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# CLT AND LIL FOR STATIONARY LINEAR PROCESSES GENERATED BY MARTINGALES

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**Abstract.** In this paper, we study a general central limit theorem and a general law of the iterated logarithm for partial sums of a generalized linear process $X_n=\sum_{j=-\infty}^{\infty}\alpha_{n-j}\varepsilon_j$, where $\{\alpha_n\}$ is a sequence of invariant random variables and $\{\varepsilon_n\}$ is a strictly stationary sequence of martingale differences.

**1. Introduction.** Let $\Omega$ be the Cartesian product $\prod_{n=-\infty}^{\infty}\boldsymbol{R}$ of copies of the real line, and let $\varepsilon_n : \Omega\to\boldsymbol{R}$ be the coordinate functions $\varepsilon_n(\omega)=\omega_n$, $\omega=\{\omega_n\}$. $\Omega$ is given the $\sigma$-field $\mathcal{F}$ which is the smallest $\sigma$-field containing all sets $\varepsilon_n^{-1}B$, with $B$ a Borel subset of $\boldsymbol{R}$. Let $T$ be the shift transformation, $T\{\omega_n\}=\{\omega_{n+1}\}$. We consider any $T$-invariant probability $P$ (i.e. $P(TA)=P(A)$ for all $A\in\mathcal{F}$) defined on $(\Omega, \mathcal{F})$. Then $\{\varepsilon_n, -\infty<n<\infty\}$ is a (strictly) stationary sequence on $(\Omega, \mathcal{F}, P)$ and any stationary sequence has the coordinate representation as just described, see, for example, Stout (1974). Let $\mathcal{I}$ denote the invariant $\sigma$-field; that is, $\mathcal{I}$ is the collection of $A\in\mathcal{F}$ such that $TA=A$. If for all $A\in\mathcal{I}$ either $P(A)=0$ or $P(A)=1$, then $P$ is said to be *ergodic*. Let $\mathcal{F}_n$ be the sub-$\sigma$-field of $\mathcal{F}$ generated by the $\varepsilon_m$; $m\leqq n$, and write $\mathcal{F}_{-\infty}=\bigcap_{n=-\infty}^{\infty}\mathcal{F}_n$. Since $\mathcal{F}_n\uparrow\mathcal{F}$ (i.e. $\bigcup_{n=-\infty}^{\infty}\mathcal{F}_n$ generates $\mathcal{F}$), from the entropic ergodic theory it follows in general that $\mathcal{I}\subset\mathcal{F}_{-\infty} \bmod P$ (which means that for every $A_1\in\mathcal{I}$, there exists $A_2\in\mathcal{F}_{-\infty}$ such that $P(A_1\Delta A_2)=0$, where $A_1\Delta A_2=A_1\cup A_2-A_1\cap A_2$), see Parry (1981), p. 68; evidently $\mathcal{I}\subset\mathcal{P}(T)$ where $\mathcal{P}(T)$ is the Pinsker $\sigma$-field. We shall say that $P$ is *weakly regular* if $\mathcal{F}_{-\infty}=\mathcal{I} \bmod P$ (i.e. $\mathcal{F}_{-\infty}\subset\mathcal{I} \bmod P$ as well). On the other hand, $P$ is called *regular* if $P(A)=0$ or $P(A)=1$ for all $A\in\mathcal{F}_{-\infty}$, see Ibragimov and Linnik (1971). Clearly $P$ is regular if and only if $P$ is ergodic and weakly regular.

In this paper we shall be concerned with a stationary process on $(\Omega, \mathcal{F}, P)$ which may be represented in the form

$$X_n=\sum_{j=-\infty}^{\infty}\alpha_{n-j}\varepsilon_j, \tag{1}$$

where the $\alpha_j$ are $\mathcal{I}$-measurable random variables such that

(2) $$\sum_{j=-\infty}^{\infty} \alpha_j^2 < \infty \text{ a.s., and}$$

(3) $$E(\varepsilon_0 | \mathcal{J}_{-1}) = 0 \text{ a.s. and } E(\varepsilon_0^2 | \mathcal{J}) < \infty \text{ a.s.}$$

Since $E(\varepsilon_n | \mathcal{F}_{n-1}) = E(\varepsilon_0 | \mathcal{F}_{-1}) \circ T^n = 0$ a.s. for all $n$, $\{\varepsilon_n, \mathcal{F}_n, -\infty < n < \infty\}$ forms a stationary sequence of (not necessarily square integrable) martingale differences. We note that the conditional expectation $E(\varepsilon_0^2 | \mathcal{J})$ is defined as an $\mathcal{J}$-measurable function even when $E\varepsilon_0^2 = \infty$, and that if the measure $\lambda(A) = \int_A \varepsilon_0^2 dP$, $A \in \mathcal{J}$, is $\sigma$-finite, then $E(\varepsilon_0^2 | \mathcal{J}) < \infty$ a.s. by the Radon-Nikodym theorem. Given any $\sigma$-field $\mathcal{C} \subset \mathcal{F}$, there exists (see Loève (1978), pages 19 ff.) a family of regular conditional probabilities $(P_\omega : \omega \in \Omega)$ on $\mathcal{F}$ induced by $\mathcal{C}$. That is, $P_\omega(A)$ is a function defined for $A \in \mathcal{F}$ and $\omega \in \Omega$ such that

(a) for each $A \in \mathcal{F}$, $P_\omega(A)$ is a version of $P(A | \mathcal{C})$, and

(b) for each $\omega \in \Omega$, $P_\omega(\cdot)$ is a probability measure on $\mathcal{F}$.

We consider only the case $\mathcal{C} = \mathcal{J}$. In this case, any $\mathcal{J}$-measurable function $X$ is $P_\omega$-almost surely equal to $X(\omega)$ for $P$-almost all $\omega \in \Omega$. Hence for any $\mathcal{F}$-measurable function $Y$ with $|EY| \leq \infty$, we have $E(Y | \mathcal{J}) = \int Y dP_\omega$ $P_\omega$-a.s. for $P$-almost all $\omega \in \Omega$. Moreover, $P_\omega$ is a $T$-invariant and ergodic measure on $(\Omega, \mathcal{F})$ and because of (3), $\{\varepsilon_n\}$ is a stationary sequence of square integrable martingale differences on $(\Omega, \mathcal{F}, P_\omega)$, at least for $P$-almost all $\omega \in \Omega$, see Eagleson (1975) and Volný (1987). In view of (2) and the remark stated here, the series in (1) converges almost surely on $(\Omega, \mathcal{F}, P_\omega)$—at least for $P$-almost all $\omega \in \Omega$, and hence $X_n$ is well defined on $(\Omega, \mathcal{F}, P)$, since $P(A) = \int P_\omega(A) P(d\omega) = 1$, where $A = [\sum_{j=-\infty}^{\infty} \alpha_{n-j} \varepsilon_j$ converges$]$.

We shall obtain in this paper a general central limit theorem for $S_n = \sum_{i=1}^n X_i$ and the corresponding iterated logarithm law which provides information on the rate of almost sure convergence of $S_n$. The ergodic version of Theorem 1 was given by Heyde (1974). Theorem 2 appears to be new even in the ergodic case.

Under the assumptions (2) and (3) define

(4) $$f(\lambda) = (2\pi)^{-1} E(\varepsilon_0^2 | \mathcal{J}) \left| \sum_{j=-\infty}^{\infty} \alpha_j e^{i\lambda j} \right|^2, \qquad -\pi \leq \lambda \leq \pi .$$

In view of the remark stated above, $f_\omega(\lambda) = [f(\lambda)](\omega)$ is a spectral density of the process $\{X_n\}$ considered on $(\Omega, \mathcal{F}, P_\omega)$ for $P$-almost all $\omega \in \Omega$.

**Theorem 1.** *Suppose that* (2) *and* (3) *hold and* $f(\lambda)$ (*given by* (4)) *is continuous at* $\lambda = 0$ *with probability one. Then* $n^{-1/2} S_n$ *converges in distribution as* $n \to \infty$ *to a limit law with characteristic function*

$$\varphi(t)=\int \exp[-\pi f(0)t^2]dP.$$

**Theorem 2.** *Suppose in addition to the assumptions of Theorem* 1 *that*

$$E(|\varepsilon_0|^r|\mathcal{I})<\infty \text{ a.s. } \textit{for some } r>2, \textit{ and}$$

$$E(\varepsilon_0^2|\mathcal{F}_{-m})\leqq CE(\varepsilon_0^2|\mathcal{I}) \text{ a.s. } \textit{for some positive integer } m \textit{ and some } C<\infty.$$

*Then*

$$\limsup_{n\to\infty} S_n/(2n\log\log n)^{1/2}=(2\pi f(0))^{1/2} \text{ a.s.}$$

The last assumption of Theorem 2 is satisfied if $P$ is weakly regular and $E(\varepsilon_0^2|\mathcal{F}_{-n})$ converges uniformly as $n\to\infty$, by means of the decreasing martingale theorem.

A process of the kind (1) is widely used in time series analysis. In this context, a stationary sequence $\{X_n\}$ will be given first and the $\varepsilon_n$ will be given as functions of the $X_n$. In this case, we shall work with the coordinate representation of $\{X_n\}$. The results still hold, since the proofs are essentially based on the existence of regular conditional probabilities. As an example, let $\{X_n\}$ be the coordinate representation of a stationary sequence. Assume that $EX_0=0$ and $E(X_0^2|\mathcal{I})<\infty$ a.s. and let $\varepsilon_n=X_n-E(X_n|\mathcal{F}_{n-1})$ be the prediction errors (here $\mathcal{F}_n$ is the $\sigma$-field generated by the $X_m$; $m\leqq n$). Then $\{\varepsilon_n\}$ is a stationary martingale difference sequence with $E(\varepsilon_0^2|\mathcal{I})<\infty$ a.s. As usual, let $(P_\omega:\omega\in\Omega)$ be a family of regular conditional probabilities on $\mathcal{F}$ induced by $\mathcal{I}$, and $E_\omega(\cdot)$ denote the expectation with respect to $P_\omega$. Then we have $\varepsilon_n=X_n-E_\omega(X_n|\mathcal{F}_{n-1})$ $P_\omega$-a.s. for $P$-almost all $\omega\in\Omega$, see Eagleson (1975). If $P$ is weakly regular, then for $P$-almost all $\omega\in\Omega$, $P_\omega$ is regular (i.e. $\{X_n\}$ is a stationary purely non-deterministic process on $(\Omega, \mathcal{F}, P_\omega)$ if $E_\omega(\varepsilon_0^2)>0$, see, for example, Hannan (1970)) and so

$$X_n=\sum_{j=-\infty}^{n}\alpha_{n-j}\varepsilon_j \qquad P_\omega\text{-a.s.}$$

where $\alpha_j=E_\omega^{-1}(\varepsilon_0^2)E_\omega(X_n\varepsilon_{n-j})$; $=0$ if $E_\omega(\varepsilon_0^2)=0$, and satisfies $\sum_{j=0}^{\infty}\alpha_j^2<\infty$. Therefore $X_n$ can be written as

$$X_n=\sum_{j=-\infty}^{n}\alpha_{n-j}\varepsilon_j \quad \text{a.s.}$$

where $\alpha_j(\omega)=[E^{-1}(\varepsilon_0^2|\mathcal{I})E(X_n\varepsilon_{n-j}|\mathcal{I})](\omega)$; $=0$ if $E(\varepsilon_0^2|\mathcal{I})(\omega)=0$, and satisfies $\sum_{j=0}^{\infty}\alpha_j^2<\infty$ a.s. Let $\mathcal{G}_n$ be the $\sigma$-field generated by the $\varepsilon_m$; $m\leqq n$. Then it is easy to see that $\mathcal{F}_n=\sigma(\mathcal{G}_n\cup\mathcal{I}) \bmod P$. Thus if in addition $P$ is ergodic, then the best linear predictor is the best predictor (both being best in the least squares sense; see Hannan and Heyde (1972)), and this remains true on $(\Omega, \mathcal{F}, P_\omega)$

for $P$-almost all $\omega\in\Omega$ even if $P$ is not ergodic.

**2. Proofs. Proof of Theorem 1.** The following lemma has proved to be of considerable use in obtaining general central limit theorems from their ergodic versions, see Volný (1984, 1987).

**Lemma.** *Let $X$ be a random variable on $(\Omega, \mathcal{F}, P)$ and set $s_n(\omega)=n^{-1/2}\sum_{i=1}^{n}X(T^i\omega)$ for $\omega\in\Omega$. Let $(P_\omega:\omega\in\Omega)$ be a family of regular conditional probabilities on $\mathcal{F}$ induced by $\mathcal{I}$. If $s_n$, considered on $(\Omega, \mathcal{F}, P_\omega)$, converges in distribution as $n\to\infty$ to a limit law with characteristic function $\varphi_\omega(t)$ for $P$-almost all $\omega\in\Omega$, then $s_n$, considered on $(\Omega, \mathcal{F}, P)$, converges in distribution as $n\to\infty$ to a limit law with characteristic function*

$$\varphi(t)=\int\varphi_\omega(t)P(d\omega).$$

As we have remarked in Section 1, at least for $P$-almost all $\omega\in\Omega$, $\{\varepsilon_n\}$ forms a stationary ergodic sequence of square integrable martingale differences on $(\Omega, \mathcal{F}, P_\omega)$, and $f_\omega(\lambda)$ is a spectral density of the process $\{X_n\}$ considered on $(\Omega, \mathcal{F}, P_\omega)$. $f(\lambda)$ being continuous at $\lambda=0$ with probability one, by a result of Heyde (1974), $n^{-1/2}S_n$, considered on $(\Omega, \mathcal{F}, P_\omega)$, converges in distribution to the normal law with mean zero and variance $2\pi f_\omega(0)$ for $P$-almost all $\omega\in\Omega$, from which and the above lemma the desired conclusion follows.

**Proof of Theorem 2.** Put

$$A=\{\omega' : \limsup_{n\to\infty} S_n(\omega')/(2n\log\log n)^{1/2}=(2\pi f_{\omega'}(0))^{1/2}\} \quad\text{and}$$

$$A_\omega=\{\omega' : \limsup_{n\to\infty} S_n(\omega')/(2n\log\log n)^{1/2}=(2\pi f_\omega(0))^{1/2}\}.$$

In view of the remark stated in Section 1, for $P$-almost all $\omega\in\Omega$, $P_\omega(A)=P_\omega(A_\omega)$. If we can show that $P_\omega(A_\omega)=1$ for $P$-almost all $\omega\in\Omega$, then $P(A)=\int P_\omega(A)P(d\omega)=1$ which is desired to obtain. Thus in order to prove Theorem 2, we may restrict ourselves to the case when $P$ is ergodic. In this case

$$E(\varepsilon_0^2\,|\,\mathcal{I})=E\varepsilon_0^2=\sigma^2 \text{ a.s.} \quad\text{and}\quad E|\varepsilon_0|^r<\infty \quad\text{for some } r>2.$$

Without loss of generality, assume that the $\alpha_j$ are constants and

$$f(\lambda)=(2\pi)^{-1}\sigma^2\left|\sum_{j=-\infty}^{\infty}\alpha_j e^{i\lambda j}\right|^2.$$

For $n\geq 1$, put

$$Y_n=\Big(\sum_{j=-\infty}^{\infty}\alpha_j\Big)\varepsilon_n,\quad Z_n=X_n-Y_n \quad\text{and}\quad T_n=\sum_{i=1}^{n}Z_i.$$

Since $|\sum_{j=-\infty}^{\infty}\alpha_j|<\infty$ and $\{\varepsilon_n\}$ has been assumed to be a stationary ergodic sequence of square integrable martingale differences, it follows from a result of Stout (1970) that

$$\limsup_{n\to\infty}\Big(\sum_{i=1}^{n} Y_i\Big)/(2n\log\log n)^{1/2}=(2\pi f(0))^{1/2} \quad \text{a. s.}$$

Thus it suffices to show that

$$\lim_{n\to\infty} T_n/(n\log\log n)^{1/2}=0 \quad \text{a. s.}$$

$T_n$ can be written as

$$T_n=\sum_{j=-\infty}^{\infty} a_{nj}\varepsilon_j ,$$

where

$$a_{nj}=\sum_{i=1-j}^{n-j}\alpha_i-\sum_{i=-\infty}^{\infty}\alpha_i \quad \text{if } 1\leqq j\leqq n \quad \text{and} \quad a_{nj}=\sum_{i=1-j}^{n-j}\alpha_i \quad \text{otherwise},$$

and it is easy to see that

$$|a_{nj}|\leqq K \quad \text{for all } n, \text{ all } j \text{ and some } K<\infty .$$

We shall prove that for all $\varepsilon>0$ and $\theta>0$

$$P[|T_n|>\varepsilon(n\log\log n)^{1/2}]=O((\log n)^{-\theta}). \tag{5}$$

Under the assumption of continuity of $f(\lambda)$ at $\lambda=0$,

$$ET_n^2=\sigma^2\sum_{j=-\infty}^{\infty} a_{nj}^2=n\delta_n , \quad \delta_n\to 0 \text{ as } n\to\infty$$

(see Heyde (1974)). We assume without loss of generality that $\delta_n>0$ for all $n$, and put $K_n=\delta_n^{1/r}$. For any $\theta>0$,

$$\begin{aligned} &P[|a_{nj}\varepsilon_j|>K_n(n/\log\log n)^{1/2} \text{ for some } j] \qquad (6)\\ &\leqq\sum_{j=-\infty}^{\infty}|a_{nj}|^r E|\varepsilon_0|^r K_n^{-r}(n/\log\log n)^{-r/2}\\ &\leqq K^{r-2}E|\varepsilon_0|^r\Big(\sum_{j=-\infty}^{\infty}a_{nj}^2\Big)K_n^{-r}(n/\log\log n)^{-r/2}\\ &=K^{r-2}\sigma^{-2}E|\varepsilon_0|^r n^{-(r-2)/2}(\log\log n)^{r/2}\\ &=o((\log n)^{-\theta}). \end{aligned}$$

Let for $n\geqq 1$ and $-\infty<j<\infty$,

$$\varepsilon_{nj}'=\varepsilon_j I[|a_{nj}\varepsilon_j|\leqq K_n(n/\log\log n)^{1/2}]$$

and let

$$\varepsilon_{nj}=\varepsilon'_{nj}-E(\varepsilon'_{nj}\,|\,\mathcal{F}_{j-m}).$$

Then $\{\varepsilon_{n,mj+l}, \mathcal{F}_{mj+l}, -\infty<j<\infty\}$ is a martingale difference sequence for each $n\geqq 1$ and $0\leqq l<m$, and

$$|a_{nj}\varepsilon_{nj}|\leqq 2K_n(n/\log\log n)^{1/2}=c_n, \quad \text{say},$$

and by the condition of the theorem and the property of measure-preserving transformation,

$$E(\varepsilon^2_{nj}\,|\,\mathcal{F}_{j-m})\leqq E(\varepsilon^2_j\,|\,\mathcal{F}_{j-m})\leqq C\sigma^2 \quad \text{a.s. for all } j.$$

Put $\lambda_n=c_n^{-1}$, $A_n=(\lambda_n^2/2)(1+\lambda_n c_n/2)=(3/4)\lambda_n^2$ and $U_{nl}=\sum_{j=-\infty}^{\infty}a_{n,mj+l}\varepsilon_{n,mj+l}$. Then for any $\varepsilon>0$,

$$P\Big[\sum_{j=-\infty}^{\infty}a_{nj}\varepsilon_{nj}>\varepsilon(n\log\log n)^{1/2}\Big] \tag{7}$$

$$\leqq \sum_{l=0}^{m-1}P[U_{nl}>\varepsilon m^{-1}(n\log\log n)^{1/2}]$$

$$=\sum_{l=0}^{m-1}P[\exp(\lambda_n U_{nl})>\exp\{\varepsilon m^{-1}\lambda_n(n\log\log n)^{1/2}\}]$$

$$\leqq \sum_{l=0}^{m-1}P[W_{nl}>\exp\{\varepsilon m^{-1}\lambda_n(n\log\log n)^{1/2}-CnA_n\delta_n\}],$$

where $W_{nl}=\exp\{\lambda_n U_{nl}-A_n\sum_{j=-\infty}^{\infty}a^2_{n,mj+l}E(\varepsilon^2_{n,mj+l}\,|\,\mathcal{F}_{m(j-1)+l})\}$. By a result of Stout (1974), Lemma 5.4.1 and the Fatou lemma, $EW_{nl}\leqq 1$, and hence for any $\theta>0$, the right-hand side of (7) is dominated by

$$m\exp\{-\varepsilon m^{-1}\lambda_n(n\log\log n)^{1/2}+CnA_n\delta_n\}$$

$$=m\exp[-\{(\varepsilon/2mK_n)-(3/16)CK_n^{\tau-2}\}(\log\log n)]$$

$$\leqq(\log n)^{-\theta}$$

for all $n$ large enough, since $K_n\to 0$ as $n\to\infty$. Repeating the same argument for $-a_{nj}\varepsilon_{nj}$ we obtain that

$$P\Big[\Big|\sum_{j=-\infty}^{\infty}a_{nj}\varepsilon_{nj}\Big|>\varepsilon(n\log\log n)^{1/2}\Big]\leqq 2(\log n)^{-\theta} \tag{8}$$

for all $n$ large enough. Using the martingale property, we have for any $\tau>0$ and $\theta>0$,

$$P\Big[\Big|\sum_{j=-\infty}^{\infty}a_{nj}E(\varepsilon'_{nj}\,|\,\mathcal{F}_{j-m})\Big|>\tau(n\log\log n)^{1/2}\Big] \tag{9}$$

$$\leqq\tau^{-1}(n\log\log n)^{-1/2}\sum_{j=-\infty}^{\infty}E[\,|a_{nj}\varepsilon_j|\,I\{|a_{nj}\varepsilon_j|>K_n(n/\log\log n)^{1/2}\}]$$

$$\leqq \tau^{-1}E|\varepsilon_0|^r(n\log\log n)^{-1/2}[K_n(n/\log\log n)^{1/2}]^{1-r}\sum_{j=-\infty}^{\infty}|a_{nj}|^r$$

$$\leqq \tau^{-1}K^{r-2}\sigma^{-2}E|\varepsilon_0|^rK_n n^{-(r-2)/2}(\log\log n)^{(r-2)/2}$$

$$=o((\log n)^{-\theta}).$$

Since $|T_n|\leqq|\sum_{j=-\infty}^{\infty}a_{nj}\varepsilon_{nj}|+|\sum_{j=-\infty}^{\infty}a_{nj}E(\varepsilon'_{nj}|\mathcal{F}_{j-m})|$ on the event $[|a_{nj}\varepsilon_{nj}]\leqq K_n(n/\log\log n)^{1/2}$ for all $j]$, we obtain (5) from (6), (8) and (9).

Fix $\theta>1$ sufficiently large and choose $0<a<1$ such that $a\theta>1$ and $(1-a)r/2>1$. Let $n_k=[\exp k^a]$ for $k\geqq 1$. Then $\sum_{k=1}^{\infty}(\log n_k)^{-\theta}<\infty$, and hence by (5)

$$\sum_{k=1}^{\infty}P[|T_{n_k}|>\varepsilon(n_k\log\log n_k)^{1/2}]<\infty.$$

Since $\varepsilon$ is arbitrary, by the Borel-Cantelli lemma

$$T_{n_k}/(n_k\log\log n_k)^{1/2}\longrightarrow 0 \text{ a.s. as } k\to\infty.$$

Let

$$M_k=\max_{n_k<n\leq n_{k+1}}|T_n-T_{n_k}|/(n_k\log\log n_k)^{1/2},\quad k\geqq 1.$$

For each $k\geqq 1$,

$$|T_n|/(n\log\log n)^{1/2}\leqq|T_{n_k}|/(n_k\log\log n_k)^{1/2}+M_k$$

for each $n_k<n\leqq n_{k+1}$. Thus it suffices to show that $M_k\to 0$ a.s. as $k\to\infty$ to complete the proof. By Burkholder's and Hölder's inequalities, there exists a constant $C_r<\infty$ depending only or $r$ such that

$$E\Big|\sum_{j=p}^{q}a_{nj}\varepsilon_j\Big|^r\leqq C_rE\Big(\sum_{j=p}^{q}a_{nj}^2\varepsilon_j^2\Big)^{r/2}$$

$$=C_rE\Big(\sum_{j=p}^{q}|a_{nj}|^{(2r-4)/r}|a_{nj}|^{4/r}\varepsilon_j^2\Big)^{r/2}$$

$$\leqq C_rE\Big\{\Big(\sum_{j=p}^{q}a_{nj}^2\Big)^{r/2-1}\sum_{j=p}^{q}a_{nj}^2|\varepsilon_j|^r\Big\}$$

$$\leqq C_r\Big(\sum_{j=-\infty}^{\infty}a_{nj}^2\Big)^{r/2}E|\varepsilon_0|^r.$$

Hence by the Fatou lemma,

$$E|T_n|^r\leqq B_1n^{r/2}\quad\text{for all}\quad n\geqq 1\quad\text{and some}\quad B_1<\infty.$$

Since $\{Z_n\}$ is stationary, there exists a constant $B_2<\infty$ such that

$$E(\max_{1\leq m\leq n}|T_m|)^r\leqq B_2n^{r/2}\quad\text{for all}\quad n\geqq 1,$$

see Serfling (1970), Theorem B. Hence we have

$$
\begin{aligned}
EM_k^r &= E(\max_{1\le m\le n_{k+1}-n_k}|T_m|)^r/(n_k\log\log n_k)^{r/2} \\
&\leqq B_2(n_{k+1}-n_k)^{r/2}/(n_k\log\log n_k)^{r/2} \\
&\leqq B_3 k^{-(1-a)r/2}(\log k)^{-r/2}
\end{aligned}
$$

for all $k$ and some $B_3<\infty$. $\sum_{k=1}^{\infty}EM_k^r<\infty$, since $(1-a)r/2>1$, and hence $M_k\to 0$ a.s. as desired.

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