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NON-COMMUTATIVE THREE DIMENSIONAL SPHERES II —NON-COMMUTATIVE HOPF FIBERING—

By

KENGO MATSUMOTO

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Summary. We deform the 3-sphere into non-commutative C^*-algebras S_Θ^3 by continuous functions Θ's. We then prove that the fixed point C^*-subalgebra of the non-commutative 3-sphere S_Θ^3 under the action of the unit circle S^1 is isomorphic to the C^*-algebra of all continuous functions on the 2-sphere. This means that the non-commutative 3-spheres have Hopf fibered structures in non-commutative sense.

1. Introduction.

In the last ten years, many operator algebraists have investigated non-commutative geometry and topology on some kind of C^*-algebras, which are sometimes called "non-commutative manifolds" cf. [1], [2], [3], [5], [10], [12], [15], $\cdots$, etc. Although the decisive definition of "non-commutative (topological) manifolds" has not been given yet, it seems to be very natural to think of them as a subcatgory of the category of the C^*-algebras as seen in many excelent works as above.

In [6], the author has deformed the ordinary 3-sphere S^3 into non-commutative C^*-algebras along one parameter. These deformed C^*-algebras are thought of non-commutative versions of S^3 and examples of a family of "non-commutative manifolds". They are realized as a one parameter family of C^*-algebras $\{S_\theta^3\}_{\theta\in\boldsymbol{R}}$ called non-commutative 3-spheres. For each parameter $\theta\in\boldsymbol{R}$, the C^*-algebra S_θ^3 becomes the universal C^*-algebra generated by two normal operators S and T satisfying the following three relations:

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$$\begin{cases} TS=e^{2\pi i\theta}ST \\ (1-T^*T)(1-S^*S)=0 \\ \|S\|=\|T\|=1\,. \end{cases} \tag{1}$$

On the other hand, as in [8, Proposition 8.1], it is also realized to be the universal C^*-algebra generated by two normal operators Z and W satisfying the following two relations:

$$\begin{cases} ZW=e^{2\pi i\theta}WZ \\ Z^*Z+W^*W=1\,. \end{cases} \tag{2}$$

In deformations of ordinary manifolds into non-commutative C^*-algebras, one parameter deformation would not always be suitable especially in deformations of higher dimensional manifolds (cf. [3], [10]). In fact, in the above non-commutative 3-spheres, one knows that they are deformed along one certain direction in the manifold S^3 so that they might be regarded as "partially" non-commutative 3-spheres in a sense. In order to deform the 3-sphere and construct "totally" non-commutative 3-spheres, it would be one way to adopt a class $\mathcal{F}$ of continuous functions as deformation parameter including one parameters as constant functions and enlarge the direction of deformations.

In this paper, we shall first generalize the construction of the above non-commutative 3-spheres $\{S_\theta^3\}_{\theta\in R}$ to provide "totally" non-commutative 3-spheres $\{S_\Theta^3\}_{\Theta\in\mathcal{F}}$ as deformation C^*-algebras of S^3 parametrized by continuous functions Θ's and describe briefly their structures as C^*-algebra (Proposition 2, Theorem 3). We shall second study the structure of these (totally) non-commutative 3-spheres from the aspect as non-commutative topological manifolds. Namely, in studying the topological structure of the ordinary 3-sphere, it is remarkable that the 3-sphere has the structure as principal S^1-bundle over S^2, so called Hopf fibered structure. We shall show that our non-commutative 3-spheres are also regarded to be non-commutative S^1-bundle over S^2 (Theorem 6). This means that they are realized as the non-commutative Hopf fibered spaces.

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2. Deformations of the 3-sphere by continuous functions.

Let $\mathcal{F}$ be the set of all real valued continuous functions on the closed interval $I=[0, 1]$. For any but fixed function Θ in $\mathcal{F}$, we consider the following operator relations for normal operators Z and W on a Hilbert space:

$$\begin{cases} ZW=e^{2\pi i\hat{\Theta}}ZW \\ Z^*Z+W^*W=1 \\ \hat{\Theta}=\Theta(Z^*Z) \end{cases} \tag{3}$$

where $\hat{\Theta}=\Theta(Z^*Z)$ means the self-adjoint operator obtained by the functional calculus of the operator Z^*Z by the continuous function Θ.

Note that, when the function Θ is constant number θ, the above relation (3) is nothing but the relation (2). Therefore the relation (3) is a generalized form of (2).

Let's construct concretely the C^*-algebra generated by two normal operators Z and W determined by only the relation (3), which will be written as S_Θ^3. This is our "totally" non-commutative 3-sphere cited in the previous section.

Let S^1 be the unit circle in the complex plane $\boldsymbol{C}$. Fix the function Θ and consider the homeomorphism α_Θ on the annulus $I\times S^1$ defined by

$$\alpha_\Theta(r, e^{2\pi i\xi})=(r, e^{2\pi i(\Theta(r)+\xi)}), \qquad r, \xi\in I=[0, 1].$$

It induces an automorphism on the C^*-algebra $C(I\times S^1)$ of all complex valued continuous functions on $I\times S^1$, which is also denoted by α_Θ or simply Θ. We fix a point r in $[0, 1]$. Let $\alpha_{\Theta(r)}$ be the automorphism on $C(S^1)$ induced by the rotation around origin with angle $\Theta(r)$. It is also simply written as $\Theta(r)$. The restriction of a function on the annulus $I\times S^1$ to the circle $\{r\}\times S^1$ at level r induces the surjection from $C(I\times S^1)$ to $C(\{r\}\times S^1)=C(S^1)$. The surjection is compatible with actions α_Θ and $\alpha_{\Theta(r)}$ so that it is extended to a surjection π_r on crossed products by the actions:

$$\pi_r: C(I\times S^1)\times_\Theta \boldsymbol{Z} \longrightarrow C(S^1)\times_{\Theta(r)}\boldsymbol{Z}, \qquad r\in[0, 1].$$

The C^*-algebra $C(S^1)\times_{\Theta(r)}\boldsymbol{Z}$ is known as non-commutative 2-torus of angle $\Theta(r)$, which is denoted by $A_{\Theta(r)}$. It is also well known that $A_{\Theta(r)}$ is generated by two unitaries $V(r)$ and $U(r)$ satisfying the following familiar relation:

$$V(r)\cdot U(r)=e^{2\pi i\Theta(r)}U(r)\cdot V(r), \qquad r\in[0, 1]$$

where $V(r)$ is the unitary coming from the positive generator of the integer group $\boldsymbol{Z}$ and $U(r)$ is the canonical unitary generator of the algebra $C(S^1)$.

Now take the homomorphisms π_0 and π_1 at boundaries of the annulus $I\times S^1$. We shall define our (totally) non-commutative 3-sphere as a deformation by the given continuous function Θ in the following way:

Definition (non-commutative 3-sphere).

$$S_\Theta^3=\{a\in C(I\times S^1)\times_\Theta \boldsymbol{Z} \mid \pi_0(a)\in C^*(V(0)),\ \pi_1(a)\in C^*(U(1))\}$$

where $C^*(V(0))$ and $C^*(U(1))$ mean the C^*-subalgebras of $A_{\Theta(0)}$ and $A_{\Theta(1)}$ generated by $V(0)$ and $U(1)$ respectively.

When the function Θ is constant real number θ, the crossed product $C(I\times S^1)\times_\theta \boldsymbol{Z}$ is isomorphic to the tensor product C^*-algebra $C(I)\otimes A_\theta$. Hence the C^*-algebra S_θ^3 is nothing but the original non-commutative 3-sphere $S_\theta^3(=L_\theta(1, 0))$ deformed by the parameter θ as seen in [8, Theorem B]. And it is the universal C^*-algebra generated by two normal operators with relations (2) and hence (1) as we stated in the previous section. Furthermore, when the function Θ is constantly zero, S_Θ^3 becomes the commutative C^*-algebra of all complex valued continuous functions on 3-sphere. Therefore the C^*-algebra S_Θ^3 is thought of a deformation of 3-sphere by the continuous function Θ.

Next, we shall briefly describe the reason why S_Θ^3 can be regarded to be the universal C^*-algebra determined by the relation (3). We shall first investigate the structure of the crossed product C^*-algebra $C(I\times S^1)\times_\Theta \boldsymbol{Z}$.

Recall that there exists a surjective homomorphism π_r from $C(I\times S^1)\times_\Theta \boldsymbol{Z}$ to the C^*-algebra $C(S^1)\times_{\Theta(r)}\boldsymbol{Z}=A_{\Theta(r)}$ for each $r\in[0, 1]$. These continuous family of surjections $\{\pi_r\}_{r\in[0,1]}$ onto $\{A_{\Theta(r)}\}_{r\in[0,1]}$ give rise to continuous cross sections from $C(I\times S^1)\times_\Theta \boldsymbol{Z}$ in fibered space $\{A_{\Theta(r)}\}_{r\in[0,1]}$. By slightly generalizing the proof of Proposition 6.4 in [8] or directly the results of [11] and [14], we have the following:

Lemma 1. *The C^*-algebra $C(I\times S^1)\times_\Theta \boldsymbol{Z}$ is realized to be the C^*-algebra consisting of continuous cross sections of continuous field $\{A_{\Theta(r)}\}_{r\in[0,1]}$ of C^*-algebras.*

Namely, the crossed product $C(I\times S^1)\times_\Theta \boldsymbol{Z}$ is regarded as an algebra consisting of continuous cross sections of non-commutative torus bundle over the closed interval.

Since the non-commutative 3-sphere S_Θ^3 is a C^*-subalgebra of $C(I\times S^1)\times_\Theta \boldsymbol{Z}$ with suitable boundary conditions as in Definition, one easily sees the following:

Proposition 2. *For each Θ in $\mathfrak{T}$, non-commutative 3-sphere S_Θ^3 is the C^*-algebra of continuous cross sections of the fibered space $\{A_{\Theta(r)}\}_{r\in[0,1]}$ over the interval $[0, 1]$ each of whose fiber is non-commutative 2-torus $A_{\Theta(r)}$ with angle $\Theta(r)$ for r in $(0, 1)$ and $C^*(V(0))$, $C^*(U(1))$ on the boundary points $\{0, 1\}$ respectively, where the C^*-algebras $C^*(V(0))$ and $C^*(U(1))$ are ones seen in the definition of S_Θ^3.*

By generalizing discussions given in the proof of Theorem C in [8], one sees that the following two cross sections Z and W generate S_Θ^3 as C^*-algebra:

$$Z(r)=\sqrt{1-r}\cdot V(r), \qquad W(r)=\sqrt{r}\cdot U(r), \qquad r\in[0, 1]$$

where $V(r)$ and $U(r)$ are the unitary generators of non-commutative 2-torus $A_{\Theta(r)}$ cited before. One then knows that the above two normal operators Z and W satisfy relation (3) and have no more operator relations than (3). Hence we have:

Theorem 3. *For each function Θ in $\mathcal{F}$, S_Θ^3 is realized to be the universal C^*-algebra with relation* (3).

The proof of Theorem 3 is completed by modifying the discussions seen in the proof of the special case of Theorem C in [8].

Remark 4. We are also able to extend the almost all discussions in [8] to the deformation C^*-algebras obtained by continuous functions. Hence we have non-commutative lens spaces $\{L_\Theta(p, q)\}_{\Theta\in\mathcal{F}}$ parametrized by continuous functions as a wider class than the class of the original non-commutative lens spaces $\{L_\theta(p, q)\}_{\theta\in\boldsymbol{R}}$ parametrized by one parameter defined in [8]. They are also realized as fixed point subalgebras of S_Θ^3 under suitable cyclic group actions.

Remark 5. The author is planning to discuss on the classification of $\{S_\Theta^3\}_{\Theta\in\mathcal{F}}$ up to isomorphism or stable isomorphism concerning about Θ's and non-commutative differential structures on S_Θ^3 in [7].

3. Non-commutative Hopf fibering.

We first recall the ordinary Hopf fibered structure of the 3-sphere S^3. We represent S^3 as the unit sphere of complex 2-plane $\boldsymbol{C}^2$, namely

$$S^3=\{(z, w)\in\boldsymbol{C}^2 \mid |z|^2+|w|^2=1\}.$$

Then, the unit circle S^1 in complex plane $\boldsymbol{C}$ acts on S^3 as in the following way:

$$\gamma_\lambda : (z, w)\in S^3 \longrightarrow (\lambda z, \lambda w)\in S^3, \qquad \lambda\in\boldsymbol{C}, \ |\lambda|=1.$$

It is well known that the orbit space of S^3 under the action γ is homeomorphic to 2-sphere S^2. That is, S^3 becomes the principal S^1-bundle over S^2. This fibered space is called Hopf fibered space.

We second consider the non-commutative version of the above discussions. Fix a continuous function Θ in $\mathcal{F}$ and represent our non-commutative 3-sphere S_Θ^3 as the universal C^*-algebra generated by two normal operators Z and W with relations (3). Bearing the above ordinary case in mind, we define the free action α of the unit circle S^1 on S_Θ^3 by

$$\alpha_\lambda(Z)=\lambda Z, \qquad \alpha_\lambda(W)=\lambda W, \qquad \lambda\in \boldsymbol{C},\ |\lambda|=1.$$

Since each α_λ preserves the relation (3), one knows that it defines an automorphism on S_Θ^3 for each $\lambda\in \boldsymbol{C}$, $|\lambda|=1$, by the universality of the C^*-algebra.

In this setting, the result is the following:

Theorem 6. *For any $\Theta\in\mathcal{T}$, we have:*

(i) *The fixed point algebra $(S_\Theta^3)^\alpha$ of S_Θ^3 under the action α is the universal C^*-algebra generated by mutually commuting normal operator M, self-adjoint operator H and the identity* 1 *satisfying the following relation:*

$$M^*M+H^2=H, \tag{4}$$

(ii) *The fixed point algebra $(S_\Theta^3)^\alpha$ is isomorphic to the commutative C^*-algebra of all complex valued continuous functions on a* 2*-sphere.*

Let's start the proof of Theorem 6.

We define the operators H, K and M in $(S_\Theta^3)^\alpha$ by

$$H=Z^*Z, \qquad K=W^*W\ (=1-H), \qquad M=ZW^*.$$

We notice the fact that the commutation relation $ZW=e^{2\pi i\hat{\Theta}}WZ$ in (3) automatically implies the relation $ZW^*=e^{-2\pi i\hat{\Theta}}W^*Z$ (cf. [4], [9]). Then the following lemma is immediate.

Lemma 7. *Both the operators H and M are fixed under the action α and satisfy relation* (4).

Now we prepare some notations and lemmas.

For an integer n and an operator X on a Hilbert space, we introduce the operator $\tilde{X}^n$ defined by

$$\tilde{X}^n=\begin{cases} X^n & (n\geqq 0)\\ X^{*(-n)} & (n\leqq 0).\end{cases}$$

For two integers j, k, we write by $j\wedge k$ the minimum of them. Hence for integers j, k, l and m, the operators Z^jZ^{*k} and W^lW^{*m} are expressed in the forms $H^{j\wedge k}\tilde{Z}^{j-k}$ and $K^{l\wedge m}\tilde{W}^{l-m}$, respectively. We also use the notation $\tilde{\lambda}^n$ for $\lambda\in\boldsymbol{C}$, $|\lambda|=1$ and $n\in\boldsymbol{Z}$ in the similar way.

It is easy to see the following formula by induction.

Lemma 8. *For any integers k, n,*

$$\tilde{Z}^k\tilde{W}^{-k}=e^{-k(k-1)\pi i\hat{\Theta}}\cdot\tilde{M}^k, \qquad \tilde{W}^n\tilde{Z}^{-n}=e^{n(n-1)\pi i\hat{\Theta}}\cdot\tilde{M}^{-n}.$$

Let $\mathcal{P}(Z, W)$ be the *-subalgebra of S_{θ}^{3} algebraically generated by Z and W. Then it is dense in S_{θ}^{3}. We restrict the action α to the subalgebra $\mathcal{P}(Z, W)$ and denote by $\mathcal{P}(Z, W)^{\alpha}$ the fixed point subalgebra of $\mathcal{P}(Z, W)$ under α. In order to know the structures of the algebras $\mathcal{P}(Z, W)^{\alpha}$ and $(S_{\theta}^{3})^{\alpha}$, the expectation E from S_{θ}^{3} to $(S_{\theta}^{3})^{\alpha}$ defined in the following plays an important role.

Let μ be the normalized Haar measure on S^1. The expectation E is defined by

$$E(A)=\int_{S^1}\alpha_{\lambda}(A)d\mu(\lambda), \qquad A\in S_{\theta}^{3}.$$

Lemma 9. *Keep the above notations. We have:*

(i) $E(\mathcal{P}(Z, W))=\mathcal{P}(Z, W)^{\alpha}$.

(ii) $\mathcal{P}(Z, W)^{\alpha}$ *is contained in the* C^**-subalgebra* $C^*(H, M)$ *generated by* H, M *and the identity* 1.

Proof. It is clear that $\mathcal{P}(Z, W)^{\alpha}$ is contained in $E(\mathcal{P}(Z, W))$. Hence it suffices to show that, for any element X of $\mathcal{P}(Z, W)$, $E(X)$ is a polynomial of Z, W and expressed by H and M. Let X be an element of $\mathcal{P}(Z, W)$. By the commutation relation in (3), X is expressed as in the following way:

$$X=\sum_{j,k,l,m}c_{j,k,l,m}Z^{j}Z^{*k}W^{l}W^{*m}$$

where each coefficient $c_{j,k,l,m}$ is a polynomial of $e^{2\pi i\hat{\theta}}$ and the indices $\{j, k, l, m\}$ run through non-negative integers finitely. As the operator ZWZ^*W^* (hence $e^{2\pi i\hat{\theta}}$) is fixed by the action α, it follows that

$$\begin{aligned}E(X)&=\sum_{j,k,l,m}c_{j,k,l,m}E(Z^{j}Z^{*k}W^{l}W^{*m})\\&=\sum_{j,k,l,m}c_{j,k,l,m}\int_{S^1}\lambda^{j-k+l-m}d\mu(\lambda)\cdot Z^{j}Z^{*k}W^{l}W^{*m}\\&=\sum_{\substack{j,k,l,m\\ j-kl+-m=0}}c_{j,k,l,m}Z^{j}Z^{*k}W^{l}W^{*m}.\end{aligned}$$

Hence $E(X)$ is contained in the algebra $\mathcal{P}(Z, W)\cap(S_{\theta}^{3})^{\alpha}=\mathcal{P}(Z, W)^{\alpha}$. Furthermore one sees that the above last polynomial is equal to the following:

$$\begin{aligned}&\sum_{\substack{j,k,l,m\\ j-k+l-m=0}}c_{j,k,l,m}H^{j\wedge k}\tilde{Z}^{j-k}K^{l\wedge m}\tilde{W}^{l-m}\\&=\sum_{\substack{j,k,l,m\\ j-k+l-m=0}}c_{j,k,l,m}e^{-(j-k)(j-k-l)\pi i\hat{\theta}}H^{j\wedge k}(1-H)^{l\wedge m}\tilde{M}^{j-k}.\end{aligned}$$

Since the operator $\hat{\theta}=\theta(H)$ is obtained by the functional calculus of H by the function θ, these coefficients $c_{j,k,l,m}e^{-(j-k)(j-k-l)\pi i\hat{\theta}}$ belong to the C^*-subalgebra generated by H and 1. Hence we conclude that the algecra $\mathcal{P}(Z, W)^{\alpha}$ is contained in the C^*-algebra $C^*(H, M)$ generated by H, M and 1. □

Since the expectation E is continuous, one knows that the algebra $\mathcal{P}(Z, W)^{\alpha}$ is dense in $(S_{\theta}^{3})^{\alpha}$ by the first part of Lemma 9. Thus the second part of Lemma 9 implies the following corollary.

Corollary 10. *The fixed point algebra* $(S_{\theta}^{3})^{\alpha}$ *coincides with the* C^**-algebra* $C^*(H, M)$ *generated by the operators* H, M *and the identity* 1 *satisfying relation* (4).

Finally we shall see the C^*-algebra $C^*(H, M)$ is isomorphic to the commutative C^*-algebra $C(S^2)$ of all complex valued continuous functions on a 2-sphere.

Final proof of Theorem 6.

We represent the C^*-algebra S_{θ}^{3} as the C^*-algebra of continuous cross sections of fibered space $\{A_{\theta(r)}\}_{r\in[0,1]}$ as in Proposition 2. Then the operators H and M may be expressed as cross sections in the following way:

$$H(r)=1-r\,,\qquad M(r)=\sqrt{r(1-r)}\cdot V(r)U(r)^*\qquad r\in[0,1]\,.$$

Hence their spectra are of the forms:

$$Sp(H)=[0,1]\,,\qquad Sp(M)=\{z\in \boldsymbol{C}\mid |z|\leqq 1\},$$

because the spectrum $Sp(V(r)\cdot U(r))=\{z\in \boldsymbol{C}\mid |z|=1\}$. Let us decompose M into $X+iY$, $X=X^*$, $Y=Y^*$. Then it is easy to see that $Sp(X)=Sp(Y)=[0, 1]$. Note that the C^*-algebra $C^*(X, Y, H)$ generated by X, Y, H and 1 is isomorphic to the C^*-algebra $C^*(H, M)$ and hence the fixed point algebra $(S_{\theta}^{3})^{\alpha}$. Since relation (4) can be changed by the following relation:

$$X^2+Y^2+(H-1/2)^2=1/4\,,$$

it is clear that the C^*-algebra $C^*(X, Y, H)$ is isomorphic to the commutative C^*-algebra of all complex valued continuous functions on a 2-sphere. Finally, the universality of the C^*-algebra $C^*(H, M)$ concerning the relation (4) is easily seen by the universality of the C^*-algebra S_{θ}^{3} or the fact that the spectra of H and M are full. This completes the proof of Theorem 6.

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Kengo Matsumoto
Department of Mathematics
Faculty of Science
Tokyo Metropolitan University
Minami-Ohsawa 1-1
Hachioji-shi
Tokyo, 192-03 Japan