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ALMOST SURE CONVERGENCE OF SEQUENCES WITH RANDOM INDICES

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Summary. Let Z_{+}^{d}, where $d\geqq 1$ is an integer, denote the positive integer d-dimensional lattice points. Let $\{Y_{\boldsymbol{n}}, \boldsymbol{n}\in Z_{+}^{d}\}$ be a set of random variables. Let $\{N_{\boldsymbol{n}}, \boldsymbol{n}\in Z_{+}^{d}\}$ be a set of Z_{+}^{d}-valued random variables. In this paper we study almost sure convergence of the random field $\{Y_{N_{\boldsymbol{n}}}, \boldsymbol{n}\in Z_{+}^{d}\}$ as $\boldsymbol{n}\rightarrow\infty$. We introduce an almost sure version of Anscombe condition and study its consequences in strong limit theorem.

1. Introduction

Let Z_{+}^{d}, where $d\geqq 1$ is an integer, denote the positive integer d-dimensional lattice points. The points in Z_{+}^{d} will be denoted by $\boldsymbol{m}, \boldsymbol{n}$, etc., or sometimes, when necessary, more explicity, by $(m_1, m_2, \cdots, m_d)$, $(n_1, n_2, \cdots, n_d)$, etc. The set Z_{+}^{d} is partially ordered by stipulating $\boldsymbol{m}\leqq\boldsymbol{n}$ iff $m_i\leqq n_i$ for each i, $1\leqq i\leqq d$. Further, $|\boldsymbol{n}|$ is used for $\prod_{i=1}^{d} n_i$ and $\boldsymbol{n}\rightarrow\infty$ means that $\min_{1\leq i\leq d} n_i\rightarrow\infty$. We write $\boldsymbol{1}$ for the point $(1, 1, \cdots, 1)$.

Let $\{Y_{\boldsymbol{n}}, \boldsymbol{n}\in Z_{+}^{d}\}$ be a random field, i.e., a set of random variables defined on a probability space $(\Omega, \mathcal{A}, P)$. Let $\{N_{\boldsymbol{n}}, \boldsymbol{n}\in Z_{+}^{d}\}$ and $\{M_{\boldsymbol{n}}, \boldsymbol{n}\in Z_{+}^{d}\}$ be sets of Z_{+}^{d}-valued random variables defined on the same probability space $(\Omega, \mathcal{A}, P)$, i.e., for every $\boldsymbol{n}\in Z_{+}^{d}$, $N_{\boldsymbol{n}}=(N_{\boldsymbol{n}}^{(1)}, \cdots, N_{\boldsymbol{n}}^{(d)})$ and $M_{\boldsymbol{n}}=(M_{\boldsymbol{n}}^{(1)}, \cdots, M_{\boldsymbol{n}}^{(d)})$ where $N_{\boldsymbol{n}}^{(i)}$ and $M_{\boldsymbol{n}}^{(i)}$, $1\leqq i\leqq d$, are positive integer-valued variables.

In the present paper we study almost sure convergence of the random field $\{Y_{N_{\boldsymbol{n}}}, \boldsymbol{n}\in Z_{+}^{d}\}$ as $\boldsymbol{n}\rightarrow\infty$. We introduce some "generalized almost sure Anscombe conditions". The introduced versions of Anscombe condition play simlar roles in the study of almost sure (a.s.) convergence of the random field $\{Y_{N_{\boldsymbol{n}}}, \boldsymbol{n}\in Z_{+}^{d}\}$, as the generalized Anscombe conditions (introduced in [1], [5] and [6]) in the study of weak convergence of $\{Y_{N_{\boldsymbol{n}}}, \boldsymbol{n}\in Z_{+}^{d}\}$. Recently many authors have been looking for sufficient conditions, concerning $\{N_{\boldsymbol{n}}, \boldsymbol{n}\in Z_{+}^{d}\}$ and $\{Y_{\boldsymbol{n}}, \boldsymbol{n}\in Z_{+}^{d}\}$, under which $Y_{N_{\boldsymbol{n}}}\rightarrow 0$ a.s. as $\boldsymbol{n}\rightarrow\infty$. Mainly the case $d=1$ have been considered, (cf.

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[2, p. 10], [3] and the references given there). The obtained results have the following form. Assume $Y_n \to 0$ a.s. as $\boldsymbol{n}\to\infty$. Then under "some additional assumptions" $Y_{N_n} \to 0$ a.s. as $\boldsymbol{n}\to\infty$. Thus the assumption $Y_n \to 0$ a.s. as $\boldsymbol{n}\to\infty$ plays a fundamental role. We do not, in general, assume that $Y_n \to 0$ a.s. as $\boldsymbol{n}\to\infty$. Our results can also be applied in the case when the random field $\{Y_n, \boldsymbol{n}\in Z_+^d\}$ does not converge almost surely. The main results, presented in this paper, are new even in the case $d=1$. We extend the main ideas presented in [1], [5] and [6].

In Section 2 "a generalized almost sure Anscombe condition" is introduced and its consequences in almost sure limit theorems with random indices are studied. In Section 3 we present short proofs of the theorems, while in Section 4 an example of application of Theorem 2 is given.

2. Almost sure versions of Anscombe condition and their applications

Let $\{N_n, \boldsymbol{n}\in Z_+^d\}$ and $\{M_n, \boldsymbol{n}\in Z_+^d\}$ be sets of Z_+^d-valued random variables. Let $\{Y_n, \boldsymbol{n}\in Z_+^d\}$ be a set of random variables.

Let $\{D(\boldsymbol{n}), \boldsymbol{n}\in Z_+^d\}$ and $\{d(\boldsymbol{n}), \boldsymbol{n}\in Z_+^d\}$ be sets of positive numbers such that $\boldsymbol{n}, \boldsymbol{m}\in Z_+^d$ and $\boldsymbol{n}\leqq\boldsymbol{m}$ imply $D(\boldsymbol{n})\leqq D(\boldsymbol{m})$.

Definition 1. A random field $\{Y_n, \boldsymbol{n}\in Z_+^d\}$ is said to satisfy an almost sure Anscombe condition

$$A(\{D(\boldsymbol{n})\}, \{M(\boldsymbol{n})\}, \{d(\boldsymbol{n})\})$$

with norming sets $\{D(\boldsymbol{n}), \boldsymbol{n}\in Z_+^d\}$ and $\{d(\boldsymbol{n}), \boldsymbol{n}\in Z_+^d\}$ iff

$$\max_{i\in G_n} |Y_i - Y_{M(n)}| \longrightarrow 0 \quad \text{a.s. as } \boldsymbol{n}\to\infty \tag{2.1}$$

where, here and in the sequel,

$$\begin{aligned} G_n &= G_n(\{D(\boldsymbol{n})\}, \{M(\boldsymbol{n})\}, \{d(\boldsymbol{n})\}) \\ &= \{\boldsymbol{k}\in Z_+^d : |D(\boldsymbol{k}) - D(M(\boldsymbol{n}))| \leqq d(M_n)\}. \end{aligned}$$

In what follows we consider the following condition concerning families $\{N_n, \boldsymbol{n}\in Z_+^d\}$ and $\{M_n, \boldsymbol{n}\in Z_+^d\}$

$$P(\limsup_{\boldsymbol{k}\to\infty} [|D(N_k) - D(M_k)| > d(M_k)]) = 0. \tag{2.2}$$

The set of all families $\{N_n, \boldsymbol{n}\in Z_+^d\}$ of Z_d^+-valued random variables satisfying the condition (2.2) we denote by $T(\{D(\boldsymbol{n})\}, \{M_n\}, \{d(\boldsymbol{n})\})$.

Let $C(\{X_n\})$ denote the set of limit points of the family $\{X_n, \boldsymbol{n}\in Z_+^d\}$.

Theorem 1. *Let $\{Y_n, \boldsymbol{n}\in Z_+^d\}$ be a family of random variables and let $\{M_n, \boldsymbol{n}\in Z_+^d\}$ and $\{N_n, \boldsymbol{n}\in Z_+^d\}$ be sets of Z_+^d-valued random variables. Assume*

(i) $C(\{Y_{M_n},\ n \in Z_+^d\}) = E$ *a.s., where* $E \subset (-\infty, \infty)$,

(ii) $\{Y_n,\ n \in Z_+^d\}$ *satisfies the condition* $A(\{D(n)\}, \{M(n)\}, \{d(n)\})$.

Then for every family $\{N_n,\ n \in Z_+^d\} \in T(\{D(n)\}, \{M(n)\}, \{d(n)\})$

(2.3) $$C(\{Y_{N_n},\ n \in Z_+^d\}) = E \qquad a.s.$$

Definition 2. A random field $\{Y_n,\ n \in Z_+^d\}$ is said to satisfy a generalized almost sure Anscombe condition ($GA(\{D(n)\}, \{M(n)\}, \{d(n)\})$) with norming sets $\{D(n),\ n \in Z_+^d\}$ and $\{d(n),\ n \in Z_+^d\}$ iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that

(2.4) $$P(\limsup_{n \to \infty} \max_{i \in G_n(\delta)} |Y_i - Y_{M_n}| > \varepsilon) = 0,$$

where

$$G_n(\delta) = G_n(\delta, \{D(n)\}, \{M(n)\}, \{d(n)\})$$
$$= \{k \in Z_+^d : |D(k) - D(M_n)| < \delta d(M_n)\}.$$

Let us observe that $G_n(1) = G_n$, so that the condition $A(\{D(n)\}, \{M(n)\}, \{d(n)\})$ implies the condition $GA(\{D(n)\}, \{M(n)\}, \{d(n)\})$.

Theorem 2. *Let* $\{Y_n,\ n \in Z_+^d\}$ *be a family of random variables and let* $\{M_n,\ n \in Z_+^d\}$ *and* $\{N_n,\ n \in Z_+^d\}$ *be sets of* Z_+^d*-valued random variables. Assume*

(i) $C(\{Y_{M_n},\ n \in Z_+^d\}) = E$ *a.s., where* $E \subset (-\infty, \infty)$,

and

(ii) $\{Y_n,\ n \in Z_+^d\}$ *satisfies the condition* $GA(\{D(n)\}, \{M(n)\}, \{d(n)\})$.

Then (2.3) *holds for every set* $\{N_n,\ n \in Z_+^d\}$ *of* Z_+^d*-valued random variables such that*

(2.5) $$(D(N_n) - D(M_n))/d(M_n) \longrightarrow 0 \qquad a.s.\ as\ n \to \infty.$$

Let us observe that the condition (2.5) is stronger than the condition (2.2) in Theorem 1, but on the other hand the condition (ii) of Theorem 2 is weaker than (ii) in Theorem 1. We also note that if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

(2.6) $$\sum_{n \in Z_+^d} P(\max_{i \in G_n(\delta)} |Y_i - Y_{M_n}| > \varepsilon) < \infty$$

then (2.4) holds too. The condition (2.6) is a weaker version of the so-called GALIL conditions introduced by A. Gut [3, p.5]. Moreover, if

(2.7) $$|N_n| / |M_n| \longrightarrow 1 \qquad \text{a.s. as } n \to \infty,$$

then (2.5) holds with $D(n) = d(n) = |n|$, $n \in Z_+^d$.

Let us now observe that the sufficient conditions presented in Theorem 1 or Theorem 2 are very close to the necessary ones (may be they even are necessary, but we were not able to prove it), since we have the following,

Theorem 3. *Let* $(\Omega, \mathcal{A}, P)$ *be a probability space such that* Ω *is the union of at most countable many atoms. Let* $\{Y_n, n \geqq 1\}$ *be a sequence of random variables and let* $\{N_n, n \geqq 1\}$ *be a sequence of positive integer-valued random variables. If for every sequence* $\{N_n, n \geqq 1\} \in T(\{D(n)\}, \{M_n\}, \{d_n\})$

$$Y_{N_n} \longrightarrow 0 \quad a.s. \text{ as } n \to \infty, \tag{2.8}$$

then

$$Y_{M_n} \longrightarrow 0 \quad a.s. \text{ as } n \to \infty, \tag{2.9}$$

and

$$\max_{i \in G_n} |Y_i - Y_{M_n}| \longrightarrow 0 \quad a.s. \text{ as } n \to \infty. \tag{2.10}$$

We remark that the results presented in Theorems 1 and 2 can also be applied to families $\{Y_n, n \in Z_+^d\}$ of random elements with values in separable Banach spaces. In such a case the symbol $|\cdot|$ denotes simply a norm of the space.

3. Proofs

Proof of Theorem 1. We have

$$Y_{N_n} = Y_{M_n} + (Y_{N_n} - Y_{M_n}), \qquad n \in Z_+^d. \tag{3.1}$$

Thus it is enough to prove that

$$Y_{N_n} - Y_{M_n} \longrightarrow 0 \quad \text{a.s. as } n \to \infty. \tag{3.2}$$

For every $\varepsilon > 0$ we obtain

$$\begin{aligned} & P\Big(\bigcap_{n \in Z_+^d} \bigcup_{k \geq n} [|Y_{N_k} - Y_{M_k}| > \varepsilon\Big) \\ & \leqq P\Big(\bigcap_{n \in Z_+^d} \bigcup_{k \geq n} [|Y_{N_k} - Y_{M_k}| > \varepsilon] \cap [|D(N_k) - D(M_k)| \leqq d(M_k)]\Big) \\ & \quad + P\Big(\bigcap_{n \in Z_+^d} \bigcup_{k \geq n} [|D(N_k) - D(M_k)| > d(M_k)]\Big) \\ & \leqq P\Big(\bigcap_{n \in Z_+^d} \bigcup_{k \geq n} [\max_{i \in G_n} |Y_i - Y_{M_k}| > \varepsilon]\Big) \\ & \quad + P\Big(\bigcap_{n \in Z_+^d} \bigcup_{k \geq n} [|D(N_k) - D(M_k)| > d(M_k)]\Big). \end{aligned} \tag{3.3}$$

Hence, by (2.1), (2.2), and (3.3), we get (3.2). But (3.2), (3.1) and (i) give (2.3).

Proof of Theorem 2. For every $\varepsilon>0$ and $\delta>0$ we have

$$(3.4)\qquad P\Big(\bigcap_{n\in Z_+^d}\bigcup_{k\geq n}[|Y_{N_k}-Y_{M_k}|>\varepsilon]\Big)$$

$$\leqq P\Big(\bigcap_{n\in Z_+^d}\bigcup_{k\geq n}[|Y_{N_k}-Y_{M_k}|>\varepsilon]\cap[|D(N_k)-D(M_k)|<\delta d(M_k)]\Big)$$

$$+P\Big(\bigcap_{n\in Z_+^d}\bigcup_{k\geq n}[|D(N_k)-D(M_k)|>\delta d(M_k)]\Big)$$

$$\leqq P\Big(\bigcap_{n\in Z_+^d}\bigcup_{k\geq n}[\max_{i\in G_n(\delta)}|Y_i-Y_{M_k}|>\varepsilon]\Big)$$

$$+P\Big(\bigcap_{n\in Z_+^d}\bigcup_{k\geq n}[|D(N_k)-D(M_k)|>\delta d(M_k)]\Big).$$

Hence, (2.5), (ii), and (3.4) yield

$$Y_{N_k}-Y_{M_k}\longrightarrow 0 \qquad \text{a.s. as } k\to\infty,$$

so that (i) and the equality

$$Y_{N_k}=Y_{M_k}+(Y_{N_k}-Y_{M_k}), \qquad k\in Z_+^d$$

end the proof of Theorem 2.

Proof of Theorem 3. It is obvious that (2.9) holds since $\{M_n,\ n\geqq 1\}\in T(\{D(n)\},\ \{M_n\},\ \{d(n)\})$. Thus it is enough to prove (2.10). But Ω is a sum of atoms so that

$$\max_{i\in G_n}|Y_i-Y_{M_n}|\longrightarrow 0 \qquad \text{a.s. as } n\to\infty$$

if and only if

$$\max_{i\in G_n}|Y_i-Y_{M_n}|\xrightarrow{P} 0 \qquad \text{as } n\to\infty$$

(cf. [Thomasian 7]). On the other hand, if the sequence $\{\max_{i\in G_n}|Y_i-Y_{M_n}|,\ n\geqq 1\}$ does not converge in probability, then similarly as in the proof of Theorem 2.2 [6] we can prove that one can define a sequence $\{\tau_n^*,\ n\geqq 1\}$ such that

$$P\Big(\limsup_{n\to\infty}[|D(\tau_n^*)-D(M_n)|>d(M_n)]\Big)=0$$

and the sequence $\{Y_{\tau_n^*},\ n\geqq 1\}$ does not converge almost surely and this contradicts (2.8). Thus the proof of Theorem 3 is completed.

4. Example

Let $\{X_n, n\geqq 1\}$ be a sequence of independent and identically distributed random variables such that $EX_1=0$ and $0<EX_1^2=\sigma^2<\infty$. Let $S_n=X_1+\cdots+X_n$, $n\geqq 1$. Then the set of cluster points of the sequence $\{S_n/(n \log\log n)^{1/2}, n\geqq 3\}$ coincides almost surely with $[-\sigma\sqrt{2}, \sigma\sqrt{2}]$, so that the sequence $\{Y_n=S_n/(n \log\log n)^{1/2}), n\geqq 3\}$ does not converge almost surely to zero. But if, for example, $M_k=2^{2^{2^k}}$, $k\geqq 1$, then by Chebyshev's inequality and Borel-Cantelli lemma

$$Y_{M_n} \longrightarrow 0 \qquad \text{a.s. as } n\to\infty .$$

Furthermore, using Kolmogorov's inequality and Borel-Cantelli lemma, one can easily prove that the sequence $\{Y_{n_k}, k\geqq 1\}$ satisfies the condition $GA(\{n\}, \{M_n\}, \{n\})$, so that by Theorem 2

$$Y_{N_n} \longrightarrow 0 \qquad \text{a.s. as } n\to\infty$$

for every sequence $\{N_n, n\geqq 1\}$ of positive integer-valued random variables such that

$$N_n/M_n \longrightarrow 1 \qquad \text{a.s. as } n\to\infty .$$

Assume now that the distribution of the random variable X_1 is continuous. Let $\{L(n), n\geqq 1\}$ be the sequence of record times, i.e., $L(1)=1$ and inductively for $n\geqq 1$

$$L(n+1)=\inf\{j>L(n): X_j>X_{L(n)}\},$$

and let $\mu(n)=\mu([1, n])$ be the number of records in the first n observations. Then (cf. [4, Chapter 4])

$$(\mathrm{Log}\ L(n))/n \longrightarrow 1 \qquad \text{a.s. as } n\to\infty$$

and

$$\mu_n/\mathrm{Log}\ n \longrightarrow 1 \qquad \text{a.s. as } n\to\infty ,$$

so that putting $N_k^{(1)}=[\mathrm{Log}\ L(n_k)]$ or $N_k^{(2)}=\mu([\exp(n_k)])$, where $n_k=M_k$, $k\geqq 1$, we get

$$N_n^{(i)}/M_n \longrightarrow 1 \qquad \text{a.s. as } n\to\infty$$

for $i=1, 2$. Thus, for $i=1, 2$, we get

$$Y_{N_n^{(i)}} \longrightarrow 0 \qquad \text{a.s. as } n\to\infty .$$

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