

A LIMIT THEOREM FOR TWO-DIMENSIONAL RANDOM WALK CONDITIONED TO STAY IN A CONE

By

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Summary. Consider a two-dimensional random walk with mean 0 and covariance $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Under some additional assumptions on the random walk the limit theorem is established for the normalized random walk which is conditioned to stay in a cone for a unit of time. The conditioned limit theorem is used to prove a tail formula for the distribution of the exit time of the random walk from a cone.

1. Introduction and the result.

Let $Z_k = (X_k, Y_k)$, $k=0, 1, 2, \dots$, $Z_0=0$, be a two-dimensional random walk with stationary independent increments which is defined on a probability space (Ω, \mathcal{F}, P) and satisfies the following condition;

$$(1.1) \quad E(Z_1)=0 \text{ and } \text{Cov}(Z_1)=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We set a sequence of the normalized random walks $\{Z^{(n)}(t)=(X^{(n)}(t), Y^{(n)}(t)), 0 \leq t < \infty\}$, $n=1, 2, \dots$ by

$$Z^{(n)}(t) = n^{-1/2}Z_k + (nt - k)n^{-1/2}(Z_{k+1} - Z_k)$$

for $k/n \leq t < (k+1)/n$, $k=0, 1, 2, \dots$. Let $C[0, \infty)$ be the space of all \mathbf{R}^2 -valued continuous functions on $[0, \infty)$ with the topology of the uniform convergence on every compact set, and let $\mathcal{C}[0, \infty)$ be the Borel field of $C[0, \infty)$. The function space $C[a, b]$ and its Borel field $\mathcal{C}[a, b]$, $0 \leq a \leq b < \infty$, will be defined similarly. Note that $Z^{(n)}(\cdot)$ is a random element with values in $C[0, \infty)$ (or sometimes we take it with values in $C[0, 1]$).

We fix a closed cone F in \mathbf{R}^2 with the angle α , $0 < \alpha < 2\pi$,

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$$F = F_{\beta, \beta+\alpha} = \{z = r(\cos\theta, \sin\theta), r \geq 0, \beta \leq \theta \leq \beta + \alpha\}$$

such that

$$(1.2) \quad -\pi/2 < \beta < \pi/2 < \beta + \alpha < 3\pi/2.$$

(The assumption (1.2) will be used in our proof of Lemma 6 in 4.) Suppose that the random walk satisfies

$$(1.3) \quad P(Z_1 \in F \setminus \{0\}) > 0.$$

On $(C[0, 1], C[0, 1])$ we introduce the law of a conditioned random walk

$$\tilde{W}^{(n)}(L) = P(Z^{(n)}(\cdot) \in L \mid Z_k \in F, 0 \leq k \leq n), L \in C[0, 1], n \geq 1.$$

By (1.3) $\tilde{W}^{(n)}$ is defined for every $n \geq 1$.

Let $|z|$ denote the Euclidean norm of z in R^2 . Let F_1 and F_2 be the cones defined by

$$F_1 = F_{\beta, \beta + \min(\alpha, \pi)} \quad \text{and} \quad F_2 = F_{\max(\beta, \beta + \alpha - \pi), \beta + \alpha}.$$

Note that $F_j \subseteq F$ ($j=1, 2$), and $F_1 = F_2 = F$ when $\alpha \leq \pi$. To give our main result we further impose the following assumptions on the random walk: There exist positive numbers $K > 1$, ν and γ for which we have, either

$$(1.4) \quad P(|Z_1| \leq K) = 1$$

or

$$(1.5) \quad \lim_{n \rightarrow \infty} \{nP(|Z_1| > Kn)/P(|Z_1| > n; Z_1 \in F_j)\} = 0, \quad j=1, 2;$$

P -almost surely

$$(1.6) \quad E(Y_1 | X_1) = 0, \quad E((Y_1)^2 | X_1) \geq \nu^2 \quad \text{and} \quad E(|Y_1|^3 | X_1) \leq \gamma.$$

Then we have the following theorem.

Theorem 1. *Suppose that the random walk satisfies (1.1), (1.3), either (1.4) or (1.5), and (1.6), where the cone F is put such that (1.2) holds. Then $\tilde{W}^{(n)}$ converges weakly in $C[0, 1]$ to the law \tilde{W} which is given in theorem A below.*

Remarks. (i) The assumption (1.5) holds if we have

$$P(|Z_1| > n) \asymp P(|Z_1| > n; Z_1 \in F_j) \asymp H_j(n) \exp\{-M_j(\log n)^2\}$$

as $n \rightarrow \infty$ ($j=1, 2$) for some positive constant M_j , where $H_j(n)$ is a positive non-increasing function on $[0, \infty)$ and the relation $f(n) \asymp g(n)$ as $n \rightarrow \infty$ stands for the following;

$$0 < \liminf_{n \rightarrow \infty} \{f(n)/g(n)\} \leq \limsup_{n \rightarrow \infty} \{f(n)/g(n)\} < \infty.$$

(ii) If the component random walks $\{X_n\}_n$ and $\{Y_n\}_n$ are mutually independent, the assumption (1.6) in Theorem 1 may be omitted. Indeed, in this case the first two in (1.6) follow from (1.1), and the third one is superfluous. See proof of Lemma 6 in 4.

Continuous analogue of Theorem 1 was given by Shimura [12] as follows. Let $\{Z(t), 0 \leq t < \infty\}$ be the two-dimensional standard Brownian motion on (Ω, \mathcal{F}, P) which starts at 0. Consider the law of a conditioned Brownian motion

$$\tilde{W}_z(L) = P(z + Z(\cdot) \in L \mid z + Z(t) \in F, 0 \leq t \leq 1), \quad L \in \mathcal{C}[0, 1]$$

for z in $\overset{\circ}{F}$, where $\overset{\circ}{F} = F \setminus \partial F$ is the interior of the set F .

Theorem A ([12], Theorem 2). *The law \tilde{W}_z converges weakly in $C[0, 1]$ as $\overset{\circ}{F} \ni z \rightarrow 0$ to a certain law \tilde{W} .*

We note that \tilde{W} is law of the conditioned Brownian motion which starts at 0 and enters at once into $\overset{\circ}{F}$, then stays there for a unit of time. Theorem A, together with the functional central limit theorem, would suggest Theorem 1 to us. Moreover it will play a key role to prove Theorem 1 in 5. By Theorem 1 and Theorem A we have the following theorem.

Theorem 2. *Under the same assumption to Theorem 1, we have*

$$(1.7) \quad P(Z_k \in F, 0 \leq k \leq n) = n^{-\pi/2\alpha} L(n),$$

where $L(n)$ is a slowly varying function at ∞ .

Spitzer [14] first observed a conditioned limit theorem of one-dimensional random walk. Iglehart [7] strengthened the result in terms of the functional central limit theorem and expressed the limit process by the Brownian meandering process. In [3] Durrett developed the conditioned limit theorem to a class of one-dimensional Markov chains. However, as far as we know, the conditioned limit theorem for a multi-dimensional random walk seems not to have been considered yet. When we extend the problem to the multi-dimensional random walk, we will encounter some difficulties to be dealt with. In the paper the most difficult point among them may be Lemma 5 in 4. We note that Theorem 1 may be refined to the limit theorem for the point process of excursions. Indeed, if we have Theorem 2, as was shown by Greenwood and Perkins [5] and [6] for the excursions from a moving boundary for a one-dimensional normalized random walk, we may conclude the following (see also [12], Theorem 1): *If the angle of the cone $\alpha > \pi/2$, the point process of the excursions in the cone for the normalized random walk converges to that for the Brownian motion in the sense of [6].* We also note that Theorem 2 is a two-

dimensional extension of a result given by Spitzer [14]. In [9] McConell considered the moments of the first exit time from a cone for $N(\geq 2)$ -dimensional random walk.

Our paper is organized as follows. In 2 we collect the preparatory materials which will be used often in the following sections. We show in 3 the tightness of the family of the laws $\tilde{W}^{(n)}$ in the space $C[0, 1]$. In 4 we show that a limit process of the normalized conditioned random walks, which starts at 0, enters instantaneously into the interior of F . In 5 we prove Theorems 1 and 2.

2. Preliminaries.

For w in $C[0, 1]$ (or, in $C[0, \infty)$) and z in R^2 , let w_z be an element of $C[0, 1]$ (or, of $C[0, \infty)$) defined by

$$w_z(t) = z + w(t).$$

Let $W_z^{(n)}$ and W_z be the probability measures on $(C[0, 1], C[0, 1])$ induced respectively by $Z_z^{(n)}$ and Z_z ;

$$W_z^{(n)}(L) = P(Z_z^{(n)} \in L) \quad \text{and} \quad W_z(L) = P(Z_z \in L), \quad L \in C[0, 1].$$

For $w \in C[0, 1]$ we put

$$\sigma^{(n)}(w) = \min\{k/n : k=0, 1, \dots, n, w(k/n) \notin F\}$$

and

$$\sigma(w) = \min\{0 \leq t \leq 1, w(t) \notin F\}$$

($\min \phi = \infty$), and introduce the transition probabilities of the normalized random walk and of the Brownian motion with absorption at F^c ;

$$\dot{p}^{(n)}(s, x, dz) = W_z^{(n)}(w(s) \in dz; \sigma^{(n)} > s)$$

and

$$\dot{p}(s, x, dz) = W_z(w(s) \in dz; \sigma > s), \quad 0 \leq s \leq 1, x, z \in F.$$

Then we have the following lemma.

Lemma 1. *Suppose that the random walk satisfies (1.1). Let sequences z, z_n in F and t, t_n in $[0, 1]$, $n=1, 2, \dots$, be such that $z_n \rightarrow z$ and $t_n \rightarrow t$. Then, as $n \rightarrow \infty$, the following hold: For every W_z -continuity set L in $C[0, 1]$,*

$$(2.1) \quad W_{z_n}^{(n)}(w(\cdot) \in L; \sigma^{(n)} > t_n) \longrightarrow W_z(w(\cdot) \in L; \sigma > t).$$

Let S be a domain in R^2 of the form $F+x$ or $(F+x) \setminus (F+x')$, where x and x' are points in R^2 and

$$F+x = \{z = z' + x : z' \in F\}.$$

Then we have

$$(2.2) \quad \dot{p}^{(n)}(t_n, z, S) \longrightarrow \dot{p}(t, z, S) \text{ uniformly in } z \text{ in } \mathbf{R}^2.$$

To prove Lemma 1 we need the following version of the functional central limit theorem which is a special case of Skorohod [13], Lemma in p. 10.

Theorem B. *We can construct a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ on which we have a sequence of random elements $\hat{Z}, \hat{Z}^{(1)}, \hat{Z}^{(2)}, \dots$ with values in $C[0, \infty)$ which satisfy*

$$\hat{Z} \stackrel{L}{=} Z, \hat{Z}^{(n)} \stackrel{L}{=} Z^{(n)} \quad (\text{identity in law})$$

and

$$\lim_{n \rightarrow \infty} \sup_{0 \leq t \leq T} |\hat{Z}^{(n)}(t) - \hat{Z}(t)| = 0 \quad \text{almost surely}$$

for every $T < \infty$.

By Theorem B the left-hand side of (2.1) equals to

$$\hat{P}(\hat{Z}_{z_n}^{(n)} \in L; \sigma^{(n)}(\hat{Z}_{z_n}^{(n)}) > t_n),$$

and, together with the winding property of the Brownian path (Ito and McKean [8], 7.11),

$$\sigma^{(n)}(\hat{Z}_{z_n}^{(n)}) \longrightarrow \sigma(\hat{Z}_z) \quad \text{almost surely.}$$

Hence, noting $\sigma(Z_z) \neq t$ almost surely, we get (2.1).

Note that, for every $z \in \mathbf{R}^2$,

$$(2.3) \quad \dot{p}^{(n)}(t_n, z, \cdot) \xrightarrow{w} \dot{p}(t, z, \cdot) \text{ in } \mathbf{R}^2$$

by a minor modification of the proof given above. So, together with $\dot{p}(t, z, \partial S) = 0$, we have (2.2) except for the uniformity. To get the uniformity, we need long but elementary steps of estimates of probabilities of events under consideration, where we use the special form of the set S and the spatial homogeneity of the random walk and the Brownian motion in addition to Theorem B. See Appendix for the detail.

Let $\tilde{W}_{z,t}^{(n)}$ and $\tilde{W}_{z,t}$ denote the laws on $(C[0, 1], C[0, 1])$ of the conditioned random walk and of the conditioned Brownian motion which are defined respectively as follows: For $0 \leq t \leq 1$ and $z \in F$

$$\tilde{W}_{z,t}^{(n)}(*) = W_z^{(n)}(*) | \sigma^{(n)} > t.$$

For $0 \leq t \leq 1$ and $z \in \mathring{F}$

$$\tilde{W}_{z,t}^{(n)}(*) = W_z(*) | \sigma > t$$

and for $z \in \partial F \setminus \{0\}$ $\tilde{W}_{z,t}$ is the weak limit of the laws $\tilde{W}_{z',s}$ as $\mathring{F} \ni z' \rightarrow z$ and $s \rightarrow t$ ([12], lemma 3). Note that, for $z \in \partial F \setminus \{0\}$ and $t > 0$, $\tilde{W}_{z,t}$ is the law of the conditioned Brownian motion which starts at z , enters at once into \mathring{F} and stays there up to time t . Then we have the following lemma.

Lemma 2. *Suppose that the random walk satisfies (1.1) and (1.3). Let the sequences z_n in F and t_n in $[0, 1]$, $n=1, 2, \dots$, be such that $z_n \rightarrow z \in F \setminus \{0\}$ and $t_n \rightarrow t \in [0, 1]$. Then $\tilde{W}_{z_n, t_n}^{(n)}$ converges weakly in $C[0, 1]$ to the law $\tilde{W}_{z, t}$.*

If $z \in F \setminus \partial F$, then $W_z(\sigma > t) > 0$ and the lemma follows directly from (2.1) in Lemma 1. Next consider the case $z \in \partial F \setminus \{0\}$. As was done in [11] and [12], we choose from the processes \hat{Z} and $\hat{Z}^{(n)}$ the meandering processes in \hat{F} \hat{Z}^\sim and $\hat{Z}^{(n)\sim}$ which give the laws $\tilde{W}_{z, t}$ and $\tilde{W}_{z_n, t_n}^{(n)}$ respectively. Here we adopt the terminology—meandering process—after Chung [2]. Then it follows from Theorem B and again from the winding property of the Brownian path we have

$$\hat{Z}^{(n)\sim}(\cdot, \hat{\omega}) \longrightarrow \hat{Z}^\sim(\cdot, \hat{\omega}) \text{ in } C[0, 1] \text{ almost surely.}$$

Hence we have the lemma. (Note that, in the latter half of the proof, we used the fact that the boundary ∂F is locally linear at z .)

Corollary to Lemma 2. *Assume (1.1) and (1.3) for the random walk. Then, for every z in $\partial F \setminus \{0\}$, we have*

$$(2.4) \quad n^{1/2} W_z^{(n)}(\sigma^{(n)} > 1) = n^{1/2} \dot{p}^{(n)}(1, z, F) \rightarrow C \text{ as } n \rightarrow \infty,$$

where C is a positive constant which depends on z and the law of Z_1 .

Proof of the corollary is given in a similar way to that of [11], corollary to Lemma 4.1, and we omit it.

For $\varepsilon > 0$ put $U_\varepsilon = \{x \in \mathbf{R}^2 : |x| < \varepsilon\}$ ε -neighbourhood of the origin. For $\varepsilon > 0$ and $J > 1$, $d_{\varepsilon, J}(n)$ denotes

$$\max\{P(|Z_1| > (1-1/J)\varepsilon n^{1/2})/P(|Z_1| > 2\varepsilon n^{1/2}/J; Z_1 \in F_j) : j=1, 2\}.$$

The following two estimates will be used in the next section.

For every $x \in U_\varepsilon^c \cap F$

$$(2.5) \quad \dot{p}^{(n)}(1, x, F) \leq f_\varepsilon n^{-1/2}, \quad n=1, 2, \dots,$$

where $U_\varepsilon^c = \mathbf{R}^2 \setminus U_\varepsilon$ and f_ε is a positive constant which may depend on ε .

For every $x \in U_{\varepsilon/J} \cap F$

$$(2.6) \quad \dot{p}^{(n)}(1/n, x, U_\varepsilon^c \cap F) / \dot{p}^{(n)}(1/n, x, (U_{\varepsilon/J})^c \cap F) \leq d_{\varepsilon, J}(n).$$

Proof of (2.5) and (2.6) may be clear if we note the condition that F is a cone (and also the formula (2.4) for (2.5)) besides the spatial homogeneity of the random walk. So we omit the detail.

3. Tightness of the $\{\tilde{W}^{(n)}\}_n$.

For $\varepsilon > 0$ and w in $C[0, 1]$ put

$$\rho_\varepsilon^{(n)}(w) = \min\{k/n : k=0, 1, \dots, n, |w(k/n)| \geq \varepsilon\}, \quad n=1, 2, \dots$$

($\min \emptyset = \infty$). The main result of the section is the following lemma.

Lemma 3. *Assume that the random walk satisfies (1.1), (1.3) and one of (1.4) and (1.5). Then, the family of probabilities $\tilde{W}^{(n)}$, $n=1, 2, \dots$, is tight in $C[0, 1]$.*

To prove Lemma 3 we will need the following lemma.

Lemma 4. *Assume that the random walk satisfies the conditions of Lemma 3. Then, for every $\varepsilon > 0$, there exists a $J > 1$ such that*

$$\lim_{n \rightarrow \infty} \tilde{W}^{(n)}(\rho_\varepsilon^{(n)} \leq 1; |w(\rho_\varepsilon^{(n)})| \geq \varepsilon) = 0.$$

Proof of Lemma 4. If (1.4) holds, then the lemma is clear, since

$$W_0^{(n)}(|w(j/n) - w((j-1)/n)| \leq Kn^{-1/2} \text{ for every } j \geq 1) = 1$$

for every $n \geq 1$. Now assume (1.5). Note that

$$\tilde{W}^{(n)}(\rho_\varepsilon^{(n)} \leq 1; |w(\rho_\varepsilon^{(n)})| \geq \varepsilon)$$

is less than

$$(3.1) \quad W^{(n)}(\rho_\varepsilon^{(n)} \leq 1; |w(\rho_\varepsilon^{(n)})| \geq \varepsilon; \sigma^{(n)} > 1) / W^{(n)}(\rho_\varepsilon^{(n)} \leq 1; \sigma^{(n)} > 1).$$

Let $1_w(L)$ be the indicator function on a set L , that is, $1_w(L) = 1$ or 0 according as $w \in L$ or $w \notin L$. The numerator of (3.1) is less than

$$\begin{aligned} & \sum_{k=1}^n \int 1_w(\min\{\rho_\varepsilon^{(n)}, \sigma^{(n)}\} > (k-1)/n) \dot{p}^{(n)}(1/n, w((k-1)/n), \\ & (U_{\varepsilon/J})^c \cap F) \{ \dot{p}^{(n)}(1/n, w((k-1)/n), U_\varepsilon^c \cap F) / \dot{p}^{(n)}(1/n, \\ & w((k-1)/n), (U_{\varepsilon/J})^c \cap F) \} W_0^{(n)}(dw). \end{aligned}$$

Here we assume the area of integral is $C[0, 1]$ when it is unspecified in it. So, by (2.6) the numerator of (3.1) is bounded by

$$d_{\varepsilon, J}(n) W_0^{(n)}(\rho_\varepsilon^{(n)} \leq 1; \rho_\varepsilon^{(n)} < \sigma^{(n)}).$$

The denominator of (3.1) is greater than

$$\sum_{k=1}^n \int 1_w(\rho_\varepsilon^{(n)} = k/n; \sigma^{(n)} > k/n) \dot{p}^{(n)}(1, w(k/n), F) W_0^{(n)}(dw),$$

which is not less than

$$f_{\varepsilon/J} n^{-1/2} W_0^{(n)}(\rho_{\varepsilon/J}^{(n)} \leq 1; \rho_{\varepsilon/J}^{(n)} < \sigma^{(n)})$$

by (2.5). By the two estimates given above,

$$\widetilde{W}^{(n)}(\rho_{\varepsilon/J}^{(n)} \leq 1; |w(\rho_{\varepsilon/J}^{(n)})| > \varepsilon) \leq (f_{\varepsilon/J})^{-1} n^{1/2} d_{\varepsilon, J}(n).$$

Since the right-hand side of the last inequality tends to zero as $n \rightarrow \infty$ by (1.5), the lemma follows.

Proof of Lemma 3. For w in $C[a, b]$, $0 \leq a \leq b < \infty$, introduce the modulus of continuity

$$\chi_w(\delta; a, b) = \sup\{|w(t) - w(s)| : a \leq s \leq t \leq b, t - s < \delta\}, \delta > 0.$$

Then the tightness of $\{\widetilde{W}^{(n)}\}_n$ follows if we have

$$(3.2) \quad \lim_{\delta \rightarrow 0+} \limsup_{n \rightarrow \infty} \widetilde{W}^{(n)}(\chi_w(\delta; 0, 1) > 5\varepsilon) = 0 \quad \text{for every } \varepsilon > 0$$

(Billingsley [1], Theorem 8.2). The following proof of (3.2) is a modification of [12], proof of Theorem 2. Take $J > 1$ as in Lemma 4. For w in the support of $\widetilde{W}^{(n)}$ with $\rho_{\varepsilon/J}^{(n)}(w) \leq 1$ and $|w(\rho_{\varepsilon/J}^{(n)})| < \varepsilon$, $\chi_w(\delta; 0, 1) > 5\varepsilon$ implies $\chi_w(\delta; \rho_{\varepsilon/J}^{(n)}, 1) > \varepsilon$, since

$$\begin{aligned} \chi_w(\delta; 0, 1) &\leq \chi_w(\delta; 0, \rho_{\varepsilon/J}^{(n)} - 1/n) + \chi_w(\delta; \rho_{\varepsilon/J}^{(n)}, 1) \\ &\quad + |w(\rho_{\varepsilon/J}^{(n)}) - w(\rho_{\varepsilon/J}^{(n)} - 1/n)|. \end{aligned}$$

Hence $\widetilde{W}^{(n)}(\chi_w(\delta; 0, 1) > 5\varepsilon)$ is bounded by

$$(3.3) \quad \begin{aligned} &\widetilde{W}^{(n)}(\rho_{\varepsilon/J}^{(n)} \leq 1; |w(\rho_{\varepsilon/J}^{(n)})| \geq \varepsilon) + \\ &\widetilde{W}^{(n)}(\rho_{\varepsilon/J}^{(n)} \leq 1; |w(\rho_{\varepsilon/J}^{(n)})| < \varepsilon; \chi_w(\delta; \rho_{\varepsilon/J}^{(n)}, 1) > \varepsilon). \end{aligned}$$

By Lemma 4 the first term in (3.3) tends to zero as $n \rightarrow \infty$. Let

$$p_{n, \delta} = \sup\{\widetilde{W}_{x, t}^{(n)}(\chi_w(\delta; 0, 1) > \varepsilon) : x \in F \cap \{\varepsilon/J \leq |z| \leq \varepsilon\}, 0 \leq t \leq 1\}.$$

By the strong Markov property of the random walk we get

$$\begin{aligned} \text{the second term in (3.3)} &= \dot{p}^{(n)}(1, 0, F)^{-1} \int 1_w(\rho_{\varepsilon/J}^{(n)} \leq 1; \rho_{\varepsilon/J}^{(n)} < \sigma^{(n)}; \\ &\varepsilon/J \leq |w(\rho_{\varepsilon/J}^{(n)})| \leq \varepsilon) \dot{p}^{(n)}(1 - \rho_{\varepsilon/J}^{(n)}, w(\rho_{\varepsilon/J}^{(n)}), F) \\ &\quad \times \widetilde{W}_{w(\rho_{\varepsilon/J}^{(n)}), 1 - \rho_{\varepsilon/J}^{(n)}}(\chi_w(\delta; 0, 1 - s) > \varepsilon)_{1, s = \rho_{\varepsilon/J}^{(n)}} W^{(n)}(dw) \leq p_{n, \delta}. \end{aligned}$$

Since $\lim_{\delta \rightarrow 0+} \limsup_{n \rightarrow \infty} p_{n, \delta} = 0$, by Lemma 2, we have the lemma.

4. Instantaneous entrance to the interior \mathring{F} .

In this section we show the following lemma.

Lemma 5. Assume that the random walk satisfies the condition of Theorem 1. Let \tilde{W} be a limit point of the sequence $\{\tilde{W}^{(n)}\}$ in $C[0, 1]$. Then, for every $t \in (0, 1]$, we have

$$(4.1) \quad \tilde{W}(w(t) \in \partial F) = 0.$$

Put

$$z_0 = (x_0, y_0) = (\cos(\beta + \alpha/2), \sin(\beta + \alpha/2)) \text{ and } F' = F + z_0.$$

Before we prove Lemma 5, we show the following lemma.

Lemma 6. Let $0 < t \leq 1$ and $t_n = k_n/n$, $0 < k_n \leq n$ with $t_n \rightarrow t$ as $n \rightarrow \infty$. Then, under the assumption to Theorem 1, we have

$$(4.2) \quad \liminf_{n \rightarrow \infty} \tilde{W}_{0, t_n}^{(n)}(w(t_n) \in F') > 0.$$

Proof. We divide the proof into two steps.

First Step. We show the following inequality: *There exists a positive constant C_0 for which we have*

$$(4.3) \quad \tilde{W}^{(n)}(w(1) \in F') \geq C_0 \tilde{W}^{(n)}(w = (w^1, w^2) : |w^1(1)| \leq 1)$$

for all large n .

Proof. Put $\mathcal{F}^x = \sigma\{X_1, X_2, \dots\}$ and $P_{\mathcal{F}^x}^*(*) = P(* | \mathcal{F}^x)$. Set $L(x) = \inf\{y : (x, y) \in F\}$, $x \in \mathbf{R}$ ($\inf \mathbf{R} = -\infty$).

By (1.2) $(x, y) \in F$ is equivalent to $y \geq L(x)$. Then we have

$$(4.4) \quad \begin{aligned} \dot{p}^{(n)}(1, \mathbf{0}, F') &\geq \int_0^1 1_{\omega}(|X^{(n)}(1)| \leq 1) P_{\mathcal{F}^x}(\sigma^{(n)}(Z^{(n)}) > 1) \\ &\times P_{\mathcal{F}^x}(Y^{(n)}(1) \geq L(X^{(n)}(1) - x_0) + y_0 | Y^{(n)}(k/n) \geq L(X^{(n)}(k/n), \\ &0 \leq k \leq n) P(d\omega). \end{aligned}$$

It follows from the independence of increments of the random walk the following holds P almost surely :

$$P_{\mathcal{F}^x}(Y_j - Y_{j-1} \leq y_j, 1 \leq j \leq m) = \prod_{j=1}^m P_{\mathcal{F}^x}(Y_j - Y_{j-1} \leq y_j)$$

for every y_j in \mathbf{R} , $1 \leq j \leq m$, $m \geq 1$. In other words we may say that $\{Y_0, Y_1, Y_2, \dots\}$ is a random walk with independent (but not stationary in general) increments with respect to the conditional probability $P_{\mathcal{F}^x} P$ almost surely. So, as in [5], Lemma 7 (a), we have, for the right-hand side of (4.4)

$$\begin{aligned} &\text{the third factor in the integrand} \\ &\geq P_{\mathcal{F}^x}(Y^{(n)}(1) \geq L(X^{(n)}(1) - x_0) + y_0) \end{aligned}$$

P almost surely. Put

$$a = \max\{0, L(x - x_0) + y_0 : |x| \leq 1\}.$$

Then, by the second assumption in (1.6), the right-hand side in the last inequality is not less than

$$J_n = P_{\mathcal{F}^X}(\nu_{\mathcal{F}^X}(Y_n)^{-1}Y_n \geq a\nu^{-1}), \quad n \geq 1,$$

where

$$\nu_{\mathcal{F}^X}(Y_n) = \{E(Y_n^2 | \mathcal{F}^X)\}^{1/2} (\geq \nu n^{1/2}).$$

By (1.6) we can apply Essen's inequality (Petrov [10], p. 111 Theorem 3) on J_n to get

$$J_n \geq (2\pi)^{-1/2} \int_{a/\nu}^{\infty} \exp(-y^2/2) dy - A\gamma\nu^{-3}n^{-1/2}, \quad n \geq 1,$$

where A is an absolute positive constant. By a series of estimates given above we have a positive constant C_0 such that

$$\dot{p}^{(n)}(1, \mathbf{0}, F') \geq C_0 \dot{p}^{(n)}(1, \mathbf{0}, [-1, 1] \times \mathbf{R})$$

for all large n . Dividing both sides of the inequality by $\dot{p}^{(n)}(1, \mathbf{0}, F)$, we have (4.3).

Here we note that, if the component random walks are mutually independent, J_n may be given by

$$J_n = P(n^{-1/2}Y_n \geq a), \quad n \geq 1.$$

Then (4.3) follows from the central limit theorem in this case (see Remark (ii) in 1).

Second Step. We show (4.2) when $t_n=1$. Suppose, on the contrary, that it would not hold, that is,

$$(4.5) \quad \liminf_{n \rightarrow \infty} \tilde{W}^{(n)}(w(1) \in F') = 0.$$

By Lemma 3 there exists a subsequence of $\{\tilde{W}^{(n)}\}_n$ which converges weakly to a law \tilde{W}' in $C[0, 1]$. It follows from (4.3) and (4.5)

$$(4.6) \quad \tilde{W}'(w = (w^1, w^2) : |w^1(1)| < 1) = 0.$$

Then, together with $\tilde{W}'(C[0, 1]) = 1$, we may choose $0 < s < 1$ and $0 < p < q < \infty$ such that

$$\tilde{W}'(p < |w(s)| < q) = \tilde{W}'(p \leq |w(s)| \leq q) > 0.$$

Take s_n from $\{k/n : 0 \leq k \leq n\}$ such that $s_n \rightarrow s$ as $n \rightarrow \infty$. By the Markov property of the random walk

$$\begin{aligned}
 & \tilde{W}^{(n)}(w=(w^1, w^2): |w^1(1)| < 1/2) \geq \dot{p}^{(n)}(1, \mathbf{0}, F)^{-1} \\
 (4.7) \quad & \times \int \mathbf{1}_w(p \leq |w(s_n)| \leq q; \sigma^{(n)} > s_n) \dot{p}^{(n)}(1-s_n, w(s_n), F) \\
 & \times \tilde{W}_{w(s_n), 1-s_n}^{(n)}(w=(w^1, w^2): |w^1(1-s_n)| < 1/2) W_0^{(n)}(dw).
 \end{aligned}$$

Note that $\tilde{W}_{z,t}$, $z \in \partial F \setminus \{0\}$ and $0 < t \leq 1$, is the law of the Brownian meandering process (see Lemma 2). Then we get

$$\tilde{W}_{z,t}(w=(w^1, w^2): |w^1(t)| < 1/3) > 0$$

for $z \in F \setminus \{0\}$ and $0 < t \leq 1$. Hence, by Lemma 2 we have

$$\begin{aligned}
 (4.8) \quad & \liminf_{n \rightarrow \infty} \inf \{ \tilde{W}_{z, 1-s_n}(w=(w^1, w^2): |w^1(1-s_n)| < 1/2) : z \in F, \\
 & p \leq |z| \leq q \} = 2C_1 > 0.
 \end{aligned}$$

By (4.7) and (4.8)

$$\tilde{W}^{(n)}(w=(w^1, w^2): |w^1(1)| < 1/2) \geq C_1 \tilde{W}^{(n)}(p \leq |w(s_n)| \leq q)$$

for all large n . Take the limit on both sides of the inequality through the subsequence to have

$$\tilde{W}'(w=(w^1, w^2): |w^1(1)| < 1) \geq C_1 \tilde{W}'(p \leq |w(s)| \leq q) > 0,$$

which contradicts with (4.6). Hence we have (4.2) when $t_n = 1$.

For the general t_n we rescale the space and time of the random walk to get

$$\tilde{W}_{0, t_n}^{(n)}(w(t_n) \in F') = \tilde{W}^{(k_n)}(w(1) \in t_n^{-1/2} F')$$

($k_n = nt_n$). Then (4.2) is reduced to the one just given above. This completes the proof of the lemma.

Proof of Lemma 5. For $\varepsilon > 0$ put

$$\varepsilon F' = \{\varepsilon z : z \in F'\} \quad \text{and} \quad \Delta_\varepsilon = F \setminus (\varepsilon F').$$

First we prove (4.1) when $t=1$, modifying the proof of [5], Lemma 9. Set $s_n = [n/2]/n$, where $[a]$ is the integral part of a . Since

$$\begin{aligned}
 & \tilde{W}^{(n)}(w(1) \in \Delta_\varepsilon) \leq K_{\varepsilon, n} = \int_F \tilde{W}_{0, n}(w(s) \in dx) \dot{p}^{(n)}(1-s_n, \mathbf{x}, \Delta_\varepsilon) \\
 & \times \left\{ \int_{F'} \tilde{W}_{0, s_n}^{(n)}(w(s_n) \in dx) \dot{p}^{(n)}(1-s_n, \mathbf{x}, F) \right\}^{-1},
 \end{aligned}$$

it is enough to show

$$(4.9) \quad \lim_{\varepsilon \rightarrow 0+} \limsup_{n \rightarrow \infty} K_{\varepsilon, n} = 0$$

for the proof. Note that, as $n \rightarrow \infty$,

$$\begin{aligned} & \dot{p}^{(n)}(1-s_n, \mathbf{x}, \Delta_\varepsilon) \longrightarrow \dot{p}(1/2, \mathbf{x}, \Delta_\varepsilon) \\ \text{and} & \dot{p}^{(n)}(1-s_n, \mathbf{x}, F) \longrightarrow \dot{p}(1/2, \mathbf{x}, F) \end{aligned}$$

uniformly in \mathbf{x} by Lemma 1. Then, together with

$$2C_2 = \inf\{\dot{p}(1/2, \mathbf{x}, F) : \mathbf{x} \in F'\} > 0,$$

we get

$$\begin{aligned} \limsup_{n \rightarrow \infty} K_{\varepsilon, n} & \leq \sup\{\dot{p}(1/2, \mathbf{x}, \Delta_\varepsilon) : \mathbf{x} \in F\} \\ & \times \{C_2 \liminf_{n \rightarrow \infty} \tilde{W}_{\theta, n}^{(n)}(w(s_n) \in F')\}^{-1}. \end{aligned}$$

Hence, (4.9) follows from Lemma 6 and from

$$\lim_{\varepsilon \rightarrow 0+} \sup\{\dot{p}(1/2, \mathbf{x}, \Delta_\varepsilon) : \mathbf{x} \in F\} = 0.$$

Next we consider the general case. Let \tilde{W}' be a limit point of $\{\tilde{W}^{(n)}\}_n$, and take its subsequence $\{\tilde{W}^{(n')}\}$ which satisfies $\tilde{W}^{(n')} \xrightarrow{w} \tilde{W}'$ and $\tilde{W}^{(n't_n')} \xrightarrow{w} \tilde{W}''$ in $C[0, 1]$. Then, for every positive bounded continuous function f on \mathbf{R}^2 , we have

$$I_{n'} = \int f(w(t_{n'})) \tilde{W}^{(n')}(dw) \longrightarrow \int f(w(t)) \tilde{W}'(dw)$$

and

$$\begin{aligned} I_{n'} \dot{p}^{(n')}(1, \mathbf{0}, F) / \dot{p}^{(n')}(t_{n'}, \mathbf{0}, F) & = \int_F f((t_{n'})^{1/2} \mathbf{z}) \\ & \times \dot{p}^{(n')}(1-t_{n'}, (t_{n'})^{1/2} \mathbf{z}, F) \tilde{W}^{(n't_n')}(w(1) \in d\mathbf{z}) \\ & \longrightarrow \int_F f(t^{1/2} \mathbf{z}) \dot{p}(1-t, t^{1/2} \mathbf{z}, F) \tilde{W}''(w(1) \in d\mathbf{z}). \end{aligned}$$

Hence, noting $\tilde{W}''(w(1) \in \hat{F}) = 1$ just proved in the first part, we get

$$(4.10) \quad \dot{p}^{(n')}(1, \mathbf{0}, F) / \dot{p}^{(n')}(t_{n'}, \mathbf{0}, F) \longrightarrow H \quad \text{as } n' \longrightarrow \infty,$$

where $0 < H \leq 1$. Thus we have the identity

$$(4.11) \quad H \int f(w(t)) \tilde{W}'(dw) = \int f(t^{1/2} w(1)) \dot{p}(1-t, t^{1/2} w(1), F) \tilde{W}''(dw)$$

for every positive bounded continuous, and so, for every bounded measurable function f on \mathbf{R}^2 . Taking $f(\mathbf{z}) = 1_{\mathbf{z} \in \partial F}$ in (4.11), we conclude $\tilde{W}'(w(t) \in \partial F) = 0$. This completes the proof of the lemma.

5. Proof of theorems.

Proof of Theorem 1. Since $\{\tilde{W}^{(n)}\}_n$ is tight by Lemma 3, the proof is complete if we have the convergence of the finite dimensional distributions. For simplicity we consider the two-dimensional distribution (the generalization

will be easy). Let f and g be bounded continuous functions on R^2 , and let $0 < \lambda < s < t \leq 1$. Take s_n and t_n from $\{k/n : 0 \leq k \leq n\}$ which satisfy $s_n \rightarrow s, t_n \rightarrow t, s_n \geq \lambda$ and $t_n \geq \lambda$. Set $\lambda_n = \max\{k/n : k/n \leq \lambda\}$. Put

$$I_\varepsilon^{(n)} = \int 1_{w(w(\lambda_n) \in \Delta_\varepsilon)} f(w(s_n))g(w(t_n))\tilde{W}^{(n)}(dw)$$

and

$$J_\varepsilon^{(n)} = \int 1_{w(w(\lambda_n) \in F \setminus \Delta_\varepsilon)} f(w(s_n))g(w(t_n))\tilde{W}^{(n)}(dw).$$

Then, by Lemma 5 $\lim_{\varepsilon \rightarrow 0+} \limsup_{n \rightarrow \infty} I_\varepsilon^{(n)} = 0$. Set

$$\phi_n(x) = \int f(w(s_n - \lambda_n))g(w(t_n - \lambda_n))\tilde{W}_{x, 1-\lambda_n}^{(n)}(dw).$$

By the Markov property

$$J_\varepsilon^{(n)} = \int 1_{w(w(\lambda_n) \in F \setminus \Delta_\varepsilon)} \phi_n(w(\lambda_n))\tilde{W}^{(n)}(dw).$$

Let

$$\phi_\lambda(x) = \int f(w(s-\lambda))g(w(t-\lambda))\tilde{W}_{x, 1-\lambda}(dw), \quad x \in \mathring{F}.$$

Note that $\phi_\lambda(\cdot)$ is a continuous function on \mathring{F} . Moreover, by Lemma 2 $\phi_n(x_n) \rightarrow \phi_\lambda(x)$ for every $x_n \in F$ and $x \in \mathring{F}$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$. Let $\tilde{W}^{(n')}$ be a weak convergent subsequence (\tilde{W}' denotes the limit law). We may assume $\tilde{W}(w(\lambda) \in \partial(F \setminus \Delta_\varepsilon)) = 0$. Then, by the continuous mapping theorem ([1] Theorem 5.5)

$$J_\varepsilon^{(n')} \rightarrow J_{\varepsilon, \lambda} = \int 1_{w(w(\lambda) \in F \setminus \Delta_\varepsilon)} \phi_\lambda(w(\lambda))\tilde{W}'(dw)$$

as $n' \rightarrow \infty$. Let $\varepsilon \rightarrow 0+$ to get

$$J_{\varepsilon, \lambda} \rightarrow I_\lambda = \int 1_{w(w(\lambda) \in \mathring{F})} \phi_\lambda(w(\lambda))\tilde{W}'(dw).$$

By the scaling property of the Brownian motion,

$$\begin{aligned} \phi_\lambda(x) &= \int f((1-\lambda)^{1/2}w((s-\lambda)/(1-\lambda))) \\ &\times g((1-\lambda)^{1/2}w((t-\lambda)/(1-\lambda)))\tilde{W}_{(1-\lambda)^{-1/2}x, 1}(dw). \end{aligned}$$

So, combining $\tilde{W}'(w : w(\lambda) \rightarrow 0 \text{ as } \lambda \rightarrow 0+) = 1$ with [12] Theorem 2, we have

$$I_\lambda \rightarrow \int f(w(s))g(w(t))\tilde{W}(dw) \text{ as } \lambda \rightarrow 0+.$$

Hence we have

$$\int f(w(s_n))g(w(t_n))\tilde{W}^{(n')}(dw) (= I_\varepsilon^{(n')} + J_\varepsilon^{(n')}) \rightarrow$$

$$\int f(w(s))g(w(t))\widehat{W}(dw) \text{ as } n' \rightarrow \infty,$$

which implies $\widehat{W}' = \widehat{W}$. This completes the proof.

Proof of Theorem 2. Let us determine the constant H in (4.10). Since $\widehat{W}' = \widehat{W}'' = \widehat{W}$ in (4.11) by Theorem 1, we have

$$H = \int_F \dot{p}((1-t)/t, \mathbf{x}, F) \widehat{W}(w(1) \in d\mathbf{x}).$$

To calculate the integral we apply the formula (3.2) and Lemma 4 in [12]. (Note that $\widehat{W}(w(1) \in d\mathbf{x})$ and $\dot{p}((1-t)/t, \mathbf{x}, F)$ are equal respectively to $\rho(1, d\mathbf{x})$ and

$$\int_F \dot{Q}(\log t^{-1}, \mathbf{x}, \mathbf{z}) \exp(-|\mathbf{z}|^2/2) d\mathbf{z}$$

in [12]. Refer also the evaluation in [5], Lemma 3 (b).) Then we get $H = t^{\pi/2\alpha}$. This, together with Theorem 1, implies

$$P(Z_k \in F, 0 \leq k \leq n) / P(Z_k \in F, 0 \leq k \leq nt) \rightarrow t^{\pi/2\alpha} \text{ as } n \rightarrow \infty$$

for every $0 < t \leq 1$. Hence (1.7) follows (see Feller [4], 8.8).

Appendix.

Here we prove (2.2) when $S = F$, that is,

$$(A.1) \quad \lim_{n \rightarrow \infty} \sup\{|\dot{p}^{(n)}(t_n, \mathbf{z}, F) - \dot{p}(t, \mathbf{z}, F)| : \mathbf{z} \in F\} = 0.$$

(Proof for another S 's will be given in a similar way.) For $c > 0$ set $F_c = cF'$. By Theorem B and by the spatial homogeneity of the random walk and the Brownian motion, we have the following: For every positive ε we may choose a positive M such that

$$\dot{p}^{(n)}(t_n, \mathbf{z}, F) > 1 - \varepsilon/2 \quad \text{and} \quad \dot{p}(t, \mathbf{z}, F) > 1 - \varepsilon/2$$

for every $\mathbf{z} \in F_M$ and $n \geq 1$. This implies

$$(A.2) \quad \sup\{|\dot{p}^{(n)}(t_n, \mathbf{z}, F) - \dot{p}(t, \mathbf{z}, F)| : \mathbf{z} \in F_M\} \leq \varepsilon \text{ for all } n \geq 1.$$

Hence we have (A.1), if, in addition to (A.2), we show

$$(A.3) \quad \lim_{n \rightarrow \infty} \sup\{|\dot{p}^{(n)}(t_n, \mathbf{z}, F) - \dot{p}(t, \mathbf{z}, F)| : \mathbf{z} \in F \setminus F_M\} = 0.$$

Suppose, on the contrary, that (A.3) would not hold, that is, we could take a sequence $\{\mathbf{z}_{n'}\}_{n'}$ in $F \setminus F_M$ such that

$$(A.4) \quad \lim_{n' \rightarrow \infty} |\dot{p}^{(n')}(\mathbf{z}_{n'}, F) - \dot{p}(t, \mathbf{z}_{n'}, F)| = \lambda > 0.$$

Case 1. Assume $\{z_{n'}\}_{n'}$ is bounded, that is, $\sup_{n'} |z_{n'}| < \infty$. Then we may choose a subsequence $\{z_{n''}\}_{n''}$ of $\{z_{n'}\}_{n'}$ such that $z_{n''} \rightarrow z''$ as $n'' \rightarrow \infty$. So, we conclude as $n'' \rightarrow \infty$

$$\dot{p}^{(n'')}(t_{n'}, z_{n'}, F) \longrightarrow \dot{p}(t, z'', F)$$

by modifying the proof of (2.1), and

$$\dot{p}(t, z_{n'}, F) \longrightarrow \dot{p}(t, z'', F),$$

which, together with (A.4), lead to a contradiction.

Case 2. Assume $\{z_{n'}\}_{n'}$ is unbounded. In this case we may choose a subsequence $\{z_{n''}\}_{n''}$ of $\{z_{n'}\}_{n'}$ which satisfies

$$|z_{n''}| \longrightarrow \infty \text{ and } \min\{|z_{n''} - z| : z \in \partial F\} \longrightarrow \mu \geq 0$$

as $n'' \rightarrow \infty$. Then, again by Theorem B and by the spatial homogeneity of the random walk and the Brownian motion, we conclude that both $\dot{p}^{(n'')}(t_{n'}, z_{n'}, F)$ and $\dot{p}(t, z_{n'}, F)$ converge to $W_{(0, \mu)}(w(s) \in F_{0, \pi}, 0 \leq s \leq t)$ as $n'' \rightarrow \infty$, which together with (A.4), again leads to a contradiction.

By the two observations given above we have (A.3), and hence, (A.1). This completes the proof.

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