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STRONG LAWS FOR THE POISSON SAMPLED STOCHASTIC PROCESSES

By

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Abstract. Let $\{X_t: t\geqq 0\}$ be a stochastic process with continuous time paramenter and $m(\cdot)$ a measure on $R_+=[0,\infty)$. For large T the value of the integral $\int_0^T X_t m(dt)$ is approximated by $\int_0^T X_t N(dt)$, where N is the Poisson random process with mean measure $m(\cdot)$. The strong law and the law of the iterated logarithm of large numbers are proved for the difference of these two integrals under very mild conditions. Any structures for $\{X_t\}$ such as stationarity, ergodicity and mixing properties are not assumed.

1. Introduction.

Let $X=\{X_t, t\geqq 0\}$ be a stochastic process with continuous time parameter $t\in R_+=[0, \infty)$ and let m be a measure on R_+ with $m(R_+)=\infty$. If X is strictly stationary, ergodic and integrable then it is possible to evaluate the value of

$$\lim_{T\to\infty} T^{-1}\int_0^T X_t dt$$

using only the values of X_t observed at regularly sampled discrete time epochs. However without stationarity assumption observations of the process at regularly sampled epochs may be of no use to find the asymptotic behavior of the integral

$$\int_0^T X_t m(dt). \tag{1.1}$$

In this paper instead of systematic sampling we employ Poisson sampling procedure discussed extensively by E. Masry [4, 5, 6, 7].

Let N be a Poisson point process with mean measure m. Suppose N is independent of $\{X_t\}$ and consider the integral

$$\int_0^T X_t N(dt), \tag{1.2}$$

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which is the sum of the observations of X_t at Poisson sampled time epochs t_i in the time interval $[0, T]$. Our main interest is the asymptotic behavior of the difference of (1.1) and (1.2). In this paper we prove the strong law of large numbers and the law of the iterated logarithm for the difference of these two integrals.

Although some results in this paper are generalized to multiparameter processes we restrict ourselves to the one-paprameter case to emphasize the idea.

2. Preliminaries.

Let m be a measure on $(R_+, \mathcal{B}_+)$, where $\mathcal{B}_+$ is the Borel $\sigma-$ algebra, and write $m(t)$ for $m([0, t])$, $t \in R_+$. Suppose that $m(t)$ is a strictly increasing continuous function on R_+ satisfying $m(0)=0$ and $m(+\infty)=+\infty$. Let $X=\{X_t, t \in R_+\}$ be a measurable stochastic process and let $N=\{N(A), A \in \mathcal{B}_+\}$ be a Poisson point process independent of X defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{M}_t$, $t>0$, be the completion of the $\sigma-$ algebra generated by $N(A)$, $A \in \mathcal{B}_+ \cap [0, t]$, and X_s, $s \in R_+$ and let $\mathcal{M}_0=\{\Omega, \phi\}$. The following well-known lemma plays the fundamental role in this paper.

Lemma 2.1. *Fix $T>0$. If $\int_0^T E|X_t| m(dt)<\infty$, then $\{(\tilde{Z}(t)=\int_0^t X_s \tilde{N}(ds), \mathcal{M}_t): 0 \leq t \leq T\}$ is a martingale where $\tilde{N}(ds)$ denotes the random measure $N(ds)-m(ds)$.*

This lemma implies that, for $0 \leq s < t \leq T$,

$$E\{\tilde{Z}(t) | \mathcal{M}_s\}=\tilde{Z}(s) \quad \text{a.s..} \tag{2.1}$$

3. The strong law of large numbers.

Define

$$T_i=\sup\{t: m(t) \leq i\} \quad i=0, 1, 2, \cdots$$

and

$$Z_i=\int_{T_{i-1}}^{T_i} X_t \tilde{N}(dt) \quad i=1, 2, \cdots.$$

Theorem 3.1. *If $\int_1^\infty m(t)^{-2} h(t)^{-2} E\{X_t^2\} m(dt)<\infty$ for some positive measurable function $h(x)$, bounded away from 0, satisfying $\inf_{T_i \leq t \leq T_{i+1}} h(t)/\tilde{h}(T_{i+1}) \geq C_0>0$ for all i, where $\tilde{h}(T)$ denotes $\sup_{0 \leq t \leq T} h(t)$, then as $T \to \infty$*

$$\frac{1}{m(T)h(T)}\left\{\int_0^T X_t N(dt)-\int_0^T X_t m(dt)\right\} \longrightarrow 0 \quad a.s. \tag{3.1}$$

Remark. The conditions in Theorem 3.1 are valid in such a situation that $E\{X_t^2\} \leq g(t)$ for all $t \geq 1$ where $g(t)$ is a nondecreasing positive function with $g(T_i)/g(T_{i+1}) \geq C_1>0$ for all $i \geq 1$. In fact $h(t)=g(t)^{1/2}$ satisfies the conditions in

Theorem 3.1. In particular, if $g(t)\leqq Km(t)^{2\alpha}$ where K and α are positive constants, then we can put $h(t)=m(t)^{\alpha-\delta}$ for any δ such that $0<\delta<\min\{\alpha, 1/2\}$.

Lemma 3.2. *For each* $\varepsilon>0$ *and* $a<b$,

$$P\Big(\sup_{a\leq s\leq b}\Big|\int_a^s X_t\tilde{N}(dt)\Big|>\varepsilon\Big)\leqq\frac{1}{\varepsilon^2}\int_a^b E\{X_t^2\}m(dt). \tag{3.2}$$

Proof of Lemma. Let $\{\Delta_n=\{s_{n0}, s_{n1}, \cdots, s_{nn_k}\}: n\geqq 1\}$ be a sequence of sets of division points of the interval $[a, b]$ such that, for each $n\geqq 1$,

$$a=s_{n0}<s_{n1}<s_{n2}<\cdots<s_{nn_k}=b,$$

$$\Delta_n\subset\Delta_{n+1}$$

and

$$\delta_n=\max\{s_{ni}-s_{n,i-1}\}\longrightarrow 0 \quad \text{as} \quad n\longrightarrow\infty.$$

By the continuity of $m(dt)$ the random process $\tilde{Z}(t)=\int_0^t X_s\tilde{N}(ds)$ has left hand limits and is right continuous. Thus we have

$$P\Big(\max_{s_{ni}\in\Delta_n}\Big|\int_a^{s_{ni}} X_t\tilde{N}(dt)\Big|>\varepsilon\Big)\uparrow P\Big(\sup_{a\leq b}\Big|\int_a^b X_t\tilde{N}(dt)\Big|>\varepsilon\Big) \tag{3.3}$$

as $n\to\infty$. On the other hand, since for each n $\Big\{\int_{s_{n,i-1}}^{s_{ni}} X_t\tilde{N}(dt), i=1, 2, \cdots, k_n\Big\}$ forms a sequence of martingale differences, by Corollary 2.1 in [1] (p. 14),

$$\begin{aligned} P\Big(\max_{s_{ni}\in\Delta_n}\Big|\int_a^{s_{ni}} X_t\tilde{N}(dt)\Big|>\varepsilon\Big) &\leqq\frac{1}{\varepsilon^2}E\Big\{\Big(\int_a^b X_t\tilde{N}(dt)\Big)^2\Big\} \\ &\leqq\frac{1}{\varepsilon^2}\int_a^b E\{X_t^2\}m(dt). \end{aligned} \tag{3.4}$$

From (3.3) and (3.4) we have (3.2). ■

Proof of Theorem. By the independence of $\{X_t\}$ and $\{N(\cdot)\}$ we have

$$\begin{aligned} E\{Z_i^2\} &=E\Big\{\Big|\int_{T_{i-1}}^{T_i} X_t\tilde{N}(dt)\Big|^2\Big\} \\ &=E\Big\{\int_{T_{i-1}}^{T_i} X_t^2 m(dt)\Big\}=\int_{T_{i-1}}^{T_i} E\{X_t^2\}m(dt). \end{aligned} \tag{3.5}$$

Moreover by the definition of $\tilde{h}(t)$ we have that for all j

$$\frac{1}{\tilde{h}(T_j)}\leqq\frac{1}{h(t)} \quad \text{for} \quad T_{j-1}\leqq t\leqq T_j. \tag{3.6}$$

Hence by the equality (3.5) and the condition in the theorem we have

$$(3.7)\qquad \sum_{j\geq 2}^{\infty}\frac{1}{m(T_j)^2\tilde{h}(T_j)^2}E\{Z_j^2\}=\sum_{j\geq 2}^{\infty}\frac{1}{m(T_j)^2\tilde{h}(T_j)^2}\int_{T_{j-1}}^{T_j}E\{X_t^2\}m(dt)$$

$$\leqq \sum_{j\geq 2}^{\infty}\int_{T_{j-1}}^{T_j}\frac{1}{m(t)^2h(t)^2}E\{X_t^2\}m(dt)=\int_{T_1}^{\infty}\frac{1}{m(t)^2h(t)^2}E\{X_t^2\}m(dt)<\infty.$$

Therefore by Theorem 2.18 in [1] as $n\to\infty$,

$$(3.8)\qquad \frac{1}{m(T_n)\tilde{h}(T_n)}\int_0^{T_n}X_t\tilde{N}(dt)\longrightarrow 0 \quad \text{a.e.},$$

so that by the assumption on $h(t)$

$$\frac{1}{m(T_n)h(T_n)}\int_0^{T_n}X_t\tilde{N}(dt)\longrightarrow 0 \quad \text{a.s..}$$

Now by Lemma 3.2 for each $j\geqq 1$ we have that for each $\varepsilon>0$

$$(3.9)\qquad P\Big(\sup_{T_{j-1}\leq s\leq T_j}\Big|\frac{1}{m(T_j)\tilde{h}(T_j)}\int_{T_{j-1}}^{s}X_t\tilde{N}(dt)\Big|>\varepsilon\Big)$$

$$\leqq \frac{1}{\varepsilon^2 m(T_j)^2\tilde{h}(T_j)^2}\int_{T_{j-1}}^{T_j}E\{X_t^2\}m(dt).$$

By the same way as in (3.7) for each $\varepsilon>0$,

$$(3.10)\qquad \sum_{j\geq 2}^{\infty}P\Big(\sup_{T_{j-1}\leq s\leq T_j}\Big|\frac{1}{m(T_j)\tilde{h}(T_j)}\int_{T_{j-1}}^{s}X_t\tilde{N}(dt)\Big|>\varepsilon\Big)$$

$$\leqq \frac{1}{\varepsilon^2}\int_{T_1}^{\infty}\frac{1}{m(t)^2h(t)^2}E\{X_t^2\}m(dt)<\infty.$$

So by the Borel-Cantelli Lemma as $j\to\infty$,

$$(3.11)\qquad \sup_{T_{j-1}\leq s\leq T_j}\Big|\frac{1}{m(T_j)\tilde{h}(T_j)}\int_{T_{j-1}}^{s}X_t\tilde{N}(dt)\Big|\longrightarrow 0 \quad \text{a.e..}$$

For $T>0$ define

$$(3.12)\qquad \langle T\rangle=\max\{T_j:\ T_i\leqq T\}.$$

Then by the assumption on $h(\cdot)$, (3.8) and (3.11) we have

$$(3.13)\qquad \Big|\frac{1}{m(T)h(T)}\int_0^{T}X_t\tilde{N}(dt)-\frac{1}{m(\langle T\rangle)h(\langle T\rangle)}\int_0^{\langle T\rangle}X_t\tilde{N}(dt)\Big|$$

$$\leqq \frac{m(\langle T\rangle)h(\langle T\rangle)}{m(T)h(T)}\cdot\frac{1}{m(\langle T\rangle h(\langle T\rangle)}\Big|\int_0^{T}X_t\tilde{N}(dt)-\int_0^{\langle T\rangle}X_t\tilde{N}(dt)\Big|$$

$$+\Big|\Big(\frac{m(\langle T\rangle)h(\langle T\rangle)}{m(T)h(T)}-1\Big)\frac{1}{m(\langle T\rangle)h(\langle T\rangle)}\int_0^{\langle T\rangle}X_t\tilde{N}(dt)\Big|$$

$$\leqq \frac{m(\langle T\rangle)h(\langle T\rangle)}{m(T)h(T)}\cdot\frac{m(\langle T\rangle)+1}{m(\langle T\rangle)}\cdot\frac{1}{m(\langle T\rangle)}\Big|\int_{\langle T\rangle}^{T}X_t\tilde{N}(dt)\Big|+o(1)$$

$$\longrightarrow 0 \quad \text{a.s..}$$

This completes the proof. ■

4. The law of the iterated logarithm.

Consider the following conditions on $\{X_t\}$ and a positive measurable function $h(x)$ which is bounded away from 0:

(A) $$\int_1^\infty \frac{1}{m(t)^2 h(t)^2} E\{X_t^2\} m(dt) < \infty,$$

(B) $$\int_1^\infty \frac{1}{m(t)^2 h(t)^4} E\{X_t^4\} m(dt) < \infty,$$

(C) $$\inf_{T_{j-1} \leq s \leq T_j} \frac{h(s)}{h(T_j)} \geqq C_0 > 0,$$

(D) $$0 < \liminf_{T\to\infty} \frac{1}{m(T)h(T)^2} \int_0^T E\{X_t^2\} m(dt)$$
$$\leqq \limsup_{T\to\infty} \frac{1}{m(T)h(T)^2} \int_0^T E\{X_t^2\} m(dt) < \infty,$$

(E) $$0 < \liminf_{T\to\infty} \frac{1}{m(T)h(T)^2} \int_0^T X_t^2 m(dt)$$
$$\leqq \limsup_{T\to\infty} \frac{1}{m(T)h(T)^2} \int_0^T X_t^2 m(dt) < \infty \quad \text{a.s.}.$$

Remark. Suppose that $E\{X_t^4\} \leqq K_1 m(t)^{4\alpha}$ and $K_2 m(t)^{2\alpha} \leqq E\{X_t^2\} \leqq K_3 m(t)^{2\alpha}$ for some $\alpha \geqq 0$ and $K_i > 0$, $(i=1, 2, 3)$. Then the conditions (A), (B), (C) and (D) are satisfied. If in addition we assume that $\{X_t\}$ satisfies some mixing condition (e.g. ϕ-mixing) with suitable mixing rate, then the condition (E) is easily checked, also.

For brevity introduce the notation:

$$\phi(t) = (2t \log\log t)^{1/2} \qquad t \geqq 3.$$

Now we state the main result.

Theorem 4.1. *Assume that there exists a function $h(x)$ satisfying the conditions* (A), (B), (C), (D) *and* (E). *Then we have*

(4.1) $$\liminf_{T\to\infty} \phi\left(\int_0^T X_t^2 (N(dt)\right)^{-1} \left\{\int_0^T X_t N(dt) - \int_0^T X_t m(dt)\right\} = -1 \quad a.s..$$

and

(4.2) $$\limsup_{T\to\infty} \phi\left(\int_0^T X_t^2 N(dt)\right)^{-1} \left\{\int_0^T X_t N(dt) - \int_0^T X_t m(dt)\right\} = +1 \quad a.s..$$

In the remainder of this section we assume that $h(x)$ is a function satisfying the conditions (A), (B), (C), (D) and (E).

Lemma 4.2. *As* $n \to \infty$,

$$\frac{1}{m(T_n)h(T_n)^2}\left\{\sum_{n=1}^{n}\left(\int_{T_{i-1}}^{T_i} X_t(\tilde{N}(dt))^2 - \int_0^{T_n} X_t^2 N(dt)\right\} \longrightarrow 0 \quad a.s.. \tag{4.3}$$

Proof. Set

$$Z_j = \int_{T_{j-1}}^{T_j} X_t \tilde{N}(dt) \qquad j=1, 2, \cdots.$$

Then we have

$$E\{Z_j^2 | \mathcal{M}_{j-1}\} = \int_{T_{j-1}}^{T_j} X_t^2 m(dt) \quad \text{a.s..} \tag{4.4}$$

Hence for $j \geqq 1$,

$$E[(Z_j^2 - E\{Z_j^2 | \mathcal{M}_{j-1}\})^2] = E\left\{\left(Z_j^2 - \int_{T_{j-1}}^{T_j} X_t^2 m(dt)\right)^2\right\} \tag{4.5}$$

$$= E\left\{\left(\int_{T_{j-1}}^{T_j} X_t \tilde{N}(dt)\right)^4\right\} - E\left\{\left(\int_{T_{j-1}}^{T_j} X_t^2 m(dt)\right)^2\right\}$$

$$= 5E\left\{\left(\int_{T_{j-1}}^{T_j} X_t^2 m(dt)\right)^2\right\} + \int_{T_{j-1}}^{T_j} E\{X_t^4\} m(dt)$$

$$\leqq 5\int_{T_{j-1}}^{T_j} E\{X_t^4\} m(dt)\{m(T_j) - m(T_{j-1})\} + \int_{T_{j-1}}^{T_j} E\{X_t^4\} m(dt)$$

(by the Schwarz inequality)

$$= 6\int_{T_{j-1}}^{T_j} E\{X_t^4\} m(dt)$$

(by the definition of the sequence $\{T_j\}$).

Thus by the conditions (B) and (C) we have

$$\sum_{j=2}^{\infty} \frac{1}{m(T_j)^2 \tilde{h}(T_j)^4} E[(Z_j^2 - E\{Z_j^2 | \mathcal{M}_{j-1}\})^2] \tag{4.6}$$

$$\leqq 6\int_{T_1}^{\infty} \frac{1}{m(t)^2 h(t)^4} E\{X_t^4\} m(dt) < \infty.$$

By Theorem 2.18 in [1] we have that, as $n \to \infty$,

$$\frac{1}{m(T_n)\tilde{h}(T_n)^2}\left\{\sum_{j=1}^{n}\left(\int_{T_{j-1}}^{T_j} X_t \tilde{N}(dt)\right)^2 - \int_0^{T_n} X_t^2 m(dt)\right\} \longrightarrow 0 \quad \text{a.s.} \tag{4.7}$$

On the other hand, by Theorem 3.1 under the conditions (B) and (C), as $T \to \infty$,

$$\frac{1}{m(T)h(T)^2}\int_0^T X_t^2 \tilde{N}(dt) \longrightarrow 0 \quad \text{a.s..} \tag{4.8}$$

Therefore combining (4.7) and (4.8) we have that, as $n \to \infty$,

$$\frac{1}{m(T_n)h(T_n)^2}\left\{\sum_{j=1}^{n}\left(\int_{T_{j-1}}^{T_j} X_t \tilde{N}(dt)\right)^2 - \int_0^{T_n} X_t^2 N(dt)\right\} \longrightarrow 0 \quad \text{a.e..} \tag{4.9}$$

This completes the proof. ■

Define

(4.10) $$U_n^2=\sum_{j=1}^{n} Z_j^2,\quad V_n^2=\int_0^{T_n} X_t^2 N(dt),\quad S_n=\int_0^{T_n} X_t \tilde{N}(dt)$$

and

$$s_n^2=E\{S_n^2\}=\int_0^{T_n} E\{X_t^2\} m(dt),\qquad n=1,2,\cdots.$$

Lemma 4.3. *As* $n\to\infty$,

(4.11) $$\frac{U_n^2}{V_n^2}\longrightarrow 1 \quad a.s..$$

Proof. By Lemma 4.2 and (E)

$$\frac{U_n^2}{V_n^2}=\frac{\sum_{i=1}^n Z_i^2}{\int_0^{T_n} X_t^2 N(dt)}=\frac{s_n^{-2}\left(\sum_{i=1}^n Z_i^2-\int_0^{T_n} X_t^2 N(dt)\right)}{s_n^{-2}\int_0^{T_n} X_t^2 N(dt)}+1\longrightarrow 1 \quad \text{a.s..}$$ ■

Lemma 4.4. *As* $n\to\infty$,

(4.12) $$\frac{V_{n+1}^2}{V_n^2}\longrightarrow 1 \quad a.s..$$

Proof. Remark that by the condition (D) there exists an $\delta>0$ such that for large n

$$\frac{s_n^2}{m(T_n)h(T_n)^2}=\frac{1}{m(T_n)h(T_n)^2}\int_0^{T_n} E\{X_t^2\} m(dt)\geqq\delta>0.$$

Hence for some positive integer j_0 we have

$$E\left\{\sum_{j=j_0}^{\infty}\frac{1}{s_j^2}\int_{T_j}^{T_{j+1}} X_t^2 N(dt)\right\}\leqq K\sum_{j=j_0}^{\infty}\frac{m(T_j)h(T_j)^2}{s_j^2}\int_{T_j}^{T_{j+1}}\frac{1}{m(t)h(t)^2}E\{X_t^2\} m(dt)$$

$$\leqq\frac{K}{\delta}\int_1^{\infty}\frac{1}{m(t)h(t)^2}E\{X_t^2\} m(dt)<\infty \quad \text{by } (A).$$

Thus we have that, as $n\to\infty$,

$$\frac{1}{s_n^2}\int_{T_n}^{T_{n+1}} X_t^2 N(dt)\longrightarrow 0 \quad \text{a.s.,}$$

from which we have

$$\frac{V_{n+1}^2}{V_n^2}\longrightarrow 1 \quad \text{a.s.,}$$

completing the proof of the lemma. ■

Lemma 4.5. *As* $n\to\infty$,

(4.13) $$0<\liminf_{n\to\infty}\frac{1}{s_n^2}\int_0^{T_n}X_t^2N(dt)\leqq\limsup_{n\to\infty}\frac{1}{s_n^2}\int_0^{T_n}X_t^2N(dt)<\infty \quad a.s..$$

Proof. Immediate from the conditions (D) and (E). ■

Lemma 4.6. *The following three statements hold with probability* 1:

(4.14) $$\phi(V_{n-1}^2)^{-1}\sum_{i=1}^{n}\{Z_iI(|Z_i|>s_i)-E[Z_iI(|Z_i|>s_i)|\mathcal{M}_{i-1}]\}\longrightarrow 0,$$

(4.15) $$V_{n-1}^{-2}\sum_{i=1}^{n}\{E[Z_i^2I(|Z_i|\leqq s_i)|\mathcal{M}_{i-1}]-[E(Z_iI(|Z_i|\leqq s_i)|\mathcal{M}_{i-1})]^2\}\longrightarrow 1$$

and

(4.16) $$\sum_{i=1}^{\infty}V_{i-1}^{-4}E[Z_i^4|\mathcal{M}_{i-1}]<\infty.$$

Proof. We will modify the arguments in [1] (pp. 123–124).

Proof of (4.14). By the condition (C), we have the following inequalities.

$$E\Big\{\sum_{j=j_0}^{\infty}s_j^{-1}|Z_j|I(|Z_j|>s_j)\Big\}\leqq\sum_{j=j_0}^{\infty}s_j^{-2}E\{Z_j^2\}$$
$$\leqq\sum_{j=j_0}^{\infty}s_j^{-2}\int_{T_{j-1}}^{T_j}E\{X_t^2\}m(dt)\leqq K\int_{j_0-1}^{\infty}\frac{1}{m(t)h(t)^2}E\{X_t^2\}m(dt)<\infty,$$

where j_0 is a sufficiently large integer and K some positive constant. So by Kronecker's lemma we have, as $n\to\infty$,

$$s_n^{-1}\sum_{i=1}^{n}Z_iI(|Z_i|>s_i)\longrightarrow 0 \quad \text{a.s.}$$

and

$$s_n^{-1}\sum_{i=1}^{n}E[Z_iI(|Z_i|>s_i)|\mathcal{M}_{i-1}]\longrightarrow 0 \quad \text{a.s..}$$

On the other hand by Lemma 4.5,

$$0<\liminf_{n\to\infty}\frac{V_{n-1}^2}{U_n^2}\leqq\limsup_{n\to\infty}\frac{V_{n-1}^2}{U_n^2}<\infty \quad \text{a.s..}$$

Thus we have proved (4.14).

Proof of (4.15).

$$\frac{1}{V_{n-1}^2}\sum_{i=1}^{n}\{E[Z_i^2I(|Z_i|<s_i)|\mathcal{M}_{i-1}]-[E(Z_iI(|Z_i|<s_i)|\mathcal{M}_{i-1}]^2\}$$
$$=\frac{U_n^2}{V_{n-1}^2}-\frac{s_n^2}{V_{n-1}^2}\Big\{\frac{1}{s_n^2}\sum_{i=1}^{n}Z_i^2I(|Z_i|>s_i)-\frac{1}{s_n^2}\sum_{i=1}^{n}[E(Z_iI(|Z_i|>s_i)|\mathcal{M}_{i-1})]^2\Big\}$$
$$+\frac{s_n^2}{V_{n-1}^2}\cdot\frac{1}{s_n^2}\sum_{i=1}^{n}\{E[Z_i^2I(|Z_i|\leqq s_i)|\mathcal{M}_{i-1}]-Z_i^2I(|Z_i|\leqq s_i)\}.$$

By Lemma 4.4 and 4.5 and the other assumptions, using the argument in [1] (p. 124), we have the result.

Proof of (4.16). By the conditions (B) and (D) we can easily obtain that

$$\sum_{i=1}^{\infty} \frac{1}{s_i^4} E\{Z_i^4\} < \infty,$$

so that, by Lemma 4.4, we have

$$\sum_{i=1}^{\infty} \frac{1}{V_i^4} E[Z_i^4 I(|Z_i| \leqq s_i) | \mathcal{M}_{i-1}] < \infty \quad \text{a. s..} \quad \blacksquare$$

Let K denote the set of absolutely continuous functions x in $C[0, 1]$ with $x(0)=0$ and whose derivatives $\dot{x}$ are such that

$$\int_0^1 \dot{x}^2(t) dt \leqq 1.$$

Define

$$\xi_n(t) = \phi(V_{n-1}^2)^{-1}\{S_i + (V_i^2 - V_{i-1}^2)^{-1}(tV_{n-1}^2 - V_{i-1}^2)Z_{i+1}\} \tag{4.17}$$

$$\text{for} \quad V_{i-1}^2 \leqq tV_{n-1}^2 < V_i^2, \quad 1 \leqq i \leqq n-2,$$

and

$$\zeta_n(t) = \phi(V_n^2)^{-1}\{S_i + (V_{i+1}^2 - V_i^2)^{-1}(tV_n^2 - V_i^2)Z_{i+1}\} \tag{4.18}$$

$$\text{for} \quad V_i^2 \leqq tV_n^2 < V_{i+1}^2, \quad 0 \leqq i \leqq n-1.$$

Lemma 4.7. *With probability* 1 *the sequence* $\{\zeta_n\}$ *is relatively compact in* $C[0, 1]$ *and the set of its limit points coincides with* K.

Proof. By Theorem 4.7 in [1] and Lemma 4.5, with probability 1, the sequence $\{\xi_n\}$ is relatively compact in $C[0, 1]$ and its set of a. s. limit points coincides with K. This combined with Lemma 4.3 gives the required result. ■

From this lemma we have

Corollary 4.8. *With probability* 1,

$$\liminf_{n\to\infty} \phi\left(\int_0^{T_n} X_t^2 N(dt)\right)^{-1} \int_0^{T_n} X_t \tilde{N}(dt) = -1 \tag{4.19}$$

and

$$\limsup_{n\to\infty} \phi\left(\int_0^{T_n} X_t^2 N(dt)\right)^{-1} \int_0^{T_n} X_t \tilde{N}(dt) = +1. \tag{4.20}$$

Proof of Theorem. We have already seen that, as $n \to \infty$,

$$\frac{\int_0^{T_{n-1}} X_t^2 N(dt)}{\int_0^{T_n} X_t^2 N(dt)} \longrightarrow 1 \quad \text{a. s.} \tag{4.21}$$

(Lemma 4.3). Hence it suffices to show that, as $n\to\infty$,

$$\left(\int_0^{T_n} X_t^2 N(dt)\right)^{-1/2} \sup_{T_{n-1}\leq s\leq T_n}\left|\int_{T_{n-1}}^{s} X_t\tilde{N}(dt)\right| \longrightarrow 0 \quad \text{a.s.,} \tag{4.22}$$

which is, by Lemma 4.4 and the condition (E), equivalent to that, as $n\to\infty$,

$$\frac{1}{\sqrt{m(T_n)h(T_n)^2}} \sup_{T_{n-1}\leq s\leq T_n}\left|\int_{T_{n-1}}^{s} X_t\tilde{N}(dt)\right| \longrightarrow 0 \quad \text{a.s..} \tag{4.23}$$

By Corollary 2.1 in [1] (p. 14) and the same argument as the proof of Lemma 3.2, we have that, for each n and $\varepsilon>0$,

$$P\left(\frac{1}{\sqrt{m(T_n)h(T_n)^2}} \sup_{T_{n-1}\leq s\leq T_n}\left|\int_{T_{n-1}}^{s} X_t\tilde{N}(dt)\right|>\varepsilon\right) \tag{4.24}$$
$$\leq\frac{K}{\varepsilon^4}\int_{T_{n-1}}^{T_n}\frac{1}{m(t)^2h(t)^4}E\{X_t^4\}m(dt),$$

where K is some positive constant, not depending on n. Hence by the condition (B) and the Borel-Cantelli lemma, we have that, for each $\varepsilon>0$,

$$P\left(\left(\int_0^{T_n} X_t^2 N(dt)^{-1/2} \sup_{T_{n-1}\leq s\leq T_n}\left|\int_{T_{n-1}}^{s} X_t(\tilde{N}(dt)\right|>\varepsilon \text{ i.o.}\right)=0. \tag{4.25}$$

This completes the proof of Theorem 4.1. ■

5. Concluding remarks.

(a) The function $h(x)$ in the conditions of Theorem 3.1 and 4.1 is not superfluous as will be seen by the following example. For any $\alpha>0$ there exists a process X satisfying the conditions from (A) through (B) with $h(x)=1\vee x^\alpha$, $x\geqq 0$. In fact let $Y(t)$, $t\geqq 0$, be a stationary process with finite fourth moments. Then the process $X(t)$, $t\geqq 0$, defined by $X(t)=h(t)Y(t)$ satisfies conditions (A), (B) and (C) with the function $h(x)$ and the Lebesgue measure m. Let η be the a.s. limit of $T^{-1}\int_0^T Y(t)^2dt$. Then integrating by parts we obtain

$$\frac{1}{m(T)h(T)^2}\int_0^T X(t)^2m(dt)=\frac{1}{T\cdot T^{2\alpha}}\int_0^T t^{2\alpha}Y(t)^2dt \tag{5.1}$$
$$=\frac{1}{T^{1+2\alpha}}T^{2\alpha}\int_0^T Y(t)^2dt-\frac{2\alpha}{T^{1+2\alpha}}\int_0^T t^{2\alpha}\left\{\frac{1}{t}\int_0^t Y(s)^2ds\right\}dt$$
$$\sim\eta-\frac{2\alpha}{2\alpha+1}\eta=\eta\left(\frac{1}{2\alpha+1}\right).$$

Thus the condition (E) is satisfied. The condition (D) can be verified similarly.

(b) For inference purpose the following pathwise central limit theorem will be useful.

Theorem 5.1. *Assume the conditions in Theorem* 4.1. *Then as* $T\to\infty$, (5.2) $$\sup_{x\in R}\left|P\Big(\frac{1}{\sqrt{\int_0^T X_t^2(N(dt)}}\int_0^T X_t\tilde{N}(dt)\leqq x\,\Big|\,X\Big)-\Phi(x)\right|\longrightarrow 0\quad a.s.,$$

where $\Phi(\cdot)$ denotes the standard normal distribution function.

Lemma 5.2. *As $T\to\infty$,*

(5.3) $$\frac{1}{m(T)^2h(T)^4}\int_0^T X_t^4 m(dt)\longrightarrow 0\quad a.s..$$

Proof of Lemma. By the condition (B),

(5.4) $$\int_1^\infty \frac{1}{m(t)^2h(t)^4}X_t^4 m(dt)<\infty\quad \text{a.s.},$$

and therefore

(5.5) $$\int_1^\infty \frac{1}{m(t)^2\tilde{h}(t)^4}X_t^4 m(dt)<\infty\,.$$

By Kronecker's lemma and the condition (C), this implies (5.3). ■

Proof of Theorem. Fix a path $\{X_t\}$ such that as $T\to\infty$,

$$\frac{1}{m(T)^2h(T)^4}\int_0^T X_t^4 m(dt)\longrightarrow 0$$

and

$$\frac{1}{m(T)h(T)}\int_0^T X_t\tilde{N}(dt)\longrightarrow 0\quad \text{a.s.}.$$

Then the random variables

$$\left\{\int_{T_{i-1}}^{T_i} X_t\tilde{N}(dt):\ i=1,2,\cdots\right\}$$

are independent with mean 0 and

(5.6) $$\tilde{s}_n^2=E\left\{\left(\int_0^{T_n}X_t\tilde{N}(dt)\right)^2\Big|X\right\}=\int_0^{T_n}X_t^2 m(dt)\,.$$

Recall $s_n^2=\int_0^{T_n}E\{X_t^2\}m(dt)$. Now we examine the Lindeberg condition. An $\varepsilon>0$ is given. Then as $n\to\infty$,

(5.7) $$\sum_{i=1}^n s_n^{-2}E\left|\int_{T_{i-1}}^{T_i}X_t\tilde{N}(dt)\right|^2 I\left(\left|\int_{T_{i-1}}^{T_i}X_t\tilde{N}(dt)\right|>\varepsilon s_n\right)$$
$$\leqq\frac{7}{\varepsilon^2 s_n^4}\int_0^{T_n}X_t^4 m(dt)\longrightarrow 0\,.$$

Thus by the conditions (D) and (E) we have that for each $x\in R$

(5.8) $$P\left(\left(\int_0^{T_n}X_t^2 m(dt)\right)^{-1/2}\int_0^{T_n}X_t\tilde{N}(dt)\leqq x\,|\,X\right)\longrightarrow\Phi(x)\,.$$

On the other hand

$$(5.9)\qquad \frac{\int_0^{T_n} X_t^2 N(dt)}{\int_0^{T_n} X_t^2 m(dt)} = \frac{\int_0^{T_n} X_t^2 \tilde{N}(dt)}{\int_0^{T_n} X_t^2 m(dt)} + 1 \longrightarrow 1 \quad \text{a.s..}$$

Therefore we have that for each $x \in R$

$$(5.10)\qquad P\left(\left(\int_0^{T_n} X_t^2 N(dt)\right)^{-1/2} \int_0^{T_n} X_t \tilde{N}(dt) \leqq x \mid X\right) \longrightarrow \Phi(x).$$

This easily implies (5.2). ■

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