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FIBRATIONS OVER CLASSIFYING SPACES

By

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Summary: Let BG and $B\,\mathrm{Aut}\,(X)$ be classifying spaces of a Lie group G with finite connected components and a topological monoid $\mathrm{Aut}(X)$ of self homotopy equivalences of finite complex X respectively. We determine a homotopy set of maps between BG and $B\,\mathrm{Aut}(X)$ which are homotopic to the constant map on skeletons. By applying this result to the canonical map $[X, BG]\to[X, B\,\mathrm{Aut}(G)]$, we give some examples of fibre bundles and fibre spaces and study the relations between the fibre bundles isomorphism and the fibre homotopy equivalence.

Introduction

A continuous map $f: X\to Y$ from a CW complex X to a topological space Y is called a phantom map, if the restriction map $f|X^n$ is homotopic to the constant map for all $n\geqq 0$. More generally, two maps $f, g: X\to Y$ are called a phantom pair, when the restriction maps $f|X^n$ and $g|X^n$ are homotopic for all $n\geqq 0$. We call the map $g: X\to Y$ an f-phantom map. In [5], we characterized a phantom pair by using the localization and completion and described a homotopy set of phantom pairs by the ordinary cohomology group under some conditions.

Let BG and $B\,\mathrm{Aut}(X)$ be classifying spaces of a Lie group G with finite connected components and a topological monoid $\mathrm{Aut}(X)$ of self homotopy equivalences of finite complex X respectively. We determine a homotopy set of maps between BG and $B\,\mathrm{Aut}(X)$ which are homotopic to the constant map on skeletons. We give uncountably many distinct fibrations $E\to BG$ with a fibre G which are fibre homotopy equivalent to the trivial bundle on skeletons for some Lie group G.

Our main results are as follows.

Theorem 2.2. *Let G be a Lie group with finite connected components and X a connected finite CW complex which is a rational H-space. Then,* $\tilde{\theta}\,(BG, B\,\mathrm{Aut}(X))$

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is weak homotopy equivalent to a product $\prod_{k\geq 0} K(A_k, k)$ *of Eilenberg-MacLane spaces where* $A_k = \prod_{j\geq i>k} H^{i-k}(BG; H^{j-i}(X; \pi_j(X)) \otimes Z^{\wedge}/Z$.

Theorem 2.4. *Let* X *be a connected CW-complex of finite type and* G *a topological monoid. Then, for any map* $f: X \to B\,\mathrm{Aut}(G)$ *the inverse image* $j_*^{-1}(f)$ *of the induced map* $j_*: [X, BG] \to [X, B\,\mathrm{Aut}(G)]$ *is the inverse image of the map* $\underleftarrow{\mathrm{Lim}}^1 \pi_1 C_f(X^n, BG) \to \underleftarrow{\mathrm{Lim}}^1 \pi_1 C_f(X^n, B\,\mathrm{Aut}(G))$. *If* X *is finite complex, the induced map* j_* *is a monomorphism. Moreover, the analogous result holds also for* $i: B\,\mathrm{Aut}_*(G) \to B\,\mathrm{Aut}(G)$, *and* $[X, BG] \cap [X, B\,\mathrm{Aut}_*(G)] = *$ *in the homotopy set* $[X, B\,\mathrm{Aut}(G)]$.

Example 3.1. Let G be a connected Lie group such as $SO(n)$ $(n>4)$, $Sp(n)$ $(n>1)$, $U(n)$ $(n>1)$, $SU(n)$ $(n>3)$ and the exceptional Lie groups G_2, F_4, E_6, E_7, E_8. There are uncountably many distinct fibrations $E \to BG$ with a fibre G which are fibre homotopy equivalent to the trivial bundle on skeletons.

1. Preliminaries

In this paper, we work in the category of compactly generated Hausdorff spaces with base points and base point preserving maps. Two maps $f, g: X \to Y$ are called a phantom pair, when the restriction maps $f|X^n$ and $g|X^n$ are homotopic for all $n \geq 0$. We call the map $g: X \to Y$ an f-phantom map. Especially, when $f = *$, a continuous map $g: X \to Y$ is called a phantom map.

We use the notations and definitions in our paper [5] and the following notations. Base point free maps are used only in the definitions of $\mathrm{Map}(X, Y)$. $\bar{C}_f(X, Y)$, $\mathrm{Aut}_1(X)$ and $\mathrm{Aut}(X)$.

$\mathrm{Map}_*(X, Y)$: the space of continuous based maps from X to Y.
$\mathrm{Map}(X, Y)$: the space of continuous base point free maps from X to Y.
$C_f(X, Y)$: the connected component of $\mathrm{Map}_*(X, Y)$ which contains $f: X \to Y$.
$\bar{C}_f(X, Y)$: the connected component of $\mathrm{Map}(X, Y)$ which contains $f: X \to Y$.
$\tilde{\theta}_f(X, Y)$: the space of f-phantom maps.
$\theta_f(X, Y)$: the set of the connected components of $\tilde{\theta}_f(X, Y)$.
$\mathrm{Aut}_*(X)$: the space of self homotopy equivalences of X in $\mathrm{Map}_*(X, X)$.
$\mathrm{Aut}(X)$: the space of self homotopy equivalences of X in $\mathrm{Map}(X, X)$.
$\mathrm{Aut}_1(X) = \mathrm{Aut}(X) \cap \bar{C}_{id}(X, X)$, $\mathrm{Aut}_{*1}(X) = \mathrm{Aut}_*(X) \cap C_{id}(X, X)$.
$\cong_w$: the weak homotopy equivalence.

For a general space X, we shall use the localization and completion of the geometric realization of the singular complex of X. This is sufficient for our purposes.

Proposition 1.1. *Let W be a connected nilpotent CW-complex of finite type and X a connected nilpotent CW complex. For any map $f: W \to X$,*

(1) *$C_f(W, X)^\wedge$ is weak homotopy equivalent to $C_f(W, X^\wedge)$ and $C_f(W^\wedge, X^\wedge)$.*
(2) *$C_f(W, X)_Q$ is weak homotopy equivalent to $C_f(W, X_Q)$ and $C_f(W_Q, X_Q)$.*

where $e^\wedge: X \to X^\wedge$ is the Sullivan completion and $l_Q: X \to X_Q$ is the rationalization. For $\bar{C}_f(W, X^\wedge)$ and $\bar{C}_f(W, X)_Q$, the analogous results hold also.

Proof. When W is a connected finite CW complex, the above statement (1) is true by Proposition 5.1, 5.4 and 7.1 of Chapter 6 in [2]. When W is a connected infinite CW-complex, the $\varprojlim^1$ groups of the homotopy groups of $C_f(W^n, X)^\wedge$ vanishes, because of Proposition 3.5 in [5]. For the statement (2), we use the analogous methods and Theorem 2.1 in [7].

Theorem 1.2. *Let G be a Lie group with finite connected components, X a connected nilpotent finite CW complex and Y a connected CW-complex of finite type. Then, for any $f: Y \to X$ where $\tilde{\theta}(X, Y) = \theta_g(X, Y)$ for $g = *$.*

(1) $\mathrm{Map}_*(BG, C_f(Y, X))$ *is equal to* $\tilde{\theta}(BG, C_f(Y, X))$.
(2) $\mathrm{Map}_*(BG, \bar{C}_f(Y, X))$ *is equal to* $\tilde{\theta}(BG, \bar{C}_f(Y, X))$.

Proof. Since $\mathrm{Map}_*(BG, X^\wedge)$ is weak contractible by Theorem 3.1 in [3], $\mathrm{Map}_*(BG, C_f(Y, X^\wedge))$ is weak contractible by Theorem C (c) in [8]. Hence, $\mathrm{Map}_*(BG, C_f(Y, X)^\wedge)$ is weak contractible by Proposition 1.1. For any map $h: BG \to C_f(Y, X)$, it satisfies $e^\wedge h \simeq *$. Then h is a phantom map by Theorem 3.6 in [5]. The second statement is analogously proved.

We define $\rho: X\rho \to X$ by the homotopy fibre of $e^\wedge: X \to X^\wedge$. If X is a connected nilpotent CW-complex of finite type, $X\rho$ is also the homotopy fibre of $l_Q: X_Q \to \hat{X}_Q$ by the arithmetic square. If X is a rational H-space of finite type, $X\rho$ is a product of Eilenberg-MacLane spaces. In this case, there is a fibre sequence $X\rho \to X \to X^\wedge \to BX\rho$ where $BX\rho$ is a classifying space of $X\rho$.

Proposition 1.3. *Let V be a connected nilpotent CW-complex of finite type which is a rational H-space. Then, for all $i \geqq 0$ and Y a connected CW-complex, the following formula holds.*

$$\pi_i(\mathrm{Map}_*(Y, BV\rho)) = \prod_{j>i} H^{j-i}(Y; \pi_j(V)) \otimes Z^\wedge/Z$$

Proposition 1.4. *Let X be a connected CW-complex which is a rational H-space of finite type. Then, the following formula holds for all $i > 0$.*

$$\pi_i(\mathrm{Aut}_{*1}(X)_Q) = \prod_{j>i} H^{j-i}(X; \pi_j(X)) \otimes Q\,.$$

Proof. For $i>0$, one obtains the following equalities by Proposition 1.1 and the properties of a rational H-space.

$$\pi_i(\mathrm{Aut}_{*1}(X)_Q)=\pi_i(\mathrm{Aut}_{*1}(X_Q))=[\Sigma^i X_Q, X_Q]$$
$$=\prod_{j>i} H^{j-i}(X_Q; \pi_j(X_Q))=\prod_{j>i} H^{j-i}(X; \pi_j(X))\otimes Q .$$

2. Main Results

Let Γ be a family of CW-complexes X such that $C_0(BG, X^{\wedge})$, $C_0(BK, X^{\wedge})$ are weak contractible for all Lie groups G with finite connected components and all connected Eilenberg-MacLane spaces K. Γ contains a finite CW-complex X, a classifying space of a Lie group and $C_f(Y, X)$ for a connected CW-complex Y.

Theorem 2.1. *Let G be a Lie group with finite connected components and V a connected nilpotent CW-complex of Γ. Then, the following results hold.*

(1) *$\tilde{\theta}(BG, V)$ is weak homotopy equivalent to $\tilde{\theta}(BK, V)$ where K is a product of Eilenberg-MacLane spaces with $G_Q\cong K_Q$. For $k\geqq 0$, $\pi_k(\tilde{\theta}(BG, V))$ is isomorphic to $A_k=\prod_{i>k} H^{i-k}(BG; \pi_{i+1}(V))\otimes Z^{\wedge}/Z$. Moreover, when V is a rational H-space, $\tilde{\theta}(BG, V)$ is weak homotopy equivalent to a product of Eilenberg-MacLane spaces $\prod_{k\geq 0} K(A_k, k)$.*

(2) *For $k\geqq 0$, $f: BG\to V$, $\pi_k(\tilde{\theta}_f(BG, V))$ is isomorphic to a quotient group of $A_k=\prod_{i>k} H^{i-k}(BG; \pi_{i+1}(V))\otimes Z^{\wedge}/Z$.*

Proof. By the principal fibration $\mathrm{Map}_*(BG, V\rho)\to\mathrm{Map}_*(BG, V)\to\mathrm{Map}_*(BG, V^{\wedge})$ and $C_0(BG, V^{\wedge})\cong_w *$, we have $\mathrm{Map}_*(BG, V\rho)\cong_w\tilde{\theta}(BG, V)$. By the properties of localization, we have $\mathrm{Map}_*(BG, V\rho)\cong_w\mathrm{Map}_*(BG_Q, V\rho)\cong_w\mathrm{Map}_*(BK_Q, V\rho)\cong_w \mathrm{Map}_*(BK, V\rho)$. By $C_0(BK, V)\cong_w *$, we have $\mathrm{Map}_*(BK, V\rho)\cong_w\tilde{\theta}(BK, V)$. Hence we have the result. The second part of (1) is easily proved by the above relation. The second statement is proved by a fibration $\tilde{\theta}_f(BG, V)\to C_{e^{\wedge}f}(BG, V^{\wedge})\to C_0(BG, BV\rho)$.

Let X be a connected nilpotent finite CW-complex which is a rational H-space. Then there are the following fibrations:

$$\mathrm{Aut}_*(X_Q)\longrightarrow \mathrm{Aut}(X_Q)\xrightarrow{ev} X_Q,\quad B\,\mathrm{Aut}_*(X)\rho\longrightarrow B\,\mathrm{Aut}(X)\rho\longrightarrow BX\rho$$

Here ev has the canonical cross section. By the fibration $B\,\mathrm{Aut}_*(X)\rho\to B\,\mathrm{Aut}(X)\rho\to BX\rho$ (X: a rational H-space), the dimensional reason of H-space and Proposition 1.3, 1.4, we have the following results.

Theorem 2.2. *Let G be a Lie group with finite connected components and X a connected finite CW complex which is a rational H-space. Then, $\tilde{\theta}(BG, B\,\mathrm{Aut}(X))$ is weak homotopy equivalent to a product $\prod_{k\geq 0} K(A_k, k)$ of Eilenberg-MacLane spaces where $A_k = \prod_{j\geq i>k} H^{i-k}(BG; H^{j-i}(X; \pi_j(X))\otimes Z^{\wedge}/Z$.*

Theorem 2.3. *Let J be a simply connected CW complex which has a finite Postnikov system and X a connected finite CW complex which is a rational H-space. Then, $\tilde{\theta}(J, B\,\mathrm{Aut}(X))$ is weak homotopy equivalent to a product of Eilenberg-MacLane spaces $\prod_{k\geq 0} K(A_k, k)$ where $A_k = \prod_{j\geq i>k} H^{i-k}(J; H^{j-i}(X; \pi_j(X))\otimes Z^{\wedge}/Z$.*

Let $j: G\to \mathrm{Aut}(G)$ be the canonical homomorphism induced by the multiplication $G\times G\to G$ where G is a topological monoid.

Theorem 2.4. *Let X be a connected CW-complex of finite type and G a topological monoid. Then, for any map $f: X\to B\,\mathrm{Aut}(G)$ the inverse image $j_*^{-1}(f)$ of the induced map $j_*: [X, BG]\to [X, B\,\mathrm{Aut}(G)]$ is the inverse image of the map $\varprojlim{}^1 \pi_1 C_f(X^n, BG)\to \varprojlim{}^1 \pi_1 C_f(X^n, B\,\mathrm{Aut}(G))$. If X is finite complex, the induced map j_* is a monomorphism. Moreover, the analogous result holds also for $i: B\,\mathrm{Aut}_*(G)\to B\,\mathrm{Aut}(G)$, and $[X, BG]\cap [X, B\,\mathrm{Aut}_*(G)] = *$ in the homotopy set $[X, B\,\mathrm{Aut}(G)]$.*

Proof. There is the following fibration where ev has the canonical cross section.

$$\mathrm{Aut}_*(G) \longrightarrow \mathrm{Aut}(G) \xrightarrow{ev} G, \qquad B\,\mathrm{Aut}_*(G) \longrightarrow B\,\mathrm{Aut}(G) \longleftarrow BG.$$

Since the homotopy group $\pi_i(B\,\mathrm{Aut}(G))$ is splited as the direct product $\pi_i(B\,\mathrm{Aut}_*(G))$ and $\pi_i(BG)$, the induced morphism j_* of cohomology groups $H^i(X, \pi_i(BB))\to H^i(X, \pi_i(B\,\mathrm{Aut}(G)))$ is monomorphism. Hence Proposition is true for a finite CW complex X by Theorem 1.1 of [5]. By Theorem 3.1 of Chapter 9 of [2] and the next diagrame, we obtain the result.

$$\begin{array}{ccccccccc} * & \longrightarrow & \varprojlim{}^1 \pi_1 C_f(X^n, BG) & \longrightarrow & [X, BG] & \longrightarrow & \varprojlim [X^n, BG] & \longrightarrow & * \\ & & \downarrow & & \downarrow & & \downarrow & & \\ * & \longrightarrow & \varprojlim{}^1 \pi_1 C_f(X^n, B\,\mathrm{Aut}(G)) & \longrightarrow & [X, B\,\mathrm{Aut}(G)] & \longrightarrow & \varprojlim [X^n, B\,\mathrm{Aut}(G)] & \longrightarrow & * \end{array}$$

Corollary 2.5. *Under the above assumptions, the map $\varprojlim{}^1 \pi_1 C_f(X^n, BG)\to \varprojlim{}^1 \pi_1 C_f(X^n, B\,\mathrm{Aut}(G))$ is 1:1 for $f=*$ or $X=BH$ a classifying space of a connected Lie group H.*

Proof. Since it holds $[\Sigma X, BG]\times[\Sigma X, B\,\mathrm{Aut}_*(G)] = [\Sigma X, B\,\mathrm{Aut}(G)]$, the $\varprojlim{}^1$ group splits. Hence the first part is true. Since it holds $\theta_f(BH, BG) = 0$

for all $f: BH \to BG$ by Theorem 2.1, the second part is true.

3. Example

In this section, we give some examples of fibre bundles and fibre spaces, and study the relations between the fibre bundle isomorphism and the fibre homotopy equivalence.

Example 3.1. Let G be a connected Lie group such as $SO(n)$ $(n>4)$, $Sp(n)$ $(n>1)$, $U(n)$ $(n>1)$, $SU(n)$ $(n>3)$ and the exceptional Lie groups G_2, F_4, E_6, E_7, E_8. We calculate $\theta(BG, B\,\mathrm{Aut}(G))$ by Theorem 2.2. For $G=SO(n)$ $(n>4)$, $Sp(n)$ $(n>1)$, $SU(n)$ $(n>3)$, we take $i=4$, $j=7$. For $G=U(n)$ $(n>1)$, we take $i=2$, $j=3$. For $G=G_2$, F_4, E_6, E_7, we take $i=8$, $j=11$. For $G=E_8$, we take $i=12$, $j=15$. Then, we have $H^i(BG, H^{j-i}(G, \pi_j(G))\otimes Z^\wedge/Z=Z^\wedge/Z$. Since fibre homotopy equivalence classes of fibrations $E\to BG$ with a fibre G are classified by $BG\to B\,\mathrm{Aut}(G)$ [6], there are uncoutably many distinct fibrations $E\to BG$ with a fibre G which are fibre homotopy equivalent to the trivial bundles on skeletons. For a general Lie group G, principal fibre bundles $E\to BG$ with a fibre G which are fibre homotopy equivalent on skeletons are isomorphic as fibre bundles by Corollary 2.5.

For $G=SO(n)$ $(n<5)$, $Sp(1)$, $U(1)$, $SU(n)$ $(n>4)$, fibrations $E\to BG$ with a fibre G which are fibre homotopy equivalent on skeletons are unique in the sense of a fibre homotopy equivalence by Theorem 2.2. For a fibration $E\to BG$ with a fibre G which is fibre homotopy equivalent to a fibre bundle on skeletons, there is a unique fibre bundle which is fibre homotopy equivalent to the fibration $E\to BG$ by Theorem 2.2 and 2.4.

Example 3.2. The cohomology ring of a symplectic group $Sp(n)$ is the exterier algebra $\Lambda(x_1, \cdots, x_n)$ deg $x_i=4i-1$ $(i=1, \cdots, n)$. Cohomology operations are calculated in §13 of [1]. For example, $P^1(x_i)=2x_1x_1-(2i+2)x_{i+1}$ for the reduced power operation mod 3. We assert that $C_{id}(BSp(n), BSp(n))$ and $C_{e^\wedge}(BSp(n), BSp(n)^\wedge)$ have non trivial homotopy groups for infinitely many dimensions. If $\pi_3(C_{id}(BSp(n), BSp(n)))$ is zero, the generator $h: S^4\to BSp(n)$ is lifted to $k: S^4\to C_{id}(BSp(n), BSp(n)))$. This case is not possible by using the above cohomology operations for the adjoint map $k^\sim: S^4\times BSp(n)\to BSp(n)$ with $k^\sim|BSp(n)\vee S^4=id\vee h$. Hence $\pi_3(C_{id}(BSp(n), BSp)n)))$ is not zero. By Proposition 1.2 or Theorem C (c) of [8], it hold $C_0(BS^1, C_{id})=\theta(BS^1, C_{id})$ and $C_0(BZ/p, C_{id})=0$. There is the homotopy spectral sequence of [4], $E_2^{st}=H^s(BS^1, \pi_t(C_{id}))$ which converges to 0. By the similar arguement of Theorem 4 of [4], we obtain the result. But $C_0(BSp(n), BSp(n)^\wedge)$ is weakly contractible by Theorem 3.1 of [3].

Example 3.3. Let G be a connected Lie group with $\pi_3(G)\neq 0$. By Theorem 2.3, $\theta(K(Z, 3), BG)$ is isomorphic to $Z^{\wedge}/Z$. Hence there are uncoutably many distinct fibre bundles $E\rightarrow BG$ with a fibre G which are the trivial bundle on skeletons. These fibre bundles are isomorphic by Theorem 2.4 and Corollary 2.5, if they are fibre homotopy equivalent.

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