal form for (g_i), hence simplifying the next prolongation step which begins with the exterior differentiation of (4.11–4.14).

Keep in mind that all forms on $U\subset M$ are written relative to a second order Frenet frame $e: U\to U(4)$. Suppose we have another second order frame

$$\tilde{e}: \tilde{U} \longrightarrow U(4).$$

We then know that e and $\tilde{e}$ are related by the formula

$$\tilde{e}=e\cdot k\,,$$

where k is a smooth local G_2-valued function. We may put

$$k=\left(e^{is},\begin{bmatrix}\pm e^{i(s+t)/2} & 0\\ 0 & e^{ir}\end{bmatrix},e^{it}\right),$$

where $s, t, r: U\cap\tilde{U}\to\boldsymbol{R}$ are smooth. Put

$$\tilde{\omega}=\tilde{e}^*\Omega\,,$$

and define tilded quantities $(\tilde{g}_i)$ by

$$d\log k_1=g_i\alpha^i=\tilde{g}_i\tilde{\alpha}^i\,, \tag{4.16}$$

where $\tilde{\omega}_0^1=\tilde{\alpha}^1+i\tilde{\alpha}^3$, and $\tilde{\omega}_0^2=\tilde{\alpha}^2+i\tilde{\alpha}^4$.

From the transformation rule

$$\tilde{\omega}=\mathrm{Ad}(k^{-1})\omega$$

we compute that

$$(\tilde{\omega}_0^1,\,\tilde{\omega}_0^2)=e^{it}(\pm e^{-i(s+t)/2}\omega_0^1,\,e^{-ir}\omega_0^2)\,.$$

It follows that we may change

$$\omega_0^1\longmapsto e^{ia}\omega_0^1,\quad \omega_0^2\longmapsto e^{ib}\omega_0^2\,, \tag{4.17}$$

where a, b are any smooth local functions. Combining (4.16) with (4.17) we see that we can change

$$g_1+ig_3\longmapsto e^{ia}(g_1+ig_3),\quad g_2+ig_4\longmapsto e^{ib}(g_2+ig_4)\,. \tag{4.18}$$

As a consequence of (4.18) we can further choose a second order frame so as to make

$$g_3=g_4=0,\quad g_1\geqq 0,\quad g_2\geqq 0\,. \tag{4.19}$$

Definition. The global functions

$$\gamma_1=(g_1)^2,\quad \gamma_2=(g_2)^2: M\longrightarrow \boldsymbol{R}$$

will be called the secondary invariants of the immersion f.

With the aid of (4.19) we can rewrite the equations in (4.11–4.14) as follows:

$$d\log k_1=g_1\alpha^1+g_2\alpha^2\,, \tag{4.20}$$

$$i(2\omega_1^1-\omega_0^0-\omega_3^3)=g_1\alpha^3+g_2\alpha^4\,, \tag{4.21}$$

$$\omega_2^1=-g_2(\alpha^1+i\alpha^3)\,. \tag{4.22}$$

Remark. The exterior equations in (4.20–4.22) are written relative to a "third order Frenet frame". Now G_2 is 3-dimensional, and consulting (4.17–4.19) we see that the structure group of the bundle of third order Frenet frames is one-dimensional. Thus we can hardly expect to improve the choice of our frame any further.

We need to exterior differentiate both sides of the equations in (4.20–4.22). We first make some preliminary calculations.

Using the Maurer-Cartan structure equations and the equations in (4.1, 4.2, 4.20, 4.21, 4.22) we calculate that

$$d\alpha^1=i(\omega_0^0-\omega_1^1)\wedge\alpha^3+g_2(\alpha^1\wedge\alpha^2-\alpha^3\wedge\alpha^4), \tag{4.23}$$

$$d\alpha^2=i(\omega_0^0-\omega_2^2)\wedge\alpha^4, \tag{4.24}$$

$$d\alpha^3=-i(\omega_0^0-\omega_1^1)\wedge\alpha^1+g_2(\alpha^1\wedge\alpha^4-\alpha^2\wedge\alpha^3), \tag{4.25}$$

$$d\alpha^4=-i(\omega_0^0-\omega_2^2)\wedge\alpha^2-2g_2\alpha_1\wedge\alpha^3, \tag{4.26}$$

$$d\omega_1^1=(1+\gamma_2-\kappa_1)\omega_0^1\wedge\bar{\omega}_0^1, \tag{4.27}$$

$$d\omega_0^0=-\omega_0^1\wedge\bar{\omega}_0^1-\omega_0^2\wedge\bar{\omega}_0^2, \tag{4.28}$$

$$d\omega_3^3=\kappa_1\omega_0^1\wedge\bar{\omega}_0^1, \tag{4.29}$$

$$d\omega_2^1=\omega_0^1\wedge\bar{\omega}_0^2+(g_2\omega_1^1-g_2\omega_2^2)\wedge\omega_0^1. \tag{4.30}$$

We also have

$$\omega_0^1\wedge\bar{\omega}_0^1=-2i\alpha^1\wedge\alpha^3, \tag{4.31}$$

$$\omega_0^2\wedge\bar{\omega}_0^2=-2i\alpha^2\wedge\alpha^4, \tag{4.32}$$

$$\omega_0^1\wedge\omega_0^2=(\alpha^1\wedge\alpha^2-\alpha^3\wedge\alpha^4)+i(\alpha^1\wedge\alpha^4-\alpha^2\wedge\alpha^3), \tag{4.33}$$

$$\omega_0^1\wedge\bar{\omega}_0^2=(\alpha^1\wedge\alpha^2+\alpha^3\wedge\alpha^4)+i(\alpha^3\wedge\alpha^2-\alpha^1\wedge\alpha^4). \tag{4.34}$$

We introduce 16 new real variables

$$(t_i;\ t_i';\ n_i;\ n_i')$$

by setting

$$dg_1=t_i\alpha^i;\ dg_2=t_i'\alpha^i;\ i(\omega_0^0-\omega_1^1)=n_i\alpha^i;\ i(\omega_0^0-\omega_2^2)=n_i'\alpha^i. \tag{4.35}$$

We assume that the secondary invariants γ_1 and γ_2 are not constant, and nowhere zero.

Consulting (4.23–4.34) we exterior differentiate both sides of the equations in (4.20–4.22), and obtain

$$\begin{aligned}0=&\,dg_1\wedge\alpha^1+dg_2\wedge\alpha^2+g_1i(\omega_0^0-\omega_1^1)\wedge\alpha^3\\&+g_1g_2(\alpha^1\wedge\alpha^2-\alpha^3\wedge\alpha^4)+g_2i(\omega_0^0-\omega_2^2)\wedge\alpha^4,\end{aligned} \tag{4.36}$$

(4.37) $(6+4\gamma_2-6\kappa_1)\alpha^1\wedge\alpha^3+2\alpha^2\wedge\alpha^4=dg_1\wedge\alpha^3+dg_2\wedge\alpha^4-g_1i(\omega_0^0-\omega_1^1)\wedge\alpha^1$
$+g_1g_2(\alpha^1\wedge\alpha^4-\alpha^2\wedge\alpha^3)-g_2i(\omega_0^0-\omega_2^2)\wedge\alpha^2-2\gamma_2\alpha^1\wedge\alpha^3,$

(4.38) $0=\alpha^1\wedge\alpha^2+\alpha^3\wedge\alpha^4+g_2i(\omega_0^0-\omega_2^2)\wedge\alpha^3+dg_2\wedge\alpha^1+\gamma_2(\alpha^1\wedge\alpha^2-\alpha^3\wedge\alpha^4),$

(4.39) $0=\alpha^3\wedge\alpha^2+\alpha^4\wedge\alpha^1-g_2i(\omega_0^0-\omega_2^2)\wedge\alpha^1+dg_2\wedge\alpha^3+\gamma_2(\alpha^1\wedge\alpha^4-\alpha\wedge^2\alpha^3).$

Substitution of (4.35) into the exterior equations in (4.36–4.39) leads to the following linear system of equations in $(t_i;\ t_i';\ n_i;\ n_i')$:

(4.40) $-t_2+t_1'+g_1g_2=0,$

(4.41) $t_3+g_1n_1=0,$

(4.42) $-t_4+g_2n_1'=0,$

(4.43) $-t_3+g_1n_2=0,$

(4.44) $-t_4'+g_2n_2'=0,$

(4.45) $-g_1n_4-g_1g_2+g_2n_3'=0,$

(4.46) $-g_1n_2+g_2n_1'=0,$

(4.47) $6+6\gamma_2-6\kappa_1-t_1-g_1n_3=0,$

(4.48) $t_1'+g_1n_4+g_1g_2=0,$

(4.49) $-t_2+g_1g_2-g_2n_3'=0,$

(4.50) $2-t_2'-g_2n_4'=0,$

(4.51) $t_4-t_3'=0,$

(4.52) $1-t_2'+\gamma_2=0,$

(4.53) $g_2n_1'-t_3'=0,$

(4.54) $t_4'=0,$

(4.55) $g_2n_2'=0,$

(4.56) $1-g_2n_4'-\gamma_2=0,$

(4.57) $g_2n_3'+t_1'=0.$

With a bit of persistence we find that the above linear system possesses exactly 3-dimensional solutions given by

(4.58) $t_i=\text{arbitrary for } i<4,$

(4.59) $t_4=t_3,$

$$t_1'=t_2-g_1g_2, \tag{4.60}$$

$$t_2'=1+\gamma_2, \tag{4.61}$$

$$t_3'=t_3, \tag{4.62}$$

$$t_4'=0, \tag{4.63}$$

$$n_1=t_3/g_1, \tag{4.64}$$

$$n_2=t_3/g_1, \tag{4.65}$$

$$n_3=(6+6\gamma_2-6\kappa_1-t_1)/g_1, \tag{4.66}$$

$$n_4=-t_2/g_1, \tag{4.67}$$

$$n_1'=t_3/g_2, \tag{4.68}$$

$$n_2'=0, \tag{4.69}$$

$$n_3'=(g_1g_2-t_2)/g_2, \tag{4.70}$$

$$n_4'=(1-\gamma_2)/g_2. \tag{4.71}$$

5. Totally parabolic surfaces with a constant secondary invariant.

In this section we will prove the following theorems.

Theorem 1. *Let $f: M \to \boldsymbol{P}^3$ be a totally parabolic surface. Then its primary invariant $\kappa=\kappa_1$ is not a constant.*

Theorem 2. *Let f be as in Theorem 1. Then the secondary invariant γ_2 is not a constant.*

Theorem 3. *Let f be as in Theorem 1. Then the secondary invariant γ_1 is a constant c if and only if $c=0$. In this case we have*

$$\kappa>1 \quad \textit{and} \quad \gamma_2=(\kappa-1)^{1/2}.$$

Theorem 4. *Let $\hat{\kappa}: M \to \boldsymbol{R}$ be any nonconstant smooth function with $\hat{\kappa}>1$. Then there exists a totally parabolic local holomorphic immersion*

$$f: U \subset M \longrightarrow \boldsymbol{P}^3$$

such that $\kappa|_U=\hat{\kappa}|_U$, and $\gamma_1|_U\equiv 0$.

Remark. i) A consequence of Theorem 1 is that a totally parabolic surface is never of constant holomorphic sectional curvature, and is never homogeneous.

ii) We will produce, in the course of proving Theorem 4, a class of non-left-invariant involutive distributions on $U(4)$ with 5-real-dimensional leaves.

Proof of Theorem 1. Let $e: U \subset M \to U(4)$ be a second order Frenet frame along f, and put as usual $\omega = e^*\Omega$. We then have

$$\omega_1^3 = k_1 \omega_0^1, \quad \omega_2^3 = 0, \quad k_1 > 0. \tag{5.1}$$

Assume that $k_1 = c$ is a constant. Exterior differentiation of both sides of the equations in (5.1) leads to the equations in (4.3, 4.4) with $d\log k_1 = 0$:

$$-(2\omega_1^1 - \omega_0^0 - \omega_3^3) \wedge \omega_0^1 - \omega_2^1 \wedge \omega_0^2 = 0, \tag{5.2}$$

$$\omega_2^1 \wedge \omega_0^1 = 0. \tag{5.3}$$

The exterior equations in (5.2, 5.3) lead to the linear system in (4.10) with $(g_i) = 0$, and we find that the only solution is given by

$$(o_i) = (s_i) = (s_i') = 0.$$

Hence

$$2\omega_1^1 - \omega_0^0 - \omega_3^3 = 0, \quad \omega_2^1 = 0. \tag{5.4}$$

Exterior differentiate both sides of the equation $\omega_2^1 = 0$ modulo equations in (4.1, 5.1, 5.4), and obtain

$$0 = -\omega_0^1 \wedge \omega_2^0.$$

But the forms ω_0^1, ω_0^2 are independent on M, and the theorem is proved. □

Given a totally parabolic surface

$$f: M \longrightarrow \boldsymbol{P}^3$$

we take a third order Frenet frame e and put $e*\Omega = \omega$. Then we have

$$d\log k_1 = g_1 \alpha^1 + g_2 \alpha^2, \tag{5.5}$$

$$i(2\omega_1^1 - \omega_0^0 - \omega_3^3) = g_1 \alpha^3 + g_2 \alpha^4, \tag{5.6}$$

$$\omega_2^1 = -g_2(\alpha^1 + i\alpha^2), \tag{5.7}$$

where

$$\omega_0^1 = \alpha^1 + i\alpha^3, \quad \omega_0^2 = \alpha^2 + i\alpha^4.$$

By Theorem 1 we know that $(g_i) \neq 0$.

Proof of Theorem 2. Suppose $g_2 \equiv 0$. Then (5.7) becomes

$$\omega_2^1 = 0,$$

and again we arrive at

$$\omega_0^1 \wedge \omega_2^0 = 0$$

violating the independence condition. We now suppose that $g_2=c>0$. Exterior differentiation of (5.5-5.7) leads to (4.40-4.57) with $(t_1')=0$. In particular, (4.52) becomes

$$1+\gamma_2=0$$

which is impossible. □

Proof of Theorems 3 and 4. Suppose $g_1=c>0$. We then obtain the equations in (4.40-4.57) with $(t_i)=0$. Rearranging thus obtained equations we have

$$t_4'=0, \tag{5.8}$$

$$n_2'=0, \tag{5.9}$$

$$t_3'=0, \tag{5.10}$$

$$g_1n_1=0, \tag{5.11}$$

$$g_2n_1'=0, \tag{5.12}$$

$$g_1n_2=0, \tag{5.13}$$

$$-g_1n_2+g_2n_1'=0, \tag{5.14}$$

$$6+6\gamma_2-6\kappa_1-g_1n_3=0, \tag{5.15}$$

$$-t_1'-g_1n_4-g_1g_2=0, \tag{5.16}$$

$$g_1g_2-g_2n_3'=0, \tag{5.17}$$

$$2-t_2'-g_2n_4'=0, \tag{5.18}$$

$$1-g_2n_4'-\gamma_2=0, \tag{5.19}$$

$$g_2n_3'+t_1'=0. \tag{5.20}$$

The equations in (5.8-5.20) represent a linear system in $(t_i'; n_i; n_i')$. Note that (5.19) implies that g_2 can not vanish. Now the equations of (5.11-5.13) imply that $n_1=n_1'=n_2=0$. In fact it is not hard to see that the linear system possesses a unique solution: it is given by

$$n_1=n_2=n_4=0, \quad n_3=(6+6\gamma_2-6\kappa_1)/g_1,$$

$$n_1'=n_2'=0, \quad n_3'=g_1, \quad n_4'=(1-\gamma_2)/g_2,$$

$$t_1'=-g_1g_2, \quad t_2'=1+\gamma_2, \quad t_3'=t_4'=0.$$

Consequently we obtain

$$ic(\omega_0^0-\omega_1^1)=(6+6\gamma_2-6\kappa_1)\alpha^3, \tag{5.21}$$

$$i(\omega_0^0-\omega_2^2)=c\alpha^3+((1-\gamma_2)/g_2)\alpha^4, \tag{5.22}$$

(5.23) $$dg_2=-cg_2\alpha^1+(1+\gamma_2)\alpha^2,$$

where $c=g_1>0$. We exterior differentiate both sides of the equation in (5.23) and obtain

(5.24) $$0=-cdg_2\wedge\alpha^1-cg_2d\alpha^1+2g_2dg_2\wedge\alpha^2+(1+\gamma_2)d\alpha^2.$$

The equations in (4.23, 4.24) in consultation with (5.21, 5.22) give

(5.25) $$d\alpha^1=g_2(\alpha^1\wedge\alpha^2-\alpha^3\wedge\alpha^4),$$

(5.26) $$d\alpha^2=c\alpha^3\wedge\alpha^4.$$

Substituting (5.23, 5.25, 5.26) into (5.24) we obtain

$$0=(c-2c\gamma_2)\alpha^1\wedge\alpha^2+(c+2c\gamma_2)\alpha^3\wedge\alpha^4.$$

It follows that

$$1-2\gamma_2=1+2\gamma_2=0$$

which is absurd. We have thus established that if γ_1 is a constant then it must be equal to zero. Suppose that $\gamma_1=c=0$. The equations in (4.20-4.22) become

(5.27) $$d\log k_1=g_2\alpha^2,$$

(5.28) $$i(2\omega_1^1-\omega_0^0-\omega_3^3)=g_2\alpha^4,$$

(5.29) $$\omega_2^1=-g_2(\alpha^1+i\alpha_3).$$

Exterior differentiation of the equations in (5.27-5.29) leads (5.8-5.20) with $g_1=0$:

(5.30) $$t_4'=0,$$

(5.31) $$n_2'=0,$$

(5.32) $$t_3'=0,$$

(5.33) $$n_1'=0,$$

(5.34) $$6+6\gamma_2-6\kappa_1=0,$$

(5.35) $$t_1'=0,$$

(5.36) $$n_3'=0,$$

(5.37) $$2-t_2'-g_2n_4'=0,$$

(5.38) $$1-g_2n_4'-\gamma_2=0.$$

Observe that g_2 is never zero. Solving the system of equations in (5.30-5.38) we obtain:

$$t_1'=t_3'=t_4'=n_1'=n_2'=n_3'=0\,, \tag{5.39}$$

$$\gamma_2=\kappa_1-1>0\,, \tag{5.40}$$

$$n_4'=(2-\kappa_1)/(\kappa_1-1)^{1/2}\,, \tag{5.41}$$

$$t_2'=\kappa_1\,. \tag{5.42}$$

Note that (n_i) are arbitrary. We consider the exterior differential system on M given by

$$\omega_0^3=\omega_2^3=0, \quad \omega_1^3=k_1\omega_0^1 \quad (k_1>1)\,, \tag{5.43}$$

$$d\log k_1-(2\omega_1^1-\omega_0^0-\omega_3^3)=(\kappa_1-1)^{1/2}\omega_0^2\,, \tag{5.44}$$

$$\omega_2^1=-(\kappa_1-1)^{1/2}\omega_0^1\,, \tag{5.45}$$

$$d(\kappa_1-1)^{1/2}=\kappa_1\operatorname{Re}\omega_0^2\,, \tag{5.46}$$

$$i(\omega_0^0-\omega_2^2)=((2-\kappa_1)/(\kappa_1-1)^{1/2})\operatorname{Im}\omega_0^2\,. \tag{5.47}$$

We now exterior differentiate the equations in (5.46, 5.47) and show that they lead to no new quadratic equations modulo the system. Exterior differentiate the equation in (5.46) and obtain

$$0=d\kappa_1\wedge\alpha^2+\kappa_1 d\alpha^2=2k_1 dk_1\wedge\alpha^2$$

since $d\alpha^2=i(\omega_0^0-\omega_2^2)\wedge\alpha^4=0$ modulo the system. Now

$$k_1 dk_1\wedge\alpha^2=\kappa_1 d\log k_1\wedge\alpha^2=0$$

by virtue of (5.44). Exterior differentiate the left side of (5.47) and obtain

$$i(d\omega_0^0-d\omega_2^2)=(2\kappa_1-4)\alpha^1\wedge\alpha^3-4\alpha^2\wedge\alpha^4\,.$$

The exterior derivative of the right hand side of (5.47) is easily seen to be equal to the above expression using the real part of the equation in (5.44) and

$$d\alpha^2=0\,, \tag{5.48}$$

$$d\alpha^4=(\kappa_1-1)^{-1/2}((2-\kappa_1)\alpha^2\wedge\alpha^4-2(\kappa_1-1)\alpha^1\wedge\alpha^3)\,. \tag{5.49}$$

We also record that

$$d\alpha^1=i(\omega_0^0-\omega_1^1)\wedge\alpha^3+g_2(\alpha^1\wedge\alpha^2-\alpha^3\wedge\alpha^4)\,, \tag{5.50}$$

$$d\alpha^3=i(\omega_0^0-\omega_1^1)\wedge\alpha^1+g_2(\alpha^1\wedge\alpha^4-\alpha^2\wedge\alpha^3)\,. \tag{5.51}$$

Let $\theta: U(4)\to\boldsymbol{R}^+$ and consider the exterior differential system, denoted by Σ_θ, given by:

$$\Omega_0^3=\Omega_2^3=0, \quad \Omega_1^3=(\cosh\theta)\Omega_0^1\,, \tag{5.52}$$

(5.53) $$d\log(\cosh\theta)-(2\Omega_1^1-\Omega_0^0-\Omega_3^3)=(\sinh\theta)\Omega_0^2,$$

(5.54) $$\Omega_2^1=-(\sinh\theta)\Omega_0^1,$$

(5.55) $$d(\sinh\theta)=(\cosh^2\theta)\operatorname{Re}\Omega_0^2,$$

(5.56) $$i(\Omega_0^0-\Omega_2^2)=(\operatorname{cosech}\theta-\sinh\theta)\operatorname{Im}\Omega_0^2.$$

The point of the long calculation leading up to the present paragraph is that the system Σ_θ on $U(4)$ is *completely integrable.* (The reader can, of course, verify this directly by closing the sytem Σ_θ with the aid of the Maurer-Cartan structure equations of $U(4)$.) We see also that a leaf of Σ_θ locally projects down to give a complex surface in P^3 with

$$\kappa_1=\cosh^2(\pi(\theta)).$$

Finally we remark that the completely integrable distribution defined by Σ_θ is *non-left-invariant.* □

Remark. The equation (5.27) shows that the real 1-form α^2 is a globally defined form on M. Now (5.48) shows that α^2 is a closed form. Hence given a totally parabolic immersion $f: M\to P^3$ there arises a de Rham cohomology class $[\alpha^2(f)]$.

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