YOKOHAMA MATHEMATICAL
JOURNAL VOL. 38, 1991

KAUFFMAN POLYNOMIALS FOR 2-BRIDGE KNOTS AND LINKS

Dedicated to Professor Masahisa Adachi on his sixtieth birthday

By

TAIZO KANENOBU

(Received May 23, 1990; Revised September 18, 1990)

Abstract. We construct arbitrarily many skein equivalent 2-bridge knots (resp. links) with the same Kauffman polynomial (resp. Kauffman and 2-variable Alexander polynomials). We also consider a similar example for a 3-braid knot or link.

In [4, Proposition 4.1], we give a method to construct a pair of 2-bridge knots or links sharing the same Kauffman polynomial, and using this we obtain the following examples: a pair of fibered, amphicheiral, skein equivalent 2-bridge knots with the same Kauffman polynomial [4, Theorem 4], and a pair of fibered, skein equivalent 2-bridge links with the same 2-variable Alexander and Kauffman polynomials [4, Theorem 6]. Also we have the following examples: arbitrarily many skein equivalent, amphicheiral, fibered 2-bridge knots [4, Theorem 1], and arbitrarily many skein equivalent, fibered 2-bridge links with the same 2-variable Alexander polynomial [4, Theorem 7]. In this paper we prove the following:

Theorem 1. *For any positive integer N, there exist* 2^N, *mutually distinct, amphicheiral, fibered 2-bridge knots, which are skein equivalent and have the same Kauffman polynomial.*

Theorem 2. *For any positive integer N, there exist* 2^N, *mutually distinct, fibered 2-bridge links, which are skein equivalent and have the same Kauffman and 2-variable Alexander polynomials.*

The numbers of pairs of 2-bridge knots and links through 20 crossings which share the same Kauffman polynomial are 58 [5] and 37 [6], respectively, where

---

1980 Mathematics Subject Classification (1985 Revision). Primary 57M25.

Key words and phrases. 2-bridge link, 3-braid link, Kauffman polynomial.

Partially supported by Grant-in-Aid for Encouragement of Young Scientists (No. 01740057), Ministry of Education, Science and Culture.

a knot (or link) and its mirror image are counted as one. Among them the two pairs of 2-bridge knots of 20 crossings cannot be explained by [4, Proposition 4.1]. Observing these pairs, we obtain Proposition 3, which is an extended version of this proposition, and prove the above theorems. In Sect. 4, we list all the above pairs putting in the form applicable to Proposition 3.

In Sect. 3, we give a method to construct many 3-braid knots or links having the same Kauffman polynomial, which is a generalization of [4, Proposition 5.5].

We refer to [8] for the definition of a skein triple and a skein equivalence. If two links L_1 and L_2 are skein equivalent, we write $L_1 \sim L_2$. We refer to [7] for the definition of the Kauffman and the L polynomials. We consider the 2-variable Conway potential function [1] rather than the 2-variable Alexander polynomial.

Acknowledgement. I would like to thank Toshio Sumi for supplying a computer program for making the tables.

1. Preliminaries

Let α be a 3-braid, $\alpha \in B_3$. We denote a 3-knit $\alpha S_2^{a_1}\alpha^{-1}S_1^{a_2}\alpha \cdots \alpha^{-1}S_1^{a_n}\alpha$ with n even and $\alpha S_2^{a_1}\alpha^{-1}S_1^{a_2}\alpha \cdots \alpha S_2^{a_n}\alpha^{-1}$ with n odd by $\alpha(a_1, a_2, \cdots, a_n)$, $a_i \in Z \cup \{\infty\}$. For $n=0$, we interpret $\alpha(a_1, a_2, \cdots, a_n)$ as α. Here S_1, S_2 are elementary 3-braids and S_1^{∞}, S_2^{∞} are braid-like elements as shown in Fig. 1. If α is a pure 3-braid and a_i is even, then we denote an oriented 2-bridge knot and an oriented 2-bridge link as shown in Fig. 2 by $K_{\alpha(a_1, a_2, \cdots, a_n)}$. The notation is the same as in [4].

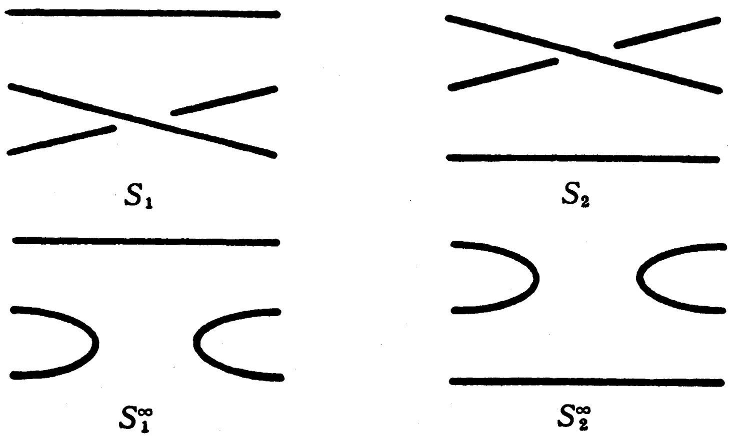
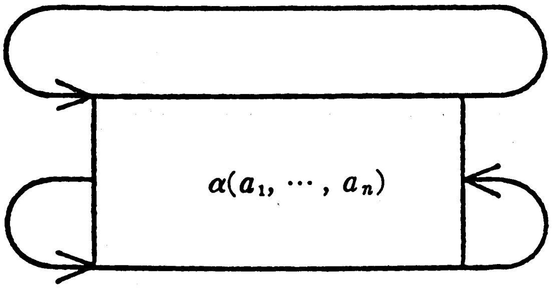
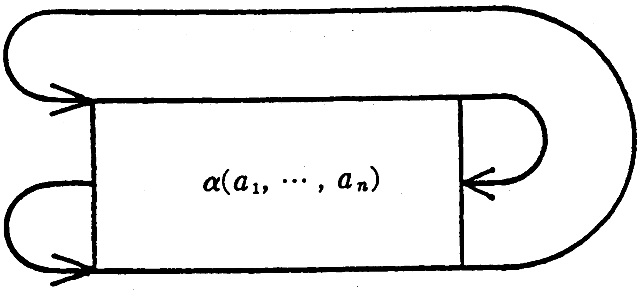


Fig. 1.



n odd



n even

Fig. 2.

Proposition 1. *If α is a pure 3-braid, then the 2-bridge knots or links $K_{\alpha(2b_1, 2b_2, \cdots, 2b_n)}$ and $K_{\alpha(2b_n, \cdots, 2b_2, 2b_1)}$ are skein equivalent.*

Proof. We prove by induction on n. The case $n=2$ is [4, Proposition 3.2]. Suppose that $K_{\alpha(2b_1, 2b_2, \cdots, 2b_n)}$ and $K_{\alpha(2b_n, \cdots, 2b_2, 2b_1)}$ are skein equivalent for $n \leqq k-1$, where $k \geqq 3$. For $\alpha = S_2^{p_1} S_1^{q_1} \cdots S_2^{p_t} S_1^{q_t}$, we define $\tilde{\alpha} = S_2^{-q_t} S_1^{-p_t} \cdots S_2^{-q_1} S_1^{-p_1}$. We have skein triples $(K_{\alpha(2b_1, \cdots, 2b_{k-2}, 2b_{k-1}-2, 2b_k)}, K_{\alpha(2b_1, \cdots, 2b_{k-2}, 2b_{k-1}, 2b_k)}, K_{\alpha(2b_1, \cdots, 2b_{k-2})} \# K_{\beta(2b_k)})$ and $(K_{\alpha(2b_k, 2b_{k-1}-2, 2b_{k-2}, \cdots, 2b_1)}, K_{\alpha(2b_k, 2b_{k-1}, 2b_{k-2}, \cdots, 2b_1)}, K_{\alpha(2b_k)} \# K_{\alpha(2b_{k-2}, \cdots, 2b_1)})$, where β is either α or $\tilde{\alpha}$ according as if k is odd or even. We can prove that $K_{\alpha(2b_k)}$ and $K_{\tilde{\alpha}(2b_k)}$ are skein equivalent by induction on b_k. If $b_{k-1}=0$, then $K_{\alpha(2b_1, \cdots, 2b_{k-2}, 2b_{k-1}, 2b_k)} = K_{\alpha(2b_1, \cdots, 2b_{k-2}+2b_k)}$ and $K_{\alpha(2b_k, 2b_{k-1}, 2b_{k-2}, \cdots, 2b_1)} = K_{\alpha(2b_k + 2b_{k-2}, \cdots, 2b_1)}$ are skein equivalent by the inductive hypothesis. Thus by induction on b_{k-1}, we can prove for the case $n=k$. This completes the proof.

Let $\nabla_{\alpha(2b_1, 2b_2, \cdots, 2b_n)}$ be the Conway potential function of the 2-bridge link $K_{\alpha(2b_1, 2b_2, \cdots, 2b_n)}$, where α is a pure 3-braid and n is odd.

Proposition 2. $\nabla_{\alpha(2b_1, 2b_2, \cdots, 2b_n)} = \nabla_{\alpha(2b_n, \cdots, 2b_2, 2b_1)}$.

Proof. Since $\nabla_{\alpha(2b_1, \cdots, 2b_{n-2}, 2b_{n-1}, 2b_n)} = \nabla_{\alpha(2b_1, \cdots, 2b_{n-2}, 2b_{n-1}-2, 2b_n)} - (t_1 - t_1^{-1})(t_2 - t_2^{-1}) \nabla_{\alpha(2b_1, \cdots, 2b_{n-2})} \nabla_{\alpha(2b_n)} = \nabla_{\alpha(2b_1, \cdots, 2b_{n-2}+2b_n)} - b_{n-1}(t_1 - t_1^{-1})(t_2 - t_2^{-1}) \nabla_{\alpha(2b_1, \cdots, 2b_{n-2})} \nabla_{\alpha(2b_n)}$ by [1, p. 338], we can prove by induction on n.

Let $\tilde{B}_3$ be the subset of the 3-braid group B_3 consisting of the 3-braids of the form $(S_2^{b_1}S_1^{b_2}\cdots)(\cdots S_2^{-b_2}S_1^{-b_1})$. Let $\Lambda_{\alpha(a_1, a_2, \cdots, a_n)}$ be the L polynomial of the unoriented diagram as shown in Fig. 3.

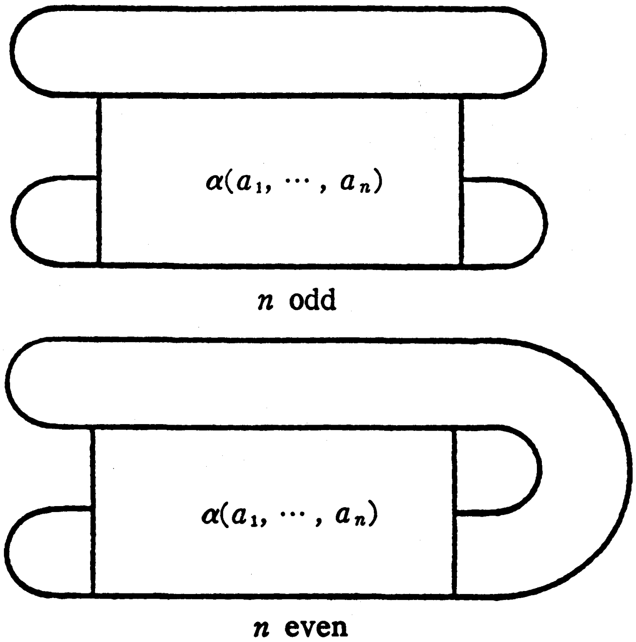


Fig. 3.

Proposition 3. *If $\beta \in \tilde{B}_3$ and $p_i \in Z$ for each i, then*

$$\Lambda_{\beta(p_1, p_2, \cdots, p_n)} = \Lambda_{\beta(p_n, \cdots, p_2, p_1)}. \tag{1}$$

Proof. We prove by induction on n. For $n=2, 3$, (1) is true by [4, Proposition 4.1]. First, supposing that (1) holds for $n \leqq k-1$, where $k \geqq 3$, we prove the following special form of (1) for $j \geqq 0$:

$$\Lambda_{\beta(p_1, \cdots, p_i, \underbrace{1, \cdots, 1}_{j}, q_i, \cdots, q_1)} = \Lambda_{\beta(q_1, \cdots, q_i, \underbrace{1, \cdots, 1}_{j}, p_i, \cdots, p_1)}, \tag{2}$$

where $k=2i+j$, $q_1=p_k$, $q_2=p_{k-1}$, $\cdots$, $q_i=p_{k-i+1}$. For $i=1$, (2) is true by [4, Proposition 4.1], so we assume that $i>1$. If either p_i or q_i is either 0 or ∞, then (2) holds by inductive hypothesis. Thus by [4, Proposition 2.2] and induction on i, (2) is proved. This also proves (1) for the case $n=k$ is even, and for the case $n=k$ is odd and $p_{(k+1)/2}=1$, and so we consider the case $n=k$ is odd and $p_{(k+1)/2} \neq 1$. If $p_{(k+1)/2}$ is either 0 or ∞, then (1) holds by inductive hypothesis, thus by [4, Proposition 2.2], (1) holds in this case. This completes the proof.

For a 3-braid β, we define a 3-braid $\beta\langle p_1, p_2, \cdots, p_n\rangle$, $p_i \in Z$, as follows:

$$\beta\langle p_1\rangle = \beta(p_1, -p_1),$$

$$\beta\langle p_1, p_2, \cdots, p_{n-1}, p_n\rangle=\beta\langle p_1, p_2, \cdots, p_{n-1}\rangle\langle p_n\rangle.$$

Note that if $\beta\in\tilde{B}_3$, then $\beta\langle p_1, p_2, \cdots, p_n\rangle\in\tilde{B}_3$. It is easy to see the following:

Lemma 1.

$$\beta\langle p_1, p_2, \cdots, p_n\rangle=\beta\langle p_1, p_2, \cdots, p_{i-1}\rangle\langle p_i, \cdots, p_n\rangle, \qquad 2\leqq i\leqq n.$$

Let $p_1, p_2, \cdots, p_n, q_1, q_2, \cdots, q_m$ be integers, $m=3^n-1$, satisfying $p_i=\varepsilon_j q_j$, where $j\equiv 0 \pmod{3^{i-1}}$, $j\not\equiv 0 \pmod{3^i}$, and

$$\varepsilon_j=\begin{cases} 1 & \text{if } j/3^{i-1}\equiv 1 \pmod 3; \\ -1 & \text{if } j/3^{i-1}\equiv 2 \pmod 3. \end{cases}$$

Then $\beta\langle p_1, p_2, \cdots, p_n\rangle = \beta(q_1, q_2, \cdots, q_m)$. Thus $\beta\langle -p_1, -p_2, \cdots, -p_n\rangle = \beta(q_m, \cdots, q_2, q_1)$. Also we have $\beta\langle p_1, p_2, \cdots, p_n\rangle(p)=\beta(q_1, q_2, \cdots, q_m, p, -q_m, \cdots, -q_2, -q_1)$ and $\beta\langle -p_1, -p_2, \cdots, -p_n\rangle(p)=\beta(-q_1, -q_2, \cdots, -q_m, p, q_m, \cdots, q_2, q_1)$. Therefore from Propositions 1-3, we obtain:

Lemma 2. (i) *If α is a pure 3-braid and $b_i, b\in Z$, then*

$$K_{\alpha<2b_1, 2b_2, \cdots, 2b_n>}\sim K_{\alpha<-2b_1, -2b_2, \cdots, -2b_n>}, \quad \textit{and}$$

$$K_{\alpha<2b_1, 2b_2, \cdots, 2b_n>(2b)}\sim K_{\alpha<-2b_1, -2b_2, \cdots, -2b_n>(2b)}.$$

(ii) *If α is a pure 3-braid and $b_i, b\in Z$, then*

$$\nabla_{\alpha<2b_1, 2b_2, \cdots, 2b_n>(2b)}=\nabla_{\alpha<-2b_1, -2b_2, \cdots, -2b_n>(2b)}.$$

(iii) *If $\beta\in\tilde{B}_3$ and $p_i, p\in Z$, then*

$$\Lambda_{\beta<p_1, p_2, \cdots, p_n>}=\Lambda_{\beta<-p_1, -p_2, \cdots, -p_n>}, \quad \textit{and}$$

$$\Lambda_{\beta<p_1, p_2, \cdots, p_n>(p)}=\Lambda_{\beta<-p_1, -p_2, \cdots, -p_n>(p)}.$$

2. Proofs of Theorems

Proof of Theorem 1. Let $\beta=(S_2^{2\delta_1}S_1^{2\delta_2}\cdots)(\cdots S_2^{-2\delta_2}S_1^{-2\delta_1})$ be a nontrivial 3-braid in $\tilde{B}_3$, $\delta_i=\pm 1$. Then all the 2^N knots in $\mathcal{K}_{\beta,N}=\{K_{\beta<2\varepsilon_1, 2\varepsilon_2, \cdots, 2\varepsilon_N>}\mid \varepsilon_i=\pm 1\}$ are mutually distinct (cf. [4, Lemma 3.1]), amphicheiral (cf. [4, Lemma 3.2]), and fibered (cf. [4, Lemma 3.3]). From Lemma 1 and Lemma 2 (i), we have:

$$\begin{aligned} K_{\beta<2\varepsilon_1, 2\varepsilon_2, \cdots, 2\varepsilon_N>} &= K_{\beta<2\varepsilon_1, 2\varepsilon_2, \cdots, 2\varepsilon_{i-1}><2\varepsilon_i, \cdots, 2\varepsilon_N>} \\ &\sim K_{\beta<2\varepsilon_1, 2\varepsilon_2, \cdots, 2\varepsilon_{i-1}><-2\varepsilon_i, -2\varepsilon_{i+1}, \cdots, -2\varepsilon_N>} \\ &= K_{\beta<2\varepsilon_1, 2\varepsilon_2, \cdots, 2\varepsilon_{i-1}, -2\varepsilon_i><-2\varepsilon_{i+1}, \cdots, -2\varepsilon_N>} \\ &\sim K_{\beta<2\varepsilon_1, 2\varepsilon_2, \cdots, 2\varepsilon_{i-1}, -2\varepsilon_i><2\varepsilon_{i+1}, \cdots, 2\varepsilon_N>} \end{aligned}$$

$$=K_{\beta<2\varepsilon_1,2\varepsilon_2,\cdots,2\varepsilon_{i-1},-2\varepsilon_i,2\varepsilon_{i+1},\cdots,2\varepsilon_N>}$$

In the same way, we can prove:

$$\Lambda_{\beta<2\varepsilon_1,2\varepsilon_2,\cdots,2\varepsilon_N>}=\Lambda_{\beta<2\varepsilon_1,2\varepsilon_2,\cdots,2\varepsilon_{i-1},-2\varepsilon_i,2\varepsilon_{i+1},\cdots,2\varepsilon_N>}$$

Thus $\mathcal{K}_{\beta,N}$ is a desired set of 2-bridge knots.

Proof of Theorem 2. Let β be as in the above. Then the set of 2-bridge links $\mathcal{L}_{\beta,N}=\{K_{\beta<2\varepsilon_1,2\varepsilon_2,\cdots,2\varepsilon_N>(2)}\mid\varepsilon_i=\pm1\}$ is a desired one. The proof is similar to the above.

Remark. In a similar way, we can prove that the 2-bridge links in the above $\mathcal{L}_{\beta,N}$ have the same dichromatic link invariant defined by Hoste and Kidwell [2], which is a extension of both the 2-variable Jones and the 2-variable Alexander polynomials. This invariant also extends the dichromatic link invariant defined by Hoste and Przytycki [3], which specializes to the Jones polynomial.

3. Closed 3-braids

Let α be a 3-braid and let $\alpha[2n;p_1,p_2,\cdots,p_{2m}]$ be the closed 3-knit $(\Delta^{2n}\alpha S_2^{p_1}\alpha^{-1}S_1^{p_2}\alpha\cdots\alpha^{-1}S_1^{p_{2m}})^\wedge$, where $\Delta^2=(S_1S_2S_1)^2$ is a generator of the center of B_3, $n\in Z$, $p_i\in Z\cup\{\infty\}$.

Proposition 4. *If α is a 3-braid and $n, p_i\in Z$, then $\alpha[2n;p_1,p_2,\cdots,p_{2m}]$ and $\alpha[2n;p_{2m},\cdots,p_2,p_1]$ are skein equivalent.*

Proof. We prove by induction on m. The case $m=1$ is [4, Proposition 5.1]. Supposing that the proposition is true for $m\leqq k-1$, $k\geqq 3$, we prove:

$$\alpha[2n;p_1,\cdots,p_i,\cdots,p_{2k}]\sim\alpha[2n;p_{2k},\cdots p_i,\cdots,p_1] \tag{3}$$

If $p_i=0$, then (3) holds by the inductive hypothesis. From the skein triples: $(\alpha[2n;p_1,\cdots,p_i,\cdots,p_{2k}],\ \alpha[2n;p_1,\cdots,p_i-2,\cdots,p_{2k}],\ \alpha[2n;p_1,\cdots,p_i-1,\cdots,p_{2k}])$, and $(\alpha[2n;p_{2k},\cdots,p_i,\cdots,p_1],\ \alpha[2n;p_{2k},\cdots,p_i-1,\cdots,p_1],\ \alpha[2n,p_{2k},\cdots,p_i-2,\cdots,p_1])$, if (3) holds for $p_i=1$, then (3) holds for all $p_i\in Z$. When $p_i=1$ for all i, (3) is trivial. Thus the proof is complete.

Proposition 5. *If $\beta\in\tilde{B}_3$, and $n, p_i\in Z$, then*

$$\Lambda_{\beta[2n;p_1,p_2,\cdots,p_{2m}]}=\Lambda_{\beta[2n;p_{2m},\cdots,p_2,p_1]} \tag{4}$$

(Note that $F_{\beta[2n;p_1,p_2,\cdots,p_{2m}]}=a^{-(6n+p_1+p_2+\cdots+p_{2m})}\Lambda_{\beta[2n;p_1,p_2,\cdots,p_{2m}]}$.)

Proof. We prove by induction on m. The case $m=1$ is trivial and the case $m=2$ is [4, Proposition 5.5]. Supposing that the proposition is true for $m\leqq k-1$, $k\geqq 3$, we prove (4) for $m=k$. If $p_i=0$, then (4) is true by the inductive hypothesis. If $p_i=\infty$, then

$$\Lambda_{\beta[2n;\, p_1, p_2, \cdots, p_{2k}]}=a^{-2n}\Lambda_{\beta\langle p_{i+1}, \cdots, p_{2k}, p_1, \cdots, p_{i-1}\rangle}$$

and

$$\Lambda_{\beta[2n;\, p_{2k}, \cdots, p_2, p_1]}=a^{-2n}\Lambda_{\beta\langle p_{i-1}, \cdots, p_1, p_{2k}, \cdots, p_{i+1}\rangle},$$

which equal by Proposition 3. Thus from [4, Proposition 2.2], if (4) holds for $p_i=1$, then (4) holds for all $p_i\in Z$. When $p_i=1$ for all i, then (4) is trivial, and so the proof is complete.

For the integers $p_1, p_2, \cdots, p_{2m}$, q, $r_1, r_2, \cdots, r_{6m}$, if

$$r_i=\begin{cases} p_{i/3} & \text{if } i\equiv 0 \pmod 3, \\ q & \text{if } i\equiv 1 \pmod 3, \\ -q & \text{if } i\equiv 2 \pmod 3, \end{cases}$$

then $\alpha\langle q\rangle[2n;\, p_1, p_2, \cdots, p_{2m}]=\alpha[2n;\, r_1, r_2, \cdots, r_{6m}]$, where $\alpha\in B_3$ and $n\in Z$. Hence $\alpha\langle -q\rangle[2n;\, p_{2m}, \cdots, p_2, p_1]=\alpha[2n;\, r_{6m}, \cdots, r_2, r_1]$. Thus if $\beta\in\tilde{B}_3$, then from Propositions 4 and 5 the following four closed 3-braids are equivalent and have the same Kauffman polynomial:

$$\beta\langle\pm q\rangle[2n;\, p_1, p_2, \cdots, p_{2m}], \qquad \beta\langle\pm q\rangle[2n;\, p_{2m}, \cdots, p_2, p_1].$$

In general, we have:

Proposition 6. *If $\beta\in\tilde{B}_3$, and $p_i, q_j\in Z$, then all the closed 3-braids in the following set are skein equivalent and have the same Kauffman polynomial*

$$\{\beta\langle\varepsilon_1 q_1, \varepsilon_2 q_2, \cdots, \varepsilon_l q_l\rangle[2n;\, p_1, p_2, \cdots, p_{2m}],$$
$$\beta\langle\varepsilon_1 q_1, \varepsilon_2 q_2, \cdots, \varepsilon_l q_l\rangle[2n;\, p_{2m}, \cdots, p_2, p_1];\ \varepsilon_j=\pm 1\}.$$

However, the classification of the above closed 3-braids is not easy.

Example. Let $\beta=S_2^2S_1^{-2}$. Then by the computer calculation, we can show the Jones polynomials of the 3-parallel links of the following four closed 3-braids are mutually distinct using [9, Corollary 4.4.9]:

$$\beta\langle\pm 1\rangle[0;\, 3, -1, 1, 1], \qquad \beta\langle\pm 1\rangle[0;\, 1, 1, -1, 3].$$

4. Tables

Tables 1 and 2 give all the pairs of 2-bridge knots and links through 20 crossings sharing the same Kauffman polynomial. We list only one member from each pair putting in the form $K_{\alpha_i\langle p_1, p_2, \cdots, p_n\rangle}$, where $\alpha_1=S_2^2S_1^{-2}$,

Table 1.

$K_{\alpha_1(p,q)}$

cr	p	q	det	
13	2	1	245	
	2	−1	255	
14	2	−2	505	P
15	4	1	495	
	4	−1	505	
	3	2	745	
	3	−2	755	
16	4	2	995	P
	4	−2	1005	P
17	6	1	745	
	6	−1	755	
	5	2	1245	
	5	−2	1255	
	4	3	1495	
	4	−3	1505	
18	6	2	1495	P
	6	−2	1505	P
	4	−4	2005	P
19	8	1	995	
	8	−1	1005	
	7	2	1745	
	7	−2	1755	
	6	3	2245	
	6	−3	2255	
	5	4	2495	
	5	−4	2505	
20	8	2	1995	P
	8	−2	2005	P
	6	4	2995	P
	6	−4	3005	P

$K_{\alpha_1(p,q,r)}$

cr	p	q	r	det	
16	1	1	−1	625	
17	2	1	1	1175	
	2	−1	−1	1225	
	2	1	−1	1275	
	3	−1	1	1325	
18	3	1	1	1775	
	3	−1	−1	1825	
	3	1	−1	1925	
	3	−1	1	1975	
	1	3	−1	1875	
	2	2	1	2425	P
	2	−2	−1	2475	P
	2	2	−1	2525	P
	2	−2	1	2575	P
19	4	1	1	2375	
	4	−1	−1	2425	
	4	1	−1	2575	
	4	−1	1	2625	
	3	1	2	3625	
	3	−1	−2	3725	
	3	1	−2	3775	
	3	−1	2	3875	
	2	3	1	3675	
	2	−3	−1	3725	
	2	3	−1	3775	
	2	−3	1	3875	
20	5	1	1	2975	
	5	−1	−1	3025	
	5	1	−1	3225	
	5	−1	1	3275	
	1	5	−1	3125	
	4	2	1	4875	P
	4	−2	−1	4925	P
	4	2	−1	5075	P
	4	−2	1	5125	P
	2	4	1	4925	P
	2	−4	−1	4975	P
	2	4	−1	5025	P
	2	−4	1	5075	P
	3	3	1	5525	
	3	−3	−1	5575	
	3	3	−1	5675	
	3	−3	1	5725	
	3	2	2	7375	P
	3	−2	−2	7475	P
	3	2	−2	7525	P
	3	−2	2	7625	P

Table 1 (continued)

$K_{\alpha_1(p,q,r,s)}$

cr	p	q	r	s	det	
20	1	1	1	−1	2995	
	1	1	−1	−1	3005	☆
	1	1	−1	1	3245	
	1	−1	1	−1	3505	☆

$K_{\alpha_2(p,q)}$

cr	p	q	det	
19	2	1	4381	
	2	−1	4407	
20	2	−2	8801	P

Table 2.

$K_{\alpha_1(p,q)}$

cr	p	q	det
12	1	−1	130
14	3	1	370
	3	−1	380
16	5	1	620
	5	−1	630
	3	−3	1130
18	7	1	870
	7	−1	880
	5	3	1870
	5	−3	1880
20	9	1	1120
	9	−1	1130
	7	3	2620
	7	−3	2630
	5	−5	3130

$K_{\alpha_1(p,q,r)}$

cr	p	q	r	det	
17	1	2	−1	1250	P
18	2	1	−2	2500	
19	1	4	−1	2500	P
	3	2	1	3650	P
	3	−2	−1	3700	
	3	2	−1	3800	
	3	−2	1	3850	
	2	2	−2	5000	P A
20	4	1	2	4850	
	4	−1	−2	4950	
	4	1	−2	5050	
	4	−1	2	5150	
	2	3	−2	7500	

$K_{\alpha_2(p,q)}$

cr	p	q	det
18	1	−1	2210
20	3	1	6578
	3	−1	6604

$K_{\alpha_3(p,q)}$

cr	p	q	det	
18	1	−1	1010	P A
19	2	1	1990	P A
	2	−1	2010	P A
20	3	1	2990	P A
	3	−1	3010	P A
	2	−2	4010	P A

$\alpha_2=S_2^2S_1^2S_2^{-2}S_1^{-2}$, $\alpha_3=S_2^3S_1^{-3}$. We also list the crossing number (=cr) and the determinants (=det). An entry "P" (resp. "A") indicates that the pair are skein equivalent (resp. have the same 2-variable Alexander polynomial) endowed with suitable orientation. An entry "☆" indicates that the pair is not applicable to [4, Proposition 4.1].

References

[1] J. H. Conway: *An enumeration of knots and links*, Computational Problems in Abstract Algebra (J. Leech, ed.), Pergamon Press 1969, 329–358.

[2] J. Hoste and M.E. Kidwell: *Dichromatic link invariants*, Trans. Amer. Math. Soc. **321** (1990), 197-229.
[3] J. Hoste and J.H. Przytycki: *An invariant of dichromatic links*, Proc. Amer. Math. Soc. **105** (1989), 1003-1007.
[4] T. Kanenobu: *Examples on polynomial invariants of knots and links* II, Osaka J. Math. **26** (1989), 465-482.
[5] T. Kanenobu and T. Sumi: *Polynomial invariants of 2-bridge knots through 22 crossings*, preprint.
[6] T. Kanenobu and T. Sumi: *Polynomial invariants of 2-bridge links through 20 crossings*, to appear in Advanced Studies in Pure Math. 20 (Aspects of low dimensional manifolds).
[7] L.H. Kauffman: *On Knots*, Ann. of Math. Studies 115, Princeton University Press, Princeton, 1987.
[8] W.B.R. Lickorish and K.C. Millett: *A polynomial invariant of oriented links*, Topology **26** (1987), 107-141.
[9] J. Murakami: *The parallel version of polynomial invariants of links*, Osaka J. Math. **26** (1989), 1-55.

Department of Mathematics
Kyushu University 33
Fukuoka 812
Japan

Current Address:
Department of Mathematics
Osaka City University
Osaka 558
Japan