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A MINIMAX OPTIMAL CONTROL PROBLEM FOR EVOLUTION INCLUSIONS

By

NIKOLAOS S. PAPAGEORGIOU*, **

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1. Introduction.

In this paper we study a minimax optimal control problem of a system governed by a nonlinear evolution inclusion. Such problems are characteristic of differential games (see Krasovski-Subbotin [11]). Under mild hypotheses on the data we establish the existence of admissible trajectories and then we prove the existence of optimal controls. We also investigate their dependence on the parameters of the problem and finally for a semilinear form of the problem, we obtain a necessary and sufficient condition for optimality. An example of a parabolic distributed parameter system is also worked out in detail.

In the past minimax control problems were studied in the context of finite dimensional systems. We refer to the book of Krasovski-Subbotin [11] and the references therein. Our formulation is more general and incorporates a large class of infinite dimensional control systems (distributed parameter systems). An additional level of generality is added by allowing the control vector field to be in general multivalued. Our existence results in this paper include as a special case the work on nonlinear optimal control of parabolic systems of Ahmed [1]. Furthermore our sensitivity analysis in section 4 appears to be the first of its kind for this general class of systems and extends to infinite dimensional systems the finite dimensional work of Langen [12] and shows how the various convergence concepts and results developed by Attouch [2] in his book, can be used to investigate the response of nonlinear systems to changes in their structure.

2. Preliminaries.

Let (Ω, Σ) be a measurable space and X a separable Banach space. Throughout this paper we will be using the following notations:

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** Presently on leave at the Florida Institute of Technology, Department of Applied Mathematics, 150 West University Blvd, Melbourne, Florida 32901-6988, U. S. A.

$$P_{f(c)}(X)=\{A\subseteq X\colon \text{nonempty, closed, (convex)}\}$$

and

$$P_{(w)k(c)}(X)=\{A\subseteq X\colon \text{nonempty, } (w\text{-})\text{compact, (convex)}\}.$$

A multifunction $F\colon \Omega\to P_f(X)$ is said to be measurable, if for every $z\in X$ $\omega\to d(z, F(\omega))=\inf\{\|z-x\|\colon x\in F(\omega)\}$ is measurable. Also a multifunction $G\colon \Omega\to 2^X\setminus\{\phi\}$ is said to be graph measurable, if $\mathrm{Gr}G=\{(\omega, x)\in\Omega\times X\colon x\in G(\omega)\}\in\Sigma\times B(X)$, with $B(X)$ being the Borel σ-field of X. For closed valued multifunctions, measurability implies graph measurability. The converse is true if there exists a complete, σ-finite measure $\mu(\cdot)$ on (Ω, Σ). For details we refer to the survey paper of Wagner [18].

Now let (Ω, Σ, μ) be a finite measure space and $F\colon \Omega\to 2^X\setminus\{\phi\}$ a multifunction. By $S_F^p(1\leqq p\leqq\infty)$ we will denote the set of all L^p-selectors of $F(\cdot)$, i.e.

$$S_F^p=\{f(\cdot)\in L^p(X)\colon f(\omega)\in F(\omega)\quad \mu\text{-a.e.}\}$$

(here $L^p(X)$ is the Lebesgue-Bochner space of all measurable functions $f\colon \Omega\to X$ s.t. $\int_\Omega\|f(\omega)\|d\mu(\omega)<\infty$). This set may be empty. It is easy to see using the Lusin-Yankov-Aumann selection theorem (see Wagner [18]), that S_F^p is nonempty if and only if $F(\cdot)$ is graph measurable and $\omega\to\inf\{\|x\|\colon x\in F(\omega)\}\in L_+^1$. So if $F(\cdot)$ is graph measurable and $\omega\to\sup\{\|x\|\colon x\in F(\omega)\}$ belongs to L_+^p (in which case we say that $F(\cdot)$ is L^p-integrably bounded), then $S_F^p\neq\phi$ (note that $L_+^p=\{f\in L^p\colon f(\cdot) \text{ is } \boldsymbol{R}_+\text{-valued}\}$).

On $P_f(X)$ we can define a (generalized) metric by setting

$$h(A, B)=\max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\}.$$

This is known in the literature as the Hausdorff metric on $P_f(X)$. Recall that since X is complete, so is $(P_f(X), h)$.

If $\{K_n\}_{n\geq 1}$ are nonempty subsets of X, we define

$$s\text{-}\underline{\lim}K_p=\{x\in X\colon x=s\text{-}\lim x_n,\ x_n\in K_n,\ n\geqq 1\}$$

(here s-indicates the strong topology on X) and

$$w\text{-}\overline{\lim}K_n=\{x\in X\colon x=w\text{-}\lim x_{n_k},\ x_{n_k}\in K_{n_k},\ n_1<n_2<\cdots<n_k<\cdots\}$$

(here w- indicates the weak topology on X). It is clear from the above definitions that we always have $s\text{-}\underline{\lim}K_n\subseteq w\text{-}\overline{\lim}K_n$. We will say that the K_n's converge to K in the Kuratowski-Mosco sense (denoted by $K_n\xrightarrow{K-M}K$) if and only if $s\text{-}\underline{\lim}K_n=K=w\text{-}\underline{\lim}K_n$. Note that in the definition of the Kuratowski-Mosco convergence we have mixed the topologies. Namely in the definition of the limit inferior we considered the strong topology and in the definition of the limit superior the sequential weak topology. This mixing makes the

Kuratowski-Mosco convergence a powerful tool in variational problems (see the book of Attouch [2]). When for both $\underline{\lim}$ and $\overline{\lim}$ we consider the strong topology, we get the well known Kuratowski convergence of sets denoted by $K_n \xrightarrow{sK} K$.

Let Y, Z be two Hausdorff topological spaces. A multifunction $G: Y \to 2^Z \setminus \{\phi\}$ is said to be upper semicontinuous (u.s.c.) (resp. lower semicontinuous (l.s.c.)), if for all $U \subseteq Z$ open $G^+(U) = \{y \in Y : G(y) \subseteq U\}$ (resp. $G^-(U) = \{y \in Y : G(y) \cap U \neq \phi\}$) is open in Y. If Z is a metric space, then $G: Y \to P_f(Z)$ is said to be Hausdorff continuous (h-continuous), if it is continuous from Y into $(P_f(Z), h)$.

Finally let V be a Banach space and $A: V \to V^*$ an operator. We say that $A(\cdot)$ is monotone if and only if $\langle Ax - Ay, x - y \rangle \geqq 0$ for all $x, y \in V$. Here by $\langle \cdot, \cdot \rangle$ we denote the duality brackets of the pair (V, V^*). We will say that $A(\cdot)$ is hemicontinuous if and only if the real functions $\lambda \to \langle A(x + \lambda y), z \rangle$ is continuous on $[0, 1]$ for all $x, y, z \in V$. If V is a finite dimensional Banach space, then a monotone hemicontinuous operator $A(\cdot)$ is continuous. If V is reflexive, then $A(\cdot)$ is maximal monotone (i.e. its graph is not properly contained in the graph of another monotone operator), has property (M) (i.e. if $x_n \xrightarrow{w} x$ in V, $A(x_n) \xrightarrow{w} r$ and $\overline{\lim} \langle A(x_n), x_n \rangle \leqq \langle r, x \rangle$, then $r = A(x)$) and is demicontinuous (i.e. if $x_n \xrightarrow{s} x$ in V, then $A(x_n) \xrightarrow{w} A(x)$ in V^*). Here $\xrightarrow{s}$ denotes convergence in the strong topology while $\xrightarrow{w}$ denotes convergence in the weak topology. For further details on those concepts we refer to Barbu [6]).

3. Existence of optimal controls.

The mathematical setting is the following. Let $T = [0, b]$ and H a separable Hilbert space. Also X is a dense linear subspace of H carrying the structure of a separable reflexive Banach space, which embeds in H continuously. Identifying H with its dual (pivot space), we have that $X \hookrightarrow H \hookrightarrow X^*$, with all embeddings continuous and dense. We will also assume that they are compact. To have a concrete example in mind consider the triple $H = L^2(0, 1)$, $X = H_0^1(0, 1)$ and $X^* = H_0^1(0, 1)^* = H^{-1}(0, 1)$. Such a triple of spaces is usually known in the literature as "Gelfand triple". By $(\cdot, \cdot)$ we will denote the inner product of the Hilbert space H and by $\langle \cdot, \cdot \rangle$ the duality brackets for the dual pair (X, X^*). The two are compatible in the sense that $\langle \cdot, \cdot \rangle|_{X \times H} = (\cdot, \cdot)$. Also by $\|\cdot\|$ (resp. $|\cdot|$, $\|\cdot\|_*$), we will denote the norm of X (resp. H, X^*). We will model the control space by another separable Banach space Y.

We will be studying a minimax optimal control problem, governed by a nonlinear evolution inclusion defined on the Gelfand triple (X, H, X^*). More specifically the problem under consideration is the following. Our system is governed by the following evolution inclusion

$$\left\{\begin{array}{ll} \dot{x}(t)+A(t, x(t))\in F(t, x(t), u(t)) \text{ a.e.}, & \\ x(0)=x_0, \quad u\in S_U^1. & \end{array}\right\}$$

Also we are given an integral cost functional $J(x, u)=\int_0^b L(t, x(t), u(t))dt$.

Because of the generality of our formulation, given an admissible control $u(\cdot)$; i.e. $u\in S_U^1$ (see section 2), the set $P(u)$ of trajectories of the evolution inclusion generated by this control, if it is nonempty, in general it will not be a singleton. Then our minimax optimization problem is the following:

$$\inf_{u\in S_U^1} \sup_{x\in P(u)} J(x, u). \quad (*)$$

Of course when for every $u\in S_U^1$, $P(u)$ is a singleton, we get a standard optimal control problem.

We will start by studying the dynamics of our system and establishing the existence of trajectories for a given admissible control $u(\cdot)\in S_U^1$.

For this we will need the following hypotheses concerning the data of the evolution inclusion describing the dynamics of the system.

$H(A)$: $A: T\times X\to X^*$ is an operator s.t.

(1) $t\to A(t, x)$ is measurable,

(2) $x\to A(t, x)$ is hemicontinuous, monotone,

(3) $\|A(t, x)\|_*\leqq a(t)+b\|x\|$ a.e., with $a(\cdot)\in L_+^2$, $b>0$,

(4) $\langle A(t, x), x\rangle\geqq c\|x\|^2$ a.e., $c>0$.

$H(F)$: $F: T\times H\times Y\to P_{fc}(H)$ is a multifunction s.t.

(1) $(t, x, u)\to F(t, x, u)$ is graph measurable,

(2) $x\to F(t, x, u)$ is u.s.c. from H into H_w, where H_w denotes the Hilbert space H with the weak topology,

(3) $|F(t, x, u)|\leqq a_1(t)+b_1(t)(|x|+\|u\|)$ a.e. with $a_1(\cdot), b_1(\cdot)\in L_+^1$.

$H(U)$: $U: T\to P_{fc}(Y)$ is a measurable multifunction s.t. $U(t)\subseteq W$ a.e. with $W\in P_{wkc}(Y)$.

For a given admissible control $u\in S_U^1$, let $P(u)$ denote the set of trajectories of (*) generated by it. In the next theorem we show that for every $u\in S_U^1$, $P(u)\neq\phi$.

Theorem 3.1. *If hypotheses $H(A)$, $H(F)$ and $H(U)$ hold, then for any $u\in S_U^1$, $P(u)\neq\phi$, $P(u)\subseteq C(T, H)$ and $P(u)$ is compact in $L^2(H)$.*

Proof. Let $W(T)=\{x(\cdot)\in L^2(X): \dot{x}\in L^2(X^*)\}$. It is well known that this is a separable reflexive Banach space endowed with the norm $\|x\|_{W(T)}=(\|x\|^2_{L^2(X^*)})^{1/2}$ and $W(T)\hookrightarrow C(T, H)$ continuously (i.e. every element in $W(T)$, after possible modification on a Lebesgue null set, is equal to an element in $C(T, H)$). In addition $W(T) \hookrightarrow L^2(H)$ compactly (see Lions [13], p. 58). Also from Barbu [6] (Theorem 4.2, p. 167, see also Lions [13]), we know that $P(u) \subseteq W(T)$.

First we will determine some a priori bounds for the elements in $P(u)$. So let $x(\cdot) \in P(u)$. We have:

$$\dot{x}(t) + A(t, x(t)) = f(t) \text{ a.e. } f \in S^2_{F(\cdot, x(\cdot), u(\cdot))}$$

$$\Longrightarrow \langle \dot{x}(t), x(t) \rangle + \langle A(t, x(t)), x(t) \rangle = \langle f(t), x(t) \rangle \text{ a.e.}$$

$$\Longrightarrow \frac{1}{2} \frac{d}{dt} |x(t)|^2 + c\|x(t)\|^2 \leqq \langle f(t), x(t) \rangle \text{ a.e..}$$

Integrating the above inequality, we get:

$$|x(t)|^2 + 2c\int_0^t \|x(s)\|^2 ds \leqq |x_0|^2 + 2\left(\int_0^t \|f(s)\|_*^2 ds\right)^{1/2} \left(\int_0^t \|x(s)\|^2 ds\right)^{1/2}. \tag{1}$$

From Cauchy's inequality with $\varepsilon > 0$, we have:

$$2\left(\int_0^t \|f(s)\|_*^2 ds\right)^{1/2} \left(\int_0^t \|x(s)\|^2 ds\right)^{1/2} \leqq \frac{1}{\varepsilon}\int_0^t \|f(s)\|_*^2 ds + \varepsilon\int_0^t \|x(s)\|^2 ds. \tag{2}$$

Choosing $\varepsilon = 2c$ and substituting (2) into (1), we get

$$|x(t)|^2 + 2c\int_0^t \|x(s)\|^2 ds \leqq |x_0|^2 + \frac{1}{2c}\int_0^t \|f(s)\|_*^2 ds + 2c\int_0^t \|x(s)\|^2 ds$$

$$\Longrightarrow |x(t)|^2 \leqq |x_0|^2 + \frac{1}{2c}\int_0^t \|f(s)\|_*^2 ds \leqq |x_0|^2 + \frac{1}{2c}\int_0^t |f(s)|^2 ds$$

$$\Longrightarrow |x(t)|^2 \leqq |x_0|^2 + \frac{1}{2c}\int_0^t (2a_1(s)^2 + 4b_1(s)^2 |x(s)|^2 + 4b_1(s)^2 |W|^2) ds,$$

where $|W| = \sup\{\|u\| : u \in W\}$. Invoking Gronwall's inequality, we get

$$|x(t)| \leqq M, \quad t \in T \quad \text{for some} \quad M > 0.$$

Using this bound, we have

$$2c\int_0^t \|x(s)\|^2 ds \leqq |x_0|^2 + 2\int_0^t |f(s)| \cdot |x(s)| ds$$

$$\leqq |x_0|^2 + M\int_0^t (a_1(s) + b(s)(M + |W|)) ds$$

$$\Longrightarrow \|x\|^2_{L^2(X)} \leqq \frac{1}{2c}[|x_0|^2 + M\|a_1\|_1 + (M + |W|)M\|b_1\|_1] = M_1^2$$

$$\Longrightarrow \|x\|_{L^2(X)} \leqq M_1.$$

Next let $p(\cdot) \in L^2(X) = L^2(X^*)^*$. We have:

$$\int_0^t \langle \dot{x}(s), p(s)\rangle ds + \int_0^t \langle A(s, x(s)), p(s)\rangle ds = \int_0^t \langle f(s), p(s)\rangle ds$$

$$\Longrightarrow \int_0^t \langle \dot{x}(s), p(s)\rangle ds \leqq \int_0^t \|A(s, x(s))\|_* \cdot \|p(s)\| ds + \int_0^t \|f(s)\|_* \cdot \|p(s)\| ds$$

$$\leqq \int_0^b (a(s) + b\|x(s)\|) \cdot \|p(s)\| ds + \int_0^t \|f(s)\|_* \cdot \|p(s)\| ds .$$

Applying the Cauchy-Schwartz inequality on the integrals of the right hand side and denoting by $\langle \cdot, \cdot \rangle_0$ the duality brackets for the pair $(L^2X, L^2(X^*))$, we get:

$$\langle \dot{x}, p\rangle_0 \leqq [\|a(\cdot) + b\|x(\cdot)\| \|_2 + \|f\|_{L^2(X^*)}] \cdot \|p\|_{L^2(X)}$$

$$\leqq (\|a\|_2 + b\|x\|_{L^2(X)}] + \|f\|_{L^2(X^*)}) \cdot \|p\|_{L^2(X)}$$

$$\leqq (\|a\|_2 + bM_1 + \|a_1\|_2 + M\|b_1\|_2) \cdot \|p\|_{L^2(X)}$$

$$\Longrightarrow \|\dot{x}\|_{L^2(X^*)} \leqq \|a\|_2 + bM_1 + \|a_1\|_2 + M\|b_1\|_2 = M_2 .$$

Now define the following new orientor field:

$$\hat{F}_u(t, x) = \begin{cases} F(t, x, u(t)) & \text{if } |x| \leqq M, \\ F\left(t, \dfrac{Mx}{|x|}, u(t)\right) & \text{if } |x| > M. \end{cases}$$

From the above definition, we see that $\hat{F}_u(t, x) = F(t, p_M(x), u(t))$, where $p_M(\cdot)$ is the M-radial retraction. So from this and the fact that $p_M(\cdot)$ is continuous, we deduce immediately that $\hat{F}_u(t, \cdot)$ is u.s.c. from H into H_w. Also let q_1: $T \times H \times H \to T \times H \times Y \times H$ be defined by $q_1(t, x, y) = (t, x, u(t), y)$. Clearly $q_1(\cdot, \cdot, \cdot)$ is measurable and so $q^{-1}(\mathrm{Gr}F) = \mathrm{Gr}F(\cdot, \cdot, u(\cdot)) \in B(T) \times B(H) \times B(H)$. Then let q_2: $T \times H \times H \to T \times H \times H$ be defined by $q_2(t, x, y) = (t, p_M(x), y)$. Clearly $q_2(\cdot, \cdot, \cdot)$ is measurable and so $q_2^{-1}(\mathrm{Gr}F(\cdot, \cdot, u(\cdot))) = \mathrm{Gr}\hat{F}_u(\cdot, \cdot) \in B(T) \times B(H) \times B(H)$; i.e., $\hat{F}_u(\cdot, \cdot)$ is graph measurable. Also we have $|\hat{F}_u(t, x)| \leqq a_1(t) + b_1(t)(M + |W|) = \gamma(t)$ a.e. with $\gamma(\cdot) \in L^2_+$.

Let $h(\cdot) \in L^2(H)$ and consider the following evolution equation:

$$\left.\begin{cases} \dot{x}(t) + A(t, x(t)) = h(t) \text{ a.e.,} \\ x(0) = x_0. \end{cases}\right\} (*)\ (h).$$

From Theorem 4.2, p. 167 of Barbu [6], we know that $(*)\,(h)$ has a unique solution $p(h)(\cdot) \in W(T)$. Let $B(\gamma) = \{h \in L^2(H) : |h(t)| \leqq \gamma(t) \text{ a.e.}\}$ and consider the multifunction R: $B(\gamma) \to P_{fc}(B(\gamma))$ defined by $R(h) = S^2_{\hat{F}_u(\cdot, p(h)(\cdot))}$. We claim that $R(\cdot)$ is u.s.c. from $L^2(H)_w$ into $L^2(H)_w$. To this end since $B(\gamma)$ is weakly compact and metrizable with the relative weak $L^2(H)$-topology, it is enough to check that $\mathrm{Gr}R$ is sequentially closed in $B(\gamma) \times B(\gamma)$ with the relative product weak topology. For this purpose let $\{(h_n, g_n)\}_{n\geq 1}\subseteq \mathrm{Gr}\,R$ s.t. $(h_n, g_n)\xrightarrow{w\times w}(h, g)$ in $B(\gamma)\times B(\gamma)$. We have $g_n(t)\in \hat{F}_u(t, p(h_n)(t))$ a.e.. Working as before, it is easy to see that $\{p(h_n)(\cdot)\}_{n\geq 1}$ is bounded in $W(T)$. So by passing to a subsequence if necessary, we may assume that $p(h_n)(\cdot)\xrightarrow{w}\eta(\cdot)$ in $W(T)$, hence $p(h_n)(\cdot)\xrightarrow{s}\eta(\cdot)$ in $L^2(H)$. We claim that $\eta(\cdot)=p(h)(\cdot)$. Set $x_n(\cdot)=p(h_n)(\cdot)$. By definition we have:

$$\dot{x}_n(t)+A(t, x_n(t))=h_n(t) \text{ a.e.}$$

$$\Longrightarrow \langle \dot{x}_n, x_n-\eta\rangle_o+\langle \hat{A}(x_n), x_n-\eta\rangle_o=\langle h_n, x_n-\eta\rangle_o,$$

where $\hat{A}: L^2(X)\to L^2(X^*)$ is defined by $\hat{A}(x)(\cdot)=A(\cdot, x(\cdot))$. Hence from Tanabe [17], p. 151, we have

$$\frac{1}{2}|x_n(b)-\eta(b)|^2+\langle \eta, x_n-\eta\rangle_o+\langle \hat{A}(x_n), x_n-\eta\rangle_o=\langle h_n, x_n-\eta\rangle_o.$$

Note that by hypothesis $H(A)$ (3) $\{\hat{A}(x_n)\}_{n\geq 1}$ is bounded in $L^2(X^*)$. So by passing to a subsequence if necessary, we may assume that $\hat{A}(x_n)\xrightarrow{w} r$ in $L^2(X^*)$. Also since $x_n\xrightarrow{s}\eta$ in $L^2(Y)$ and by considering the continuous versions of those functions, we have $(1/2)|x_n(b)-\eta(b)|^2\to 0$. Also $\langle \eta, x_n-\eta\rangle_o\to 0$ and $\langle h_n, x_n-\eta\rangle_o=(h, x_n-\eta)_{L^2(H)}\to 0$. Thus $\overline{\lim}\langle \hat{A}(x_n), x_n-\eta\rangle_o=0$. But $\hat{A}(\cdot)$ is monotone, hemicontinuous, since so is $A(t, \cdot)$. Hence it has property (M). Therefore $r=\hat{A}(\eta)$; i.e. $\hat{A}(x_n)\xrightarrow{w}\hat{A}(\eta)$ in $L^2(X^*)$.

Now let $q\in L^2(X)$. We have for every $n\geq 1$:

$$\langle \dot{x}_n, q\rangle_o+\langle \hat{A}(x_n), q\rangle_o=\langle h_n, q\rangle_o.$$

Therefore in the limit as $n\to\infty$ we get

$$\langle \eta, q\rangle_o+\langle \hat{A}(\eta), q\rangle_o=\langle h, q\rangle_o.$$

Since $q\in L^2(X)$ was arbitrary, we deduce that

$$\dot{\eta}(t)+A(t, \eta(t))=h(t) \text{ a.e., } \eta(0)=x_0\Longrightarrow \eta(\cdot)=p(h)(\cdot).$$

Since $g_n\xrightarrow{w} g$ in $B(\gamma)_w$, using Theorem 3.1 of [16], we have

$$g(t)\in\overline{\mathrm{conv}}\, w\text{-}\overline{\lim}\{g_n(t)\}_{n\geq 1}\subseteq\overline{\mathrm{conv}}\, w\text{-}\overline{\lim}\hat{F}_u(t, p(h_n)(t)).$$

But $p(h_n)\xrightarrow{w} p(h)$ in $W(T)$ and $W(T)\hookrightarrow L^2(H)$ compactly (see Lions [13], Theorem 5.1, p. 58). So $p(h_n)\xrightarrow{s} p(h)$ in $L^2(H)$ and by passing to a subsequence if necessary, we may assume that $p(h_n)(t)\xrightarrow{s} p(h)(t)$ a.e. in H. Then since $\hat{F}_u(t, \cdot)$ is u.s.c. from H into H_w, we have $w\text{-}\overline{\lim}\hat{F}_u(t, p(h_n)(t))\subseteq\hat{F}_u(t, p(h)(t))$ a.e.. Therefore $g(t)\in\hat{F}_u(t, p(h)(t))$ a.e. $\Rightarrow (h, g)\in\mathrm{Gr}\,R\Rightarrow R(\cdot)$ is u.s.c. from $L^2(H)_w$ into $L^2(H)_w$ as claimed.

Apply the Kakutani-KyFan fixed point theorem to get $h\in B(\gamma)$ s. t. $h\in R(h)$. Then $h(t)\in \hat{F}_u(t, p(h)(t))$ a.e.. So if $x(\cdot)=p(h)(\cdot)$, we have

$$\dot{x}(t)+A(t, x(t))=h(t) \text{ a.e.,} \qquad x(0)=x_0, \qquad h\in S^2_{\hat{F}_u(\cdot, x(\cdot))}.$$

As in the beginning of the proof we get:

$$\frac{d}{dt}|x(t)|^2+2c\|x(t)\|^2\leqq 2c\langle h(t), x(t)\rangle \text{ a.e.}$$

$$\Longrightarrow |x(t)|^2+2c\int_0^t\|x(s)\|^2ds\leqq |x_0|^2+2\Big(\int_0^t\|h(s)\|_*^2ds\Big)\Big(\int_0^t\|x(s)\|^2ds\Big).$$

Applying Cauchy's inequality with $\varepsilon=2c$, we get:

$$|x(t)|^2\geqq |x_0|^2+\frac{1}{2c}\int_0^t\|h(s)\|_*^2ds\leqq |x_0|^2+\frac{1}{2c}\int_0^t|h(s)|^2ds.$$

From the definition of $\hat{F}_u(\cdot, \cdot)$, we have that

$$|h(s)|\leqq|\hat{F}_u(t, x(s))|\leqq a_1(s)+b_1(s)(|x(s)|+|W|).$$

So finally we have

$$|x(t)|^2\leqq|x_0|^2+\frac{1}{2c}\int_0^t(2a_1(s)+4b_1(s)^2|x(s)|^2+4b_1(s)^2|W|^2)ds$$

and as before, through Gronwall's inequality, we get

$$|x(t)|\leqq M\Longrightarrow F(t, x(t), u(t)=\hat{F}_u(t, x(t))\Longrightarrow x(\cdot)\in P(u).$$

Finally note that $P(u)$ is bounded in $W(T)$, hence relatively compact in $L^2(H)$ (since $W(T)\hookrightarrow L^2(H)$ compactly). Furthermore working as above we can easily check that $P(u)$ is closed in $L^2(H)$, hence compact in it. Q.E.D.

Remarks. **(3.1)** Hypothesis $H(A)$ (2) is more general than that of Ahmed [1] (see Lemma 3.3 in that paper), who essentially considered parabolic systems which have a linear prinicipal part in their partial differential operator.
(3.2) If X is a Hilbert space, then $P(u)$ is compact in $C(T, H)$. This follows from the fact that in this case $W(T)\hookrightarrow C(T, H)$ compactly.

Now that we have established the existence of admissible trajectories for (*), we turn our attention to the solution of the minimax control problem. We were able to solve it for systems where the differential operator $A(t, x)$ is linear in x and for multifunctions F independent in x and "convex" in u. Such multifunctions are important in optimization (see Aubin [3], Ioffe [10] in non-linear analysis (see Ioffe [10]) and in game theorem (see Krasovski-Subbotin [11] and Aubin [4]). So the minimax problem has the following special form:

$$\left.\begin{cases} J(x,\, u)=\displaystyle\int_0^b L(t,\, x(t),\, u(t))dt \to \inf_{u\in S_U^1} \sup_{x\in P(u)} \\ \text{s.t. } \dot{x}(t)+A(t)x(t)\in F(t,\, u(t)) \text{ a.e.}, \\ \qquad x(0)=x_0, \qquad u(\cdot)\in S_U^1. \end{cases}\right\} (**)$$

We will need the following hypotheses on the data of (**):

$H(A)_1$: $A: T\times X\to X^*$ is an operator s.t.

(1) $t\to A(t)x$ is measurable,

(2) $x\to A(t)x$ is linear and $\|A(t)x\|_*\leqq b\|x\|$ a.e., $b>0$, (i.e. $A(t)(\cdot)$ is continuous),

(3) $\langle A(t)x,\, x\rangle \geqq c\|x\|^2$ a.e., $c>0$, (i.e., $A(t)(\cdot)$ is strongly monotone).

$H(F)_1$: $F: T\times Y\to P_{fc}(H)$ is a multifunction s.t.

(1) $(t,\, u)\to F(t,\, u)$ is graph measurable,

(2) $u\to F(t,\, u)$ is h-continuous and for all $\lambda\in[0,\, 1]$ and all $u_1,\, u_2\in Y$ we have $F(t,\, \lambda u_1+(1-\lambda)u_2)\subseteqq \lambda F(t,\, u_1)+(1-\lambda)F(t,\, u_2)$ a.e.,

(3) $|F(t,\, u)|=\sup\{\|y\|:\, y\in F(t,\, u)\}\leqq a_1(t)+b_1(t)\|u\|$ a.e. with $a_1(\cdot),\, b_1(\cdot)\in L_+^2$.

Also we will need the following hypothesis concerning the cost integrand.

$H(L)$: $L: T\times H\times Y\to \bar{\boldsymbol{R}}=\boldsymbol{R}\cup\{+\infty\}$ is an integrand s.t.

(1) $(t,\, x,\, u)\to L(t,\, x,\, u)$ is measurable,

(2) $(x,\, u)\to L(t,\, x,\, u)$ is l.s.c.,

(3) $L(t,\, x,\, \cdot)$ is convex,

(4) $\phi_1(t)-M(|x|+\|u\|)\leqq L(t,\, x,\, u)$ a.e. with $\phi_1(\cdot)\in L^1$, $M\geqq 0$.

Remark. (3.3) As the work of Ioffe [10] has demonstrated, positively homogeneous multifunctions satisfying $H(F_1)$ (2) are the right multivalued analog of bounded linear operators.

(3.4) Because of the special form of the dynamics, the multifunction $u\to P(u)$ satisfies $P(\lambda u_1+(1-\lambda)u_2)\subseteqq \lambda P(u_1)+(1-\lambda)P(u_2)$ for all $\lambda\in[0,\, 1]$, $u_1,\, u_2\in L^2(Y)$.

Under the above hypotheses, we can state and prove the following existence result for problem (**).

Theorom 3.2. *If hypotheses $H(A)_1$, $H(F)_1$, $H(L)$ and $H(U)$ hold, then problem (**) admits a nonempty, closed, convex set of optimal controls.*

Proof. It is easy to check using Fatou's lemma that $(x,\, u)\to J(x,\, u)$ is l.s.c. on $L^2(H)\times L^2(Y)$. Also it is convex in u. Consider the multifunction $P: S_U^1\to 2^{L^2(H)}$ defined by $u\to P(u)$. From Theorem 3.1 and Remark (3.3) we know that $P(\cdot)$ is $P_{kc}(L^2(H))$-valued. We claim that $P(\cdot)$ is l.s.c.. From Delahaye-Denel [9], we know that it suffices to show that for every $u_n\xrightarrow{s}u$ in $S_U^1\Rightarrow P(u)\subseteqq s\text{-}\underline{\lim}P(u_n)$. To this end, let $x(\cdot)\in P(u)$. By definition we have:

$$\dot{x}(t)+A(t, x(t))=g(t) \text{ a.e.}, \quad x(0)=x_0, \quad g\in S^2_{F(\cdot, x(\cdot))}.$$

Observe that

$$\int_0^b \sup_{y\in F(t, u_n(t))} d(y, F(t, u(t)))dt$$

$$= \sup_{y(\cdot)\in S^2_{F(\cdot, u_n(\cdot))}} \int_0^b d(y(t), F(t, u(t)))dt = \sup_{y(\cdot)\in S^2_{F(\cdot, u_n(\cdot))}} d(y, S^2_{F(\cdot, u(\cdot))}).$$

Similarly

$$\int_0^b \sup_{y\in F(t, u(t))} d(y, F(t, u_n(t)))dt = \sup_{y(\cdot)\in S^2_{F(\cdot, u(\cdot))}} d(y, S^2_{F(\cdot, u_n(\cdot))}).$$

Hence we deduce that

$$h(S^2_{F(\cdot, u_n(\cdot))}, S^2_{F(\cdot, u(\cdot))})=\int_0^b h(F(t, u_n(t)), F(t, u(t)))dt \to 0 \text{ as } n\to\infty.$$

Thus we can find $g_n\in S^2_{F(\cdot, u_n(\cdot))}$ s.t. $g_n \xrightarrow{s} g$ in $L^2(H)$. Consider the following evolution equations:

$$\left\{\begin{array}{c} \dot{x}_n(t)+A(t)x_n(t)=g_n(t) \text{ a.e.}, \\ x_n(0)=x_0. \end{array}\right\}$$

Then $x_n(\cdot)=p(g_n)(\cdot)$. From the proof of Theorem 3.1 we know that $p(\cdot)$ is continuous from S^1_U into $L^2(H)$. Hence $p(g_n) \xrightarrow{s} p(g)$ in $L^2(H) \Rightarrow x_n \to x=p(g)$ and since clearly $x_n(\cdot)\in P(u_n)$, we have $P(u)\subseteq s\text{-}\underline{\lim}P(u_n)$ and so $P(\cdot)$ is l.s.c.. Now let $K(u)=\sup_{x\in P(u)} J(x, u)$. Using Remark 3.4 and the convexity of $J(x, \cdot)$ we can easily check that $K(\cdot)$ is convex. Also from Theorem 1, p. 122 of Berge [7], we know that $K(\cdot)$ is l.s.c. on $L^1(Y)$ and is also w-l.s.c., because it is convex. Then since S^1_U is w-compact in $L^1(Y)$ (see Proposition 3.1 of [15]), from Weierstrass' theorem we conclude that $\inf_{\in u S^1_U} K(u)$ admits a solution and the set of solutions is clearly closed, convex. Q.E.D.

4. Variational stability.

In this section we will examine the dependence of the optimal controls on the data of the problem. Such a stability analysis is useful from both the theoretical and applied viewpoints. It produces useful continuous dependence results, robust computational schemes and provides information about the best possible mathematical model, since it tells us what tolerances are permitted in the specification of the data.

So we will be examining the following sequence of minimax control problems:

$$\left.\begin{cases} J_n(x_n, u_n)=\int_0^b L_n(t, x_n(t), u_n(t))dt \to \inf_u \sup_x \\ \text{s.t. } \dot{x}_n(t)+A_n(t, x_n(t))\in F(t, x_n(t), u_n(t)) \text{ a.e.,} \\ \qquad x_n(0)=x_{on}, \qquad u_n(t)\in U_n(t) \text{ a.e.} \end{cases}\right\} (*)_n$$

and the limit problem is (*).

Let $\hat{U}_n$ be the set of optimal controls of $(*)_n$ and $\hat{U}$ the set of optimal controls of (*). We will assume that $\hat{U}_n, \hat{U}\neq\phi$, $n\geqq 1$. Of course if the hypotheses of Theorem 3.2 are satisfied, then this is automatically the case.

Recall that $\hat{A}_n: L^2(X)\to L^2(X^*)$ is defined by $\hat{A}_n(x)(t)=A_n(t, x(t))$ and similarly $\hat{A}: L^2(X)\to L^2(X^*)$. These are the realizations (liftings) of the operators $A_n(\cdot, \cdot)$, $A(\cdot, \cdot)$ on $L^2(X)$. To prove our variational stability result we will need the following hypotheses on the problems:

$H(A)_2$: $A_n, A: T\times X\to X^*$ are operators satisfying $H(A)_1$ (1)—(4) uniformly in $n\geqq 1$ and

(5) $\hat{A}_n x_n \xrightarrow{s} \hat{A}x$ in $L^2(X^*)$ as $n\to\infty$ for every $x_n\xrightarrow{s} x$ in $L^2(X)$.

$H(F)_2$: $F_n, F: T\times H\times Y\to P_{fc}(H)$ are multifunctions s.t.

(1) $(t, x, u)\to F_n(t, x, u)$, $F(t, x, u)$ are graph measurable,

(2) $h(F_n(t, x, u), F_n(t, y, u))\leqq k(t)|x-y|$ a.e. for all $u\in W$ and with $k(\cdot)\in L^1_+$,

(3) $|F_n(t, x, u)|\leqq a_1(t)+b_1(t)(|x|+\|u\|)$ a.e. with $a_1(\cdot), b_1(\cdot)\in L^2_+$,

(4) if $u_n\xrightarrow{s} u$ in W, then $F_n(t, x, u_n)\xrightarrow{h} F(t, x, u)$ for all $(t, x)\in T\times H$.

$H(U)_1$: $U_n, U: T\to P_{fc}(Y)$ are measurable multifunctions s.t. $U_n(t), U(t)\subseteq W$ a.e. with $W\in P_{wkc}(Y)$ and $U_n(t)\xrightarrow{K-M} U(t)$ a.e..

$H(L)_1$: $L_n, L: T\times H\times Y\to \boldsymbol{R}$ are integrands s.t.

(1) $t\to L_n(t, x, u)$ are measurable,

(2) $(x, u)\to L_n(t, x, u)$ are continuous, convex,

(3) $L_n(t, \cdot, \cdot)\xrightarrow{u} L(t, \cdot, \cdot)$ a.e. (where u means uniformly on compact subsets of $H\times Y$)

(4) $|L_n(t, x, u)|\leqq\phi(t)+r(|x|^2+\|u\|^2)$ a.e. with $\phi(\cdot)\in L^1_+$, $r>0$.

Assume that the control space Y is a separable, reflexive Banach space. We have the following variational stability result:

Theorem 4.1. *If hypotheses $H(A)_2$, $H(F)_2$, $H(U)_1$, $H(L)_1$ hold and $x_{on}\xrightarrow{w} x_o$ in H, then s-$\overline{\lim}\hat{U}_n\subseteq\hat{U}$ in $L^1(Y)$.*

Proof. Let $S_n=\{(x, u)\in L^2(H)\times S^1_W$: admissible pairs for $(*)_n\}$ and $S=\{(x, u)\in L^2(H)\times S^1_W$: admissible pairs for $(*)\}$. Our claim is that $S_n\xrightarrow{sK} S$ as $n\to\infty$. To this end let $(x, u)\in S$. Let $u_n(t)=\text{proj}(u(t); U_n(t))$. Because of hypothesis $H(U)_1$, $u_n(\cdot)$ is measurable and so clearly $u_n\in S^1_{U_n}$. Also since $U_n(t)$

$\xrightarrow{K-M} U(t)$ a.e., from Theorem 3.33 of Attouch [2], we have $u_n(t) \xrightarrow{s} u(t)$ a.e.. So from hypothesis $H(F)_2$ (4), we get:

$$h(F_n(t, x, u_n(t)), F(t, x, u(t))) \to 0 \quad \text{a.e.}.$$

Also $h(F_n(t, x, u_n(t)), F_n(t, y, u_n(t))) \leqq k(t)|x-y|$ a.e. for all $n \geqq 1$ and all $(x, y) \in H \times H$. By definition we have

$$\dot{x}(t)+A(t, x(t))=g(t) \quad \text{a.e.}, \quad x(0)=x_0, \quad g \in S^2_{F(\cdot, x(\cdot), u(\cdot))}.$$

Set $m_n(t)=\operatorname{proj}(g(t); F_n(t, x(t), u_n(t)))$ and $f_n(t, z)=\operatorname{proj}(m_n(t); F_n(t, z, u_n(t)))$. Observe that $m_n(t)=f_n(t, x_n(t))$. Consider the following evolution equation

$$\dot{x}_n(t)+A_n(t, x_n(t))=f_n(t, x_n(t)) \text{ a.e.}, \quad x_n(0)=x_{on}.$$

Recalling that $f_n(t, \cdot)$ is continuous (see Attouch [2], Theorem 3.33, p. 322), from Theorem 3.1, we know that the above Cauchy problem has a solution $x_n(\cdot) \in W(T)$. From the a priori estimates of the proof of Theorem 3.1, we know that $\{x_n\}_{n \geq 1}$ is bounded in $W(T)$. Hence by passing to a subsequence if necessary we may assume that $x_n \xrightarrow{w} \hat{x}$ in $W(T)$. Then we have

$$\langle \dot{x}_n - \dot{x}(t), x_n(t)-x(t)\rangle + \langle A_n(t, x_n(t))-A(t, x(t)), x_n(t)-x(t)\rangle$$
$$\leqq \langle f_n(t, x_n(t))-g(t), x_n(t)-x(t)\rangle \text{ a.e.}.$$

Integrating we have:

$$|x_n(t)-x(t)|^2 \leqq |x_{on}-x_0|^2 + 2\int_0^t \langle A_n(s, x_n(s))-A_n(s, x(s)), x_n(s)-x(s)\rangle ds$$
$$+2\int_0^t \langle A_n(s, x(s))-A(s, x(s)), x_n(s)-x(s)\rangle ds$$
$$+2\int_0^t \langle f_n(s, x_n(s))-g(s), x_n(s)-x(s)\rangle ds$$
$$\Longrightarrow |x_n(t)-x(t)|^2 \leqq |x_{on}-x_0|^2 + 2\langle \hat{A}_n x - \hat{A} x, x_n - x\rangle_0$$
$$+2\int_0^t \langle f_n(s, x_n(s))-g(s), x_n(s)-x(s)\rangle ds.$$

Observe that

$$\int_0^t \langle f_n(s, x_n(s))-g(s), x_n(s)-x(s)\rangle ds$$
$$\leqq \int_0^t |f_n(s, x_n(s))-f_n(s, x(s))| \cdot |x_n(s)-x(s)| ds$$
$$+\int_0^t |f(s, x(s))-g(s)| \cdot |x_n(s)-x(s)| ds.$$

Recalling that $\|x_n\|_{L^2(X)} \leqq M_1$, we have

$$|x_n(t)-x(t)|^2 \leqq |x_{on}-x_0|^2+2M_1\|\hat{A}_n x-\hat{A}x\|_{L^2(X^*)}$$
$$+2\int_0^t [|f_n(s, x_n(s))-f_n(s, x(s))|+|f(s, x(s))-g(s)|]\cdot|x_n(s)-x(s)|ds.$$

Invoking Lemma A.5, p. 157 of Brezis [8], we get

$$|x_n(t)-x(t)| \leqq |x_{on}-x_0|+2(M_1\|\hat{A}_n x-\hat{A}x\|_{L^2(X^*)})^{1/2}$$
$$+\int_0^t (|f_n(s, x_n(s))-f(s, x(s))|+|f(s, x(s))|+|f(s, x(s))-g(s)|)ds. \quad \mathbf{(1)}$$

Note that

$$|f_n(s, x_n(s))-f_n(s, x(s))|=d(m_n(s), F_n(s, x(s)))$$
$$\leqq h(F_n(s, x_n(s)), F_n(s, x(s))) \leqq k(s)|x_n(s)-x(s)|$$

and

$$|f_n(s, x(s))-g(s)| \leqq h(F_n(s, x(s)), F(s, x(s))) \to 0 \text{ as } n\to\infty.$$

Since $x_n \xrightarrow{w} \hat{x}$ in $W(T)$ and $W(T) \hookrightarrow L^2(H)$ compactly, we may assume that $x_n(t) \xrightarrow{s} x(t)$ in H. So since $\hat{A}_n x \xrightarrow{s} \hat{A}x$ in $L^2(X^*)$, by passing to the limit as $n\to\infty$ in (1) we get:

$$|\hat{x}(t)-x(t)| \leqq \int_0^t k(s)|\hat{x}(s)-x(s)|ds.$$

An application of Gronwall's inequality gives us $\hat{x}=x$. Hence every subsequence of $\{x_n\}_{n\geq 1}$ has a further subsequence converging strongly in $L^2(H)$ to $x(\cdot)$ (recall $W(T) \hookrightarrow L^2(H)$ compactly). Thus $x_n \xrightarrow{s} x$ in $L^2(H)$. Since $(x_n, u_n)\in S_n$ and $(x_n, u_n) \xrightarrow{S\times S} (x, u)$ in $L^2(H)\times S_W^1$, we conclude that

$$S \subseteq s\text{-}\underline{\lim} S_n. \quad \mathbf{(2)}$$

Next let $(x_n, u_n)\in S_n$, $n\geqq 1$, with $(x_n, u_n) \xrightarrow{S\times S} (x, u)$ in $L^2(H)\times S_W^1$. By passing to a subsequence if necessary we may assume that $x_n(t) \xrightarrow{s} x(t)$ a.e. in H and $u_n(t) \xrightarrow{s} u(t)$ a.e. in Y. By definition $x_n(\cdot)=p(r_n(\cdot))$ with $r_n\in S^2_{F_n(\cdot, x_n(\cdot), u_n(\cdot))}$. Using the a priori bounds and the growth hypothesis on $F_n(\cdot, \cdot, \cdot)$, we get that $\overline{\{r_n(\cdot)\}}^w_{n\geq 1}$ is bounded in $L^2(H)$, hence sequentially w-compact. Thus we may assume that $r_n \xrightarrow{w} r$ in $L^2(H)$. Note that

$$h(F_n(t, x_n(t), u_n(t)), F(t, x(t), u(t)))$$
$$\leqq h(F_n(t, x_n(t), u_n(t)), F_n(t, x(t), u_n(t)))+h(F_n(t, x(t), u_n(t)), F(t, x(t), u(t)))$$
$$\leqq k(t)|x_n(t)-x(t)|+h(F_n(t, x(t), u_n(t)), F(t, x(t), u(t))) \to 0 \text{ a.e.}$$
$$\Longrightarrow S^2_{F_n(\cdot, x_n(\cdot), u_n(\cdot))} \xrightarrow{h} S^2_{F(\cdot, x(\cdot), u(\cdot))}$$

$$\Longrightarrow r \in S^2_{F(\cdot, x(\cdot), u(\cdot))}.$$

Also we have:

$$\dot{x}_n(t) + A_n(t, x_n(t)) = r_n(t) \text{ a.e.}$$
$$\Longrightarrow \langle \dot{x}_n, q\rangle_0 + \langle \hat{A}_n(x_n), q\rangle_0 = \langle r_n, q\rangle_0 \text{ for } q \in L^2(X)$$
$$\Longrightarrow \langle \dot{x}, q\rangle_0 + \langle \hat{A}(x), q\rangle_0 = \langle r, q\rangle_0.$$

Since $q \in L^2(X)$ was arbitrary we conclude that

$$\dot{x}(t) + A(t, x(t)) = r(t) \text{ a.e.}, \quad x(0) = x_0, \quad r \in S^2_{F(\cdot, x(\cdot), u(\cdot))}$$
$$\Longrightarrow (x, u) \in S$$
$$\Longrightarrow s\text{-}\overline{\lim} S_n \subseteq S. \tag{3}$$

From (2) and (3) above we conclude that $S_n \xrightarrow{sK} S$. Also from hypothesis $H(L)_1$ we have that $J_n \xrightarrow{s} J$ uniformly on compact subsets of $L^2(H) \times L^2(Y)$. Then invoking Theorems 2.11 and 2.15 of Attouch [2], we get that

$$\phi_n(u) = \sup_{x \in L^2(H)} [J_n(x, u) = \delta_{S_n}(x, u)] \longrightarrow \sup_{x \in L^2(H)} [J(x, u) = \delta_S(x, u)] = \phi(u)$$

where $\delta_{S_n}(x, u) = 0$ if $(x, u) \in S_n$, $+\infty$ if $(x, u) \notin S_n$. Similarly for $\delta_S(\cdot, \cdot)$.

Next let $u_n \xrightarrow{s} u$ in $L^1(Y)$. Because of the compactness of $P(u_n)$ and $P(u)$ in $L^2(H)$ and the continuity of the cost functionals $J_n(\cdot, u_n)$ and $J(\cdot, u)$, we can find $x_n \in P(u_n)$ and $x \in P(u)$ s.t. $\phi_n(u_n) = J_n(x_n, u_n)$ and $\phi(u) = J(x, u)$. We may assume as before that $x_n \xrightarrow{s} x$ in $L^2(H)$. Then we have:

$$\lim J_n(x_n, u_n) = J(x, u) \Longrightarrow \lim \phi_n(u_n) = \phi(u).$$

Thus invoking Theorem 2.5 of Langen [12] we get that $s\text{-}\overline{\lim} \hat{U}_n \subseteq \hat{U}$ in $L^1(Y)$.

Q. E. D.

5. A necessary and sufficient condition for optimality.

In this section we derive a necessary and sufficient condition for a control function $\hat{u}(\cdot) \in S^1_U$ to be optimal. To get this, we will consider a special version of problem (*) with linear terminal cost and semilinear dynamics. So the minimax control problem under consideration is the following:

$$\left\{ \begin{array}{c} (c, x(b)) \to \inf\limits_u \sup\limits_x \\ \text{s.t. } \dot{x}(t) + A(t)x(t) \in F(t, u(t)) \text{ a.e.}, \\ x(0) = x_0, \quad u \in S^1_U. \end{array} \right\} (***)$$

We will need the following hypotheses on the data of (***).

$H(A)_3$: $A: T\times X\to X^*$ is an operator s.t.

(1) $\|A(t')x-A(t)x\|_*\leqq k|t'-t|^{\alpha}\|x\|$, $\alpha\in(0, 1]$,

(2) $A(t)(\cdot)$ is linear and $\|A(t)x\|_*\leqq b\|x\|$ a.e., $b>0$,

(3) $\langle A(t)x, x\rangle\geqq c\|x\|^2$ a.e..

From Tanabe [17] (section 5.4), we know that $\{A(t)\}_{t\in T}$ generates a strongly continuous evolution operator $S: \Delta=\{(t, s): 0\leqq s\leqq t\leqq b\}\to\mathcal{L}(H)$, with respect to which a trajectory of (***) can be written as

$$x(t)=S(t, 0)x_0+\int_0^t S(t, s)f(s)ds, \quad t\in T, \quad f\in S^2_{F(\cdot, u(\cdot))}.$$

If we also assume hypotheses $H(F)_1$ and $H(U)$, from Theorem 4.1, we know that (***) admits an optimal control.

Theorem 5.1. *If hypotheses $H(A)_3$, $H(F)_1$ and $H(U)$ hold, then $\hat{u}\in S^1_U$ is an optimal control for* (***) *if and only if*

$$\min_{u\in U(t)}\max_{y\in F(t, u)}(y, S(b, t)^*c)=\max_{y\in F(t, u(t))}(y, S(b, t)^*c) \quad \text{a.e..}$$

Proof. Observe that the attainable set at time b is given by

$$R(b)=\{x(b)=S(b, 0)x_0+\int_0^b S(b, s)g(s)ds, \quad g\in S^2_{F(\cdot, u(\cdot))}, \quad u(\cdot)\in S^1_U\}.$$

So $\hat{u}\in S^1_U$ will be an optimal control for the minimax problem (***) if and only if

$$\inf_{U\in S^1_U}\sup_{g\in S^2_{F(\cdot, u(\cdot))}}(c, x(b))=(c, \hat{x}(b)) \tag{1}$$

where

$$\hat{x}(b)=S(b, 0)x_0+\int_0^b S(b, s)\hat{g}(s)ds, \; g\in S^2_{F(\cdot, \hat{u}(\cdot))}.$$

Then we have:

$$\begin{aligned}\inf_{u\in S^1_U}\sup_{g\in S^2_{F(\cdot, u(\cdot))}}(c, x(b))&=\inf_{u\in S^1_U}\sup_{g\in S^2_{F(\cdot, u(\cdot))}}(c, S(b, 0)x_0+\int_0^b S(b, s)g(s)ds)\\ &=(c, S(b, 0)x_0)+\inf_{u\in S^1_U}\sup_{g\in S^2_{F(\cdot, u(\cdot))}}(c, \int_0^b S(b, s)g(s)ds).\end{aligned}$$

Note that the summand $(c, S(b, 0)x_0)$ will be cancelled by the corresponding term in the right hand side of (**1**). Then we have:

$$\begin{aligned}&\inf_{u\in S^1_U}\sup_{g\in S^2_{F(\cdot, u(\cdot))}}\int_0^b(c, S(b, s)g(s))ds\\ &=\inf_{u\in S^1_U}\sup_{g\in S^2_{F(\cdot, u(\cdot))}}\int_0^b(S(b, s)^*c, g(s))ds\\ &=\inf_{u\in S^1_U}\sup_{y\in F(s, u(s))}\int_0^b(y, S(b, s)^*c)ds\end{aligned}$$

$$= \inf_{u\in S_U^1} \int_0^b \sigma_{F(s,\,u(s))}(S(b,\ s)^*c)ds$$

$$= \int_0^b \inf_{u\in U(s)} \sigma_{F(s,\,u(s))}(S(b,\ s)^*c)ds,$$

here $\sigma_{F(s,\,u(s))}(\cdot)$ denotes the support function of $F(s,\ u(s))$. So going back in **(1)** we get:

$\hat{u}(\cdot)\in S_U^1$ solves the minimax control problem (***) if and only if

$$\int_0^b \inf_{u\in U(s)} \sigma_{F(s,\,u)}(S(b,\ s)^*c)ds = \int_0^b \sigma_{F(s,\,u(s))}(S(b,\ s)^*c)ds$$

$$\Longleftrightarrow \inf_{u\in U(s)} \sigma_{F(s,\,u)}(S(b,\ s)^*c)\sigma_{F(s,\,u(s))}(S(b,\ s)^*c) \quad \text{a.e.}$$

$$\Longleftrightarrow \min_{u\in U(s)} \max_{y\in F(s,\,u)} (y,\ S(b,\ s)^*c) = \max_{y\in F(s,\,\hat{u}(s))} (y,\ S(b,\ s)^*c) \quad \text{a.e..} \qquad \text{Q.E.D.}$$

6. An example.

We will conclude with an example illustrating the applicability of our work. So let $T=[0,\ b]$ and Z a bounded domain in $\boldsymbol{R}^n$ with smooth boundary $\partial Z=\Gamma$. We will consider the following minimax distributed parameter control system.

$$\left\{\begin{array}{l} J(x,\ u)=\int_0^b\int_Z L(t,\ z,\ x(t,\ z),\ u(t,\ z))dzdt \to \inf_u \sup_x \\ \text{s.t.}\ \dfrac{\partial x(t,\ z)}{\partial t} + \sum_{|\alpha|,\,|\beta|\le m} (-1)^{|\alpha|} D^\alpha(a_{\alpha\beta}(t,\ z)D^\beta x(t,\ z)) \\ \qquad\qquad \in F(t,\ z,\ u(t,\ z)) \text{ on } T\times Z, \\ \qquad D^\gamma x(t,\ z)=0 \text{ on } T\times\Gamma \text{ for } |\gamma|\leqq m-1, \\ \qquad x(0,\ z)=x_0(z) \text{ on } \{0\}\times Z, \\ \qquad \int_Z |u(t,\ z)|^2 dz \leqq M. \end{array}\right\} \quad (****)$$

We will need the following hypotheses concerning the data of (****).

$H(a)$: $a_{\alpha\beta}: T\times Z \to \boldsymbol{R}$ are functions in $L^\infty(T\times Z)$ s.t.

(1) $\sum_{|\alpha|,\,|\beta|\le m} a_{\alpha\beta}(t,\ z)x_\alpha x_\beta \geqq c\|x\|^2$ a.e., $c>0$,

(2) $|a_{\alpha\beta}(t,\ z)|\leqq b(z)$ a.e. with $b(\cdot)\in L^\infty(Z)$.

$H(F)_3$: $F(t,\ z,\ u)=[b_1(t,\ z)u,\ b_2(t,\ z)u]\subseteq \boldsymbol{R}$ with $b_1\leqq b_2$ a.e. $b_1(t,\ \cdot),\ b_2(t,\ \cdot)\in L^\infty(Z)$ and $t\to \|b_1(t,\ \cdot)\|_\infty,\ \|b_2(t,\ \cdot)\|_\infty \in L_+^2$.

$H(L)_2$: $L: T\times Z\times \boldsymbol{R}\times \boldsymbol{R} \to \boldsymbol{R}$ is an integrand s.t.

(1) $L(\cdot,\ \cdot,\ \cdot,\ \cdot)$ is measurable,

(2) $L(t,\ z,\ \cdot,\ \cdot)$ is l.s.c. and convex in u,

(3) $\phi(t,\ z)-M(z)(|x|+\|u\|)\leqq L(t,\ z,\ x,\ u)$ a.e. with $\phi(\cdot,\ \cdot)\in L^1(T\times Z)$, $M(\cdot)\in L^1(Z)$.

Theorem 6.1. *If hypotheses $H(a)$, $H(F)_3$ and $H(L)$ hold, then* (****) *admits a minimax optimal control.*

Proof. Consider the Gelfand triple consisting of the spaces $X=H_0^m(Z)$, $H=L^2(Z)$ and $X^*=(H_0^m(Z))^*=H^{-m}(Z)$. All the embeddings are continuous and dense and furthermore by the Sobolev-Kondrachov-Rellich embedding theorem they are also compact. On $H_0^m(Z)\times H_0^m(Z)$ consider the following time dependent Dirichlet form:

$$a(t, x, y)=\int_Z \sum_{|\alpha|, |\beta|\leq m} a_{\alpha\beta}(t, z)D^\beta x(z)D^\alpha y(z)dz.$$

From Fubini's theorem, we see that $t\to a(t, x, y)$ is measurable. Also from the Cauchy-Schwartz inequality, we get

$$\left|\int_Z a_{\alpha\beta}(t, z)D^\beta x(z)D^\alpha y(z)dz\right|\leq\|a_{\alpha\beta}\|_\infty\left(\int_Z D^\beta x(z)^2dz\right)^{1/2}\left(\int_Z D^\beta y(z)^2dz\right)^{1/2}$$

$$\Longrightarrow |a(t, x, y)|\leq N_m\|x\|_{H_0^m(Z)}\cdot\|y\|_{H_0^m(Z)}.$$

Let $A(t)\colon H_0^m(Z)\to H^{-m}(Z)$ be the linear operator defined by $a(t, x, y)=\langle A(t)x, y\rangle$. Then clearly $A(t)(\cdot)\in\mathcal{L}(X, X^*)$, $t\to A(t)x$ is measurable and because of hypothesis $H(a)$ (1) we have

$$\langle A(t)x, x\rangle=\sum_{|\alpha|, |\beta|\leq m}\int_Z a_{\alpha\beta}(t, z)D^\beta x(z)D^\alpha y(z)dz$$

$$\leq c\sum_{|\alpha|\leq m}\int_Z D^\alpha x(z)^2dz=c\|x\|_{H_0^m(Z)}.$$

Hence $A(t)x$ satisfies hypothesis $H(A)_1$.

Next let $\hat{F}\colon T\times L^2(Z)\to P_{fc}(L^2(Z))$ be defined by

$$\hat{F}(t, u)=\{y\in L^2(Z)\colon b_1(t, z)u(z)\leq y(z)\leq b_2(t, z)u(z) \text{ a.e.}\}.$$

It is easy to check that $\hat{F}(\cdot, \cdot)$ satisfies hypothesis $H(F)_1$. Also let $\hat{L}\colon T\times L^2(Z)\times L^2(Z)\to\bar{\boldsymbol{R}}$ be defined by

$$\hat{L}(t, x, u)=\int_Z L(t, z, x(z), u(z))dz.$$

From Lemma 2 of Balder [5], we know we can find $L_k\colon T\times Z\times\boldsymbol{R}\times\boldsymbol{R}\to\boldsymbol{R}$. Caratheodory functions s.t. $L_k\uparrow L$ and $\phi(t, z)-M(z)(|x|+\|u\|)\leq L_k(t, z, x, u)\leq k$. Set $\hat{L}_k(t, x, u)=\int_Z L_k(t, z, x(z), u(z))dz$. Then $\hat{L}_k(\cdot, \cdot, \cdot)$ is measurable in t and continuous in (x, u) (i.e., a Caratheodory function), hence it is jointly measurable. Furthermore the monotone convergence theorem tells us that $\hat{L}_k\uparrow\hat{L}\Rightarrow\hat{L}$ is jointly measurable. Also an application of Fatou's lemma tells us that $\hat{L}(t, \cdot, \cdot)$ is l.s.c. and it is also convex in u. Furthermore note that $\hat{\phi}(t)-\hat{M}(\|x\|_{L^2(Z)}+\|u\|_{L^2(Z)})\leq\hat{L}(t, x, u)$ a.e. with $\hat{\phi}(t)=\|\phi(t, \cdot)\|_1\hat{M}=\|M(\cdot)\|_1$. Let $\hat{x}_0=$

$x_0(\cdot) \in L^2(Z)$.

Finally let $Y = L^2(Z)$ and set $U(t) = W = \{u \in L^2(Z) : \|u\|^2_{L^2(Z)} \leqq M\} \in P_{wkc}(L^2(Z))$. Then (****) is equivalent to the following abstract minimax control problem defined on (X, H, X^*, Y):

$$\left\{\begin{array}{l} \hat{J}(x, u) = \int_0^b \hat{L}(t, x(t), u(t))dt \to \inf_u \sup_x \\ \text{s.t. } \dot{x}(t) + A(t)x(t) \in \hat{F}(t, u(t)) \quad \text{a.e.,} \\ \qquad x(0) = \hat{x}_0, \quad u(t) \in W \quad \text{a.e..} \end{array}\right\} (****)'$$

Invoking Theorem 3.2 we get that (****)′, hence (****) too, admits a minimax optimal control $\hat{u} \in L^2(T \times Z)$. Q.E.D.

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National Technical University
Department of Mathematical Sciences
Athens 157-73, GREECE