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COMPLETE FINITE REPRESENTABILITY IN C^*-ALGEBRAS

By

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Introduction

The notion of finite representability in Banach spaces was introduced by R. C. James [9] in connection with the study of superreflexivity. The connection between finite representability and ultrapowers was independently observed by C. W. Henson, L. C. Moore [8] and J. Stern [15]. The explicit definition of ultraproducts of Banach spaces was introduced by D. Dacunha-Castelle and J. L. Krivine [5] and detailed study of ultraproducts led to various applications in the local theory of Banach spaces. A comprehensive account of all this is the survey paper of S. Heinrich [7] and it is convenient to take this as our reference rather than the original sources. Ultraproducts of C^*-algebras were defined in [5] but the theory of finite representations has been confined to the setting of Banach spaces and $(1+\varepsilon)$-isomorphisms, i.e. bounded linear maps T with $\|T\|\leqq 1+\varepsilon$ and $\|T^{-1}\|\leqq 1+\varepsilon$ on finite dimensional subspaces. The purpose of this paper is to study this notion in the setting of C^*-algebras and $(1+\varepsilon)$-isomorphisms T on finite dimensional subalgebras or subspaces, that are *-homomorphisms, completely positive maps or completely bounded maps with $\|T\|_{cb}\leqq 1+\varepsilon$ and $\|T^{-1}\|_{cb}\leqq 1+\varepsilon$. Completely positive maps were introduced by W. F. Stinespring [16] as an algebraic setting for studying the existence of dilations. Every *-homomorphism is completely positive. On the other hand it has been shown by M. D. Choi and E. Christensen [4] that for any $\varepsilon>0$, there exists a pair of C^*-algebras $\mathcal{A}_\varepsilon$ and $\mathcal{B}_\varepsilon$ possessing a completely positive $(1+\varepsilon)$-isomorphism T on $\mathcal{A}_\varepsilon$ onto $\mathcal{B}_\varepsilon$ with a completely positive inverse but there is no one-one *-homomorphism on $\mathcal{A}_\varepsilon$ onto $\mathcal{B}_\varepsilon$.

Completely bounded maps were introduced by Arveson [2], V.I. Paulsen's monograph [14] gives a good account of such maps. In fact the theory of completely positive maps has been developed more by Quantum mechanicists because of its use in the Quantum theory of open systems i.e. irreversible processes, (see [6, 13, 1]).

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Let $L(E)$ be the C^*-algebra of bounded operators on a Hilbert space E to itself. For any subspace (respectively *-subspace, *-subalgebra) X of $L(E)$, $M_n(X)$ will denote the corresponding subspace (respectively *-subspace, *-subalgebra) of the matrix algebra $M_n(L(E))$. Then X is closed in $L(E)$ if and only if $M_n(X)$ is closed in $M_n(L(E))$.

Let F be a Hilbert space. Let Y be a subspace of $L(F)$. For a linear map ϕ on X to Y let ϕ_n denote the map on $M_n(X)$ to $M_n(Y)$, defined by applying ϕ element by element to each matrix over X. If X and Y are *-subspaces (respectively *-subalgebras) and ϕ is a *-map (respectively *-homomorphism) then so is ϕ_n. In case X and Y are *-subspaces and ϕ_n is positive we say that ϕ is n-positive and if ϕ_m is m-positive for each $m \in N$ then ϕ is called completely positive [16]. ϕ is called completely contractive if each ϕ_n is a contraction [2]. A scalar multiple ψ of such a ϕ satisfies $\|\psi\|_{cb} = \sup_n \|\psi_n\| < \infty$ and such maps ψ are called completely bounded. ϕ is called a complete isometry if each ϕ_n is an isometry. If ϕ is a *-homomorphism on a C^*-algebra X to a C^*-algebra Y then ϕ is completely positive and completely contractive. On the other hand if ϕ is a completely positive map on an operator system X (i.e. a *-subspace X of $L(E)$ containing the identity l_E of $L(E)$) to $L(F)$ then ϕ is completely bounded and $\|\phi(l_E)\| = \|\phi\| = \|\phi\|_{cb}$ and ϕ has an extension to a completely positive map ψ on $L(E)$ to $L(F)$ ([2]). The latter result is known as Arveson's extension theorem, and it motivated E.G. Effros, C.E. Lance, A. Connes and others to consider the injective operator systems i.e. the class of operator systems with this extension property. M.D. Choi and E.G. Effros [3] have called an operator system $\mathcal{A} \subset L(F)$ injective if for any operator systems $\mathcal{B} \subset \mathcal{C} \subset L(E)$ any completely positive map ϕ of $\mathcal{B}$ to $\mathcal{A}$, there is a completely positive extension ϕ_1 of ϕ to $\mathcal{C}$ such that $\phi_1: \mathcal{C} \to \mathcal{A}$ and have proved that if $\mathcal{A} \subset L(F)$ is an injective operator system then there is an identity preserving completely positive 1-1 map ϕ of $\mathcal{A}$ onto an essentially unique unital C^*-algebra such that its inverse is also completely positive and thus $\|\phi\|_{cb} = \|\phi^{-1}\|_{cb} = 1$ so that ϕ is a complete isometry. As noted in the proof of the uniqueness part of this result ([3], p 165–166) any identity preserving order isomorphism of unital C^*-algebras that is 2-positive also, is a 1-1 *-homomorphism and therefore, an identity preserving complete isometry between unital C^*-algebras is a 1-1 *-homomorphism.

In Section 2, we develop the necessary theory of ultraproducts of C^*-algebras and in Section 3, we give our main results on finite representations.

2. Ultraproducts in C^*-algebras

Let $(E_i)_{i \in I}$ be a family of Banach spaces and $\mathcal{U}$, a non trivial ultrafilter on I. Let $l_\infty(I, E_i)$ denote the Banach space of bounded functions $x = (x_i)$ in

$\prod_{i\in I} E_i$ with the norm $\|(x_i)\|=\sup_i\|x_i\|$ and let $N_{\mathcal{U}}$ be the subspace of those (x_i) for which $\lim_{\mathcal{U}}\|x_i\|=0$. The ultraproduct $(E_i)_{\mathcal{U}}$ is the quotient space $l_\infty(I, E_i)/N_{\mathcal{U}}$. Its elements are the equivalence classes $(x_i)_{\mathcal{U}}$ of elements $(x_i)\in l_\infty(I, E_i)$ and $\|(x_i)_{\mathcal{U}}\|=\lim_{\mathcal{U}}\|x_i\|$. If E_i's are Banach algebras (respectively C^*-algebras) then so is $(E_i)_{\mathcal{U}}$ under the operations induced by pointwise operations (cf. [5], [7] Proposition 3.1).

For Banach spaces E and F let E' denote the dual of E and $L(E, F)$ denote the Banach space of bounded linear operators on E to F with the operator norm. $L(E, E)$ will be denoted by $L(E)$. Then as in §2 [7], for another family $(F_i)_{i\in I}$ of Banach spaces, $(L(E_i, F_i))_{\mathcal{U}}$ can be identified with a closed subspace of $L((E_i)_{\mathcal{U}}, (F_i)_{\mathcal{U}})$ via

$$(T_i)_{\mathcal{U}}(x_i)_{\mathcal{U}}=(T_i x_i)_{\mathcal{U}}\,.$$

Further it follows by Lemma 7.4 [7] that for a finite dimensional space M,

$$(L(M, F_i))_{\mathcal{U}}=L(M, (F_i)_{\mathcal{U}})\,.$$

In fact, this follows from the observation that for a finite dimensional space M and $M_i=M$ for each i, we have that $x_0=\lim_{\mathcal{U}} x_i$ exists for each $(x_i)\in l_\infty(I, M)$ and thus $(M_i)_{\mathcal{U}}$ can be identified with M via $(x_i)_{\mathcal{U}}\to\lim_{\mathcal{U}} x_i$. This also gives

$$(L(E_i, M))_{\mathcal{U}}\subset L((E_i)_{\mathcal{U}}, M)\,.$$

In particular, $(E_i')_{\mathcal{U}}\subset(E_i)_{\mathcal{U}}'$ via

$$(f_i)_{\mathcal{U}}(x_i)_{\mathcal{U}}=\lim_{\mathcal{U}} f_i(x_i)\,.$$

If $(E_i)_{\mathcal{U}}$ is reflexive, then $(E_i')_{\mathcal{U}}=(E_i)_{\mathcal{U}}'$. ([8], cf. [7], Proposition 7.1).

For $T\in L(X, Y)$, let T' denote the adjoint operator on Y' to X'. In case X and Y are Hilbert spaces, let T^* be the adjoint operator induced on Y to X by T', when X is identified with X' and Y with Y' via Riesz representation theorem i.e. $\langle Tx, y\rangle=\langle x, T^*y\rangle$ for $x\in X$ and $y\in Y$. Our next result may be compared with Theorem 1.2 and other results in [12].

Proposition 2.1. (i) *For* $(T_i)_{\mathcal{U}}\in(L(E_i, F_i))_{\mathcal{U}}$ *and* $(f_i)_{\mathcal{U}}\in(F_i')_{\mathcal{U}}$, *we have* $(T_i)_{\mathcal{U}}'(f_i)_{\mathcal{U}}=(T_i'f_i)_{\mathcal{U}}$. *If* $(F_i)_{\mathcal{U}}$ *is reflexive then* $(T_i)_{\mathcal{U}}'=(T_i')_{\mathcal{U}}$.

(ii) *Suppose that for each* i, E_i *is a Hilbert space. Then*

(a) $(E_i)_{\mathcal{U}}$ *is a Hilbert space and for* $(x_i)_{\mathcal{U}}, (y_i)_{\mathcal{U}}\in(E_i)_{\mathcal{U}}$, $\langle(x_i)_{\mathcal{U}}, (y_i)_{\mathcal{U}}\rangle=\lim_{\mathcal{U}}\langle x_i, y_i\rangle$.

(b) $(L(E_i))_{\mathcal{U}}$ *is a* C^*-*subalgebra of* $L((E_i)_{\mathcal{U}})$ *and for* $(T_i)_{\mathcal{U}}\in(L(E_i))_{\mathcal{U}}$, $C((T_i)_{\mathcal{U}})\subset(C(T_i))_{\mathcal{U}}$, *where for an operator* T *on a Hilbert space* H, $C(T)$ *denotes the* C^*-*algebra generated by* T. *In particular, for* $(T_i)_{\mathcal{U}}, (S_i)_{\mathcal{U}}\in(L(E_i))_{\mathcal{U}}$, *we* *have* $(T_i)_{\mathcal{U}}(S_i)_{\mathcal{U}}=(T_iS_i)_{\mathcal{U}}$, $(T_i)^*_{\mathcal{U}}=(T^*_i)_{\mathcal{U}}$ *and* $(T_i)_{\mathcal{U}}$ *is positive if and only if*

$$(T_i)_{\mathcal{U}}=(V^*_iV_i)_{\mathcal{U}}=(W^2_i)_{\mathcal{U}}=(W_i)^2_{\mathcal{U}}$$

for some $(V_i)_{\mathcal{U}}\in C((T_i)_{\mathcal{U}})$ *and* $(W_i)_{\mathcal{U}}\in(C(T_i))_{\mathcal{U}}$ *with* W_i *positive for each* i.

Proof. (i) The first part follows from the observation that for $(x_i)_{\mathcal{U}}\in(E_i)_{\mathcal{U}}$, we have

$$\begin{aligned}[(T_i)'_{\mathcal{U}}(f_i)_{\mathcal{U}}](x_i)_{\mathcal{U}}&=(f_i)_{\mathcal{U}}[(T_i)_{\mathcal{U}}(x_i)_{\mathcal{U}}]=(f_i)_{\mathcal{U}}(T_ix_i)_{\mathcal{U}}\\&=\lim_{\mathcal{U}}f_i(T_ix_i)=\lim_{\mathcal{U}}(T'_if_i)(x_i)\\&=(T'_if_i)_{\mathcal{U}}(x_i)_{\mathcal{U}}.\end{aligned}$$

The second part is now immediate from the first part since $(F'_i)_{\mathcal{U}}=(F_i)'_{\mathcal{U}}$ in case $(F_i)_{\mathcal{U}}$ is reflexive.

(ii) The first part of (a) follows from the fact that a Banach space is a Hilbert space if and only if its norm satisfies the parallelogram law. Indeed, for $(x_i)_{\mathcal{U}}, (y_i)_{\mathcal{U}}\in(E_i)_{\mathcal{U}}$, we have

$$\begin{aligned}&\|(x_i)_{\mathcal{U}}+(y_i)_{\mathcal{U}}\|^2+\|(x_i)_{\mathcal{U}}-(y_i)_{\mathcal{U}}\|^2\\&=\lim_{\mathcal{U}}\|x_i+y_i\|^2+\lim_{\mathcal{U}}\|x_i-y_i\|^2\\&=\lim_{\mathcal{U}}[\|x_i+y_i\|^2+\|x_i-y_i\|^2]\\&=\lim_{\mathcal{U}}2[\|x_i\|^2+\|y_i\|^2]\\&=2[\lim_{\mathcal{U}}\|x_i\|^2+\lim_{\mathcal{U}}\|y_i\|^2]\\&=2\|(x_i)_{\mathcal{U}}\|^2+2\|(y_i)_{\mathcal{U}}\|^2.\end{aligned}$$

The second part is now an immediate consequence of the polarization identity.

The first part of (b) is immediate from (i) and the fact that $(C(T_i))_{\mathcal{U}}$ is a C^*-algebra containing $(T_i)_{\mathcal{U}}$. Further since $C((T_i)_{\mathcal{U}})$ is also a C^*-algebra containing $(T_i)_{\mathcal{U}}$, the latter is positive if and only if $(T_i)_{\mathcal{U}}=(V_i)^*_{\mathcal{U}}(V_i)_{\mathcal{U}}$ for some $(V_i)_{\mathcal{U}}\in C((T_i)_{\mathcal{U}})$. The proof can be completed by putting $W_i=(V^*_iV_i)^{1/2}$.

Let $n\in\boldsymbol{N}$. For a Hilbert space E, let E^n denote the Hilbert space of n-tuples $x=(\xi_j)_{1\le j\le n}$ from E with the inner product

$$\langle x, y\rangle=\langle(\xi_j)_{1\le j\le n}, (\eta_j)_{1\le j\le n}\rangle=\sum_{j=1}^{n}\langle\xi_j, \eta_j\rangle.$$

Then the matrix algebra $M_n(L(E))$ of $n\times n$ matrices with elements in $L(E)$ can be made into a C^*-algebra if we identify it with $L(E^n)$ via $(a_{jk})_{1\le j,k\le n}\rightarrow$ the operator A given by

$$(Ax)=\left(\sum_{k=1}^{n} a_{jk}\xi_k\right)_{1\le j\le n} \quad \text{for} \quad x=(\xi_j)_{1\le j\le n}\in E^n.$$

We have the following results:

Proposition 2.2. *Let $n\in N$. Let for each i, E_i be a Hilbert space. Then*
(i) *the Hilbert space $(E_i^n)_{\mathcal{U}}$ can be identified with the Hilbert space $(E_i)_{\mathcal{U}}^n$ via*

$$(x_i)_{\mathcal{U}}=((\xi_{ij})_{1\le j\le n})_{\mathcal{U}} \longrightarrow ((\xi_{ij})_{\mathcal{U}})_{1\le j\le n}.$$

(ii) *the C^*-algebra $M_n((L(E_i))_{\mathcal{U}})$ can be identified with $(M_n(L(E_i)))_{\mathcal{U}}$ via*

$$((T_{j,k,i})_{\mathcal{U}})_{1\le j,k\le n} \longrightarrow ((T_{j,k,i})_{1\le j,k\le n})_{\mathcal{U}}.$$

Proof. (i) We first note that for $(x_i)\in l_\infty(I, E_i^n)$ with $x_i=(\xi_{ij})_{1\le j\le n}$ for $i\in I$, we have

$$\lim_{\mathcal{U}}\|x_i\|^2=\lim_{\mathcal{U}}\sum_{j=1}^{n}\|\xi_{i,j}\|^2=\sum_{j=1}^{n}\lim_{\mathcal{U}}\|\xi_{i,j}\|^2.$$

So the map $(x_i)_{\mathcal{U}}\to((\xi_{ij})_{\mathcal{U}})_{1\le j\le n}$ is well defined on $(E_i^n)_{\mathcal{U}}$ and it satisfies

$$\|(x_i)_{\mathcal{U}}\|=\|((\xi_{i,j})_{\mathcal{U}})_{1\le j\le n}\|.$$

This map is now easily seen to be linear and onto $(E_i)_{\mathcal{U}}^n$.

(ii) We first note that because of (i), the C^*-algebra $L((E_i^n)_{\mathcal{U}})$ can be identified with the C^*-algebra $L((E_i)_{\mathcal{U}}^n)=M_n(L((E_i)_{\mathcal{U}}))$.

Since $(L(E_i))_{\mathcal{U}}$ is a C^*-subalgebra of $L((E_i)_{\mathcal{U}})$, we have that the C^*-algebra $M_n((L(E_i))_{\mathcal{U}})$ can be identified with a C^*-subalgebra of $M_n(L((E_i)_{\mathcal{U}}))$ and consequently with a C^*-subalgebra of $L((E_i^n)_{\mathcal{U}})$. For $(T_{jk})_{1\le j,k\le n}=((T_{j,k,i})_{\mathcal{U}})_{1\le j,k\le n}\in M_n((L(E_i))_{\mathcal{U}})$, the corresponding element of $L((E_i^n)_{\mathcal{U}})$ is given by $((T_{j,k,i})_{1\le j,k\le n})_{\mathcal{U}}$.

Corollary 2.3. *$(T_{jk})_{1\le j,k\le n}=((T_{j,k,i})_{\mathcal{U}})_{1\le j,k\le n}\in M_n((L(E_i))_{\mathcal{U}})$ is positive if and only if for each i there is $(S_{j,k,i})_{1\le j,k\le n}\in M_n(L(E_i))$ such that*

$$(T_{jk})_{1\le j,k\le n}=\left(\left(\sum_{l=1}^{n} S_{l,j,i}^{*}S_{l,k,i}\right)_{\mathcal{U}}\right)_{1\le j,k\le n}.$$

Proof. We have only to apply Proposition 2.1(ii) (b) with E_i replaced by E_i^n for each i.

Proposition 2.4. *Let for each i, E_i and F_i be Hilbert spaces, X_i and Y_i be closed subspaces of $L(E_i)$ and $L(F_i)$ respectively and $\phi_i: X_i\to Y_i$ be a linear map. Suppose that $\sup_i\|\phi_i\|<\infty$. Let ϕ denote $(\phi_i)_{\mathcal{U}}: (X_i)_{\mathcal{U}}\to(Y_i)_{\mathcal{U}}$.*

(i) *If each of the X_i and Y_i are *-subalgebras and each ϕ_i is a *-homomorphism then ϕ is a *-homomorphism.*

(ii) *If each* ϕ_i *is completely bounded with* $\sup_i \|\phi_i\|_{cb} < \infty$ *then* ϕ *is completely bounded and* $\|\phi\|_{cb} \leqq \lim_{\mathcal{U}} \|\phi_i\|_{cb}$, *equality occurs if* $\mathcal{U}$ *is countably complete.*

In particular, if each ϕ_i *is a complete contraction then so is* ϕ.

(iii) *If there is an* $I_0 \in \mathcal{U}$ *such that for each* i *in* I_0, ϕ_i *is a complete isometry then so is* ϕ.

(iv) *If each of the* X_i *is a C*-subalgebra of* $L(E_i)$ *or an operator system and each* ϕ_i *is completely positive then* ϕ *is completely positive.*

Proof. (i) is immediate from the definition of $(\phi_i)_{\mathcal{U}}$.

(ii) Let $n \in \boldsymbol{N}$. Let $T = (T_{jk})_{1 \le j, k \le n} \in M_n((X_i)_{\mathcal{U}})$. By Proposition 2.2(ii) above we have that

$$\begin{aligned}
\|\phi_n(T)\| &= \lim_{\mathcal{U}} \|(\phi_n(T))_i\| \\
&= \lim_{\mathcal{U}} \|\phi_{in}((T_{j,k,i})_{1 \le j, k \le n})\| \\
&\leqq \lim_{\mathcal{U}} (\|\phi_{in}\| \cdot \|(T_{j,k,i})_{1 \le j, k \le n}\|) \\
&\leqq \lim_{\mathcal{U}} \|\phi_i\|_{cb} \lim_{\mathcal{U}} \|(T_{j,k,i})_{1 \le j, k \le n}\| \\
&= (\lim_{\mathcal{U}} \|\phi_i\|_{cb}) \|T\|.
\end{aligned}$$

Therefore, ϕ is completely bounded and $\|\phi\|_{cb} \leqq \lim_{\mathcal{U}} \|\phi_i\|_{cb}$.

Now suppose that $\mathcal{U}$ is countably complete and $\|\phi\|_{cb} < \lim_{\mathcal{U}} \|\phi_i\|_{cb}$. Choose any r such that $\|\phi\|_{cb} < r < \lim_{\mathcal{U}} \|\phi_i\|_{cb}$. Then there is an $I_0 \in \mathcal{U}$ such that $\|\phi_i\|_{cb} > r$ for each i in I_0. Now for each i in I_0 there is a least $n_i \in \boldsymbol{N}$ such that $\|\phi_{in_i}\| > r$. So there exists an $A_i \in M_{n_i}(X_i)$ such that $\|A_i\| = 1$ but $\|\phi_{in_i}(A_i)\| > r$. For $k \in \boldsymbol{N}$, let $I_k = \{i \in I_0 : n_i > k\}$. Then $\{I_k\}$ is a decreasing sequence of sets in I with $\cap I_k = \phi$. Since $\mathcal{U}$ is countably complete there is a k_0 such that $I_{k_0} \notin \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter, $J = \{i \in I_0 : n_i \leqq k_0\} = I_0 \setminus I_{k_0}$ is in $\mathcal{U}$. For $i \in J$, let $B_i = (B_{i,j,k})_{1 \le j, k \le k_0}$ be the member of $M_{k_0}(X_i)$ obtained from A_i by putting in zeros at the vacant places i.e. $B_{i,j,k} = 0$ for $j > n_i$ or $k > n_i$ and $B_{i,j,k} = A_{i,j,k}$ for $1 \leqq j$, $k \leqq n_i$, where $A_i = (A_{i,j,k})_{1 \le j, k \le n_i}$. For $i \in I \setminus J$, let B_i be the zero $k_0 \times k_0$ matrix in $M_{k_0}(X_i)$. Then $(B_i)_{\mathcal{U}} \in (M_{k_0}(X_i))_{\mathcal{U}}$ and

$$\|(B_i)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|B_i\| = \lim_{\mathcal{U}} \|A_i\| = 1.$$

By Proposition 2.2(ii) again

$$B = ((B_{i,j,k})_{1 \le j, k \le n})_{\mathcal{U}} \in M_{k_0}((X_i)_{\mathcal{U}}) \quad \text{and} \quad \|B\| = 1.$$

Also

$$\|\phi_{k_0}(B)\| = \lim_{\mathcal{U}} \|\phi_{ik_0}(B_{i,j,k})_{1 \le j, k \le k_0}\|$$

$$=\lim_{\mathcal{U}}\|\phi_{in_i}(A_{i,j,k})_{1\leq j,k\leq n_i}\|$$

$$\geqq r,$$

so that $\|\phi_{k_0}\|\geqq r$, which, in turn, gives that $\|\phi\|_{cb}\geqq r>\|\phi\|_{cb}$. This contradiction establishes the required equality.

(iii) Suppose that $I_0\in\mathcal{U}$ is such that for each $i\in I_0$, ϕ_i is a complete isometry. Then for $n\in N$,

$$A=(A_{jk})_{1\leq j,k\leq n}=((A_{j,k,i})_{\mathcal{U}})_{1\leq j,k\leq n}\in M_n((X_i)_{\mathcal{U}})$$

we have that

$$\|\phi_n(A)\|=\lim_{\mathcal{U}}\|\phi_{in}(A_{j,k,i})_{1\leq j,k\leq n}\|$$

$$=\lim_{\mathcal{U}}\|(A_{j,k,i})_{1\leq j,k\leq n}\|$$

$$=\|A\|, \quad \text{by Proposition 2.2(ii), again.}$$

So, ϕ is a complete isometry.

(iv) We have to prove that for $n\in N$, $T=(T_{j,k})_{1\leq j,k\leq n}\in M_n((X_i)_{\mathcal{U}})$ positive, $\phi_n(T)=(\phi(T_{j,k}))_{1\leq j,k\leq n}\in M_n((Y_i)_{\mathcal{U}})$ is positive. By Proposition 2.2, T can be identified with

$$((T_{j,k,i})_{1\leq j,k\leq n})_{\mathcal{U}}\in(M_n(X_i))_{\mathcal{U}}\subset(L(E_i^n))_{\mathcal{U}},$$

where $(T_{j,k})_{1\leq j,k\leq n}=((T_{j,k,i})_{\mathcal{U}})_{1\leq j,k\leq n}$. By Proposition 2.1, $T=(V_i^*V_i)_{\mathcal{U}}$, where for each i, $V_i\in C((T_{j,k,i})_{1\leq j,k\leq n})$. Let $i\in I$. If X_i is a C^*-subalgebra of $L(E_i)$, then $V_i\in M_n(X_i)$ and thus $V_i^*V_i\in M_n(X_i)$ is positive. Since ϕ_i is completely positive we have that ϕ_{in} is positive and, therefore, $S_i=\phi_{in}(V_i^*V_i)$ is positive. For notational convenience we write $Z_i=X_i$ and $\psi_i=\phi_i$ in this case. On the other hand if X_i is not a $C*$-algebra but an operator system then by Arveson's Extension Theorem, ϕ_i has an extension to a completely positive map ψ_i on $Z_i=L(E_i)$ to $L(F_i)$ with $\|\psi_i\|_{cb}=\|\phi_i\|_{cb}=\|\phi_i(1_{E_i})\|$ and therefore, $S_i=\psi_{in}(V_i^*V_i)$ is positive. Now in all cases

$$\phi_n(T)=(\phi_{in}(T_{j,k,i})_{1\leq j,k\leq n})_{\mathcal{U}}=(\psi_{in}(T_{j,k,i})_{1\leq j,k\leq n})_{\mathcal{U}}$$

$$=(\psi_{in}(V_i^*V_i))_{\mathcal{U}}=(S_i)_{\mathcal{U}}\in(L(F_i^n))_{\mathcal{U}}.$$

So by Proposition 2.1, $\phi_n(T)$ is positive.

Remark 2.5. (a) Even though we assume axiom of choice throughout to avoid entering into the discussion of various axioms in set theory like Axiom of choice, Ultrafilter Principle and Axiom of constructibility, it may be pointed out that every ultrafilter on the set of natural numbers is countably incomplete and the least cardinal carrying a countably complete ultrafilter is inaccessible (see for instance [15]).

(b) If in (iv) above, each X_i is an operator system, then

$$\|\phi\|_{cb}=\|\phi(1_{(E_i)_{\mathcal{U}}})\|=\|(\phi_i(1_{E_i}))_{\mathcal{U}}\|$$
$$=\lim_{\mathcal{U}}\|\phi_i(1_{E_i})\|=\lim_{\mathcal{U}}\|\phi_i\|_{cb}.$$

Thus equality in (i) above can occur without the stringent condition of countable completeness of $\mathcal{U}$.

3. The Local Structure of Ultraproducts of C^*-algebra

In this section we shall study the local structure of ultraproducts of C^*-algebras. We shall find certain conditions under which a closed subspace (*-subspace or *-subalgebra) of $L(F)$, F being a Hilbert space, is completely isometric to a subspace (respectively *-subspace, *-subalgebra) of an ultraproduct of C^*-algebras. Throughout the section unless otherwise stated, for Hilbert spaces E and F, X and Y are closed subspaces of $L(F)$ and $L(E)$ respectively. For a 1-1 linear map $\phi: X\to Y$, X will be taken as an operator system or a C^*-subalgebra of $L(F)$ whenever ϕ is taken to be completely positive. Accordingly, X is taken as a C^*-subalgebra of $L(F)$ whenever ϕ is taken to be a *-homomorphism. With this understanding we define the following notions:

Definition 3.1. Let $\varepsilon>0$ and ϕ be a $(1+\varepsilon)$-isomorphism on X onto Y i.e. ϕ is a 1-1 linear map on X onto Y with $\|\phi\|\leqq 1+\varepsilon$, $\|\phi^{-1}\|\leqq 1+\varepsilon$.

(i) ϕ will be called a *complete* $(1+\varepsilon)$-isomorphism if $\|\phi\|_{cb}\leqq 1+\varepsilon$ and $\|\phi^{-1}\|_{cb}\leqq 1+\varepsilon$. In this case, the spaces X and Y are said to be completely $(1+\varepsilon)$-isomorphic.

(ii) ϕ will be called a *completely positive* $(1+\varepsilon)$-isomorphism if ϕ and ϕ^{-1} are completely positive. In this case X and Y are said to be completely order $(1+\varepsilon)$-isomorphic.

Remark 3.2. (a) Each 1-1 *-homomorphism on X onto Y is clearly a completely positive $(1+\varepsilon)$-isomorphism [16], but the converse is not true. In fact Choi and Christensen [4] have shown that for each $\varepsilon>0$, there exist C^*-algebras $\mathcal{A}_\varepsilon$ and $\mathcal{B}_\varepsilon$ and completely positive, 1-1, onto maps $\phi_\varepsilon: \mathcal{A}_\varepsilon\to\mathcal{B}_\varepsilon$ and $\phi_\varepsilon^{-1}: \mathcal{B}_\varepsilon\to\mathcal{A}_\varepsilon$ such that $\|\phi_\varepsilon\|\leqq 1+\varepsilon$ and $\|\phi_\varepsilon^{-1}\|\leqq 1+\varepsilon$ ([4], Theorem 3.5), but there exists no *-homomorphism on $\mathcal{A}_\varepsilon$ onto $\mathcal{B}_\varepsilon$ ([4], Theorem 3.3).

(b) A completely positive $(1+\varepsilon)$-isomorphism on X onto Y is a complete $(1+\varepsilon)$-isomorphism if both X and Y are operator systems.

We begin with a variant of Proposition 6.2 [7], the formulation and the basic part of the proof being similar.

Theorem 3.3. *For a Hilbert space F, let X be a closed subspace of $L(F)$*

and let $\mathcal{B}$ be a family of C^*-algebras. Let $\mathcal{M}_X$ be a family of closed subspaces of X ordered by set-inclusion such that for any finite subset (equivalently finite dimensional subspace) S of X, there exists $M\in\mathcal{M}_X$ with $S\subseteq M$. Suppose that for each $\varepsilon>0$ and each $M\in\mathcal{M}_X$, there is $B\in\mathcal{B}$ such that M is completely $(1+\varepsilon)$-isomorphic (respectively completely order $(1+\varepsilon)$-isomorphic) to a subspace of B. Then there is an ultrafilter $\mathcal{U}$ on an index set I and a map from I into $\mathcal{B}$ (assigning to each i, a C^*-algebra $E_i\in\mathcal{B}$) so that X is completely isometric (respectively completely order isometrically isomorphic) to a subspace of $(E_i)_{\mathcal{U}}$.

Proof. Let I be the collection of all pairs (M, ε) where $M\in\mathcal{M}_X$ and $\varepsilon>0$. The set I can be partially ordered by the relation

$$(M_1, \varepsilon_1)\prec(M_2, \varepsilon_2) \quad \text{if} \quad M_1\subset M_2 \quad \text{and} \quad \varepsilon_1\geqq\varepsilon_2 .$$

Then the filter $\mathcal{F}$ associated with this order consists of all sets $I_0\subset I$ for which there is an element (M_0, ε_0) of I with

$$I_0=\{(M, \varepsilon): (M_0, \varepsilon_0)\prec(M, \varepsilon)\} .$$

Let now $\mathcal{U}$ be an ultrafilter containing this order filter $\mathcal{F}$. Now $\mathcal{B}$ is a family of C^*-algebras such that for each $i=(M_i, \varepsilon_i)\in I$, there exists an $E_i\in\mathcal{B}$ such that M_i is completely $(1+\varepsilon_i)$-isomorphic (respectively completely order $(1+\varepsilon_i)$-isomorphic) to a subspace N_i of E_i. Define $\iota: X\to(E_i)_{\mathcal{U}}$ as follows:

$$\text{For } x\in X, \text{ we put } \iota x=(y_i)_{\mathcal{U}}, \qquad y_i=\begin{cases} \phi_i(x) & \text{if } x\in M_i \\ 0 & \text{if } x\notin M_i . \end{cases}$$

Then ι is a linear isometry. Let $n\in\boldsymbol{N}$ and $\varepsilon_0>0$. Consider an element $A=(a_{pq})_{1\leq p,q\leq n}\in M_n(X)$. Consider any subspace $M_0\in\mathcal{M}_X$ containing a_{pq}, $1\leqq p, q\leqq n$. Then

$$I_0=\{(M, \varepsilon): (M_0, \varepsilon_0)\prec(M, \varepsilon)\}\in\mathcal{U} ,$$

$\mathcal{U}_0=\{J\cap I_0: J\in\mathcal{U}\}$ is an ultrafilter on I_0 and $(E_i)_{\mathcal{U}_0}$ can be identified with $(E_i)_{\mathcal{U}}$ by identifying $(x_i)_{i\in I_0}$ with $(z_i)_{i\in I}$, where $z_i=x_i$ for i in I_0 and zero otherwise. For $i\in I_0$, ϕ_i and ϕ_i^{-1} are completely bounded with $\|\phi_i\|_{cb}\leqq 1+\varepsilon_i\leqq 1+\varepsilon_0$ and $\|\phi_i^{-1}\|_{cb}\leqq 1+\varepsilon_i\leqq 1+\varepsilon_0$, (respectively completely positive with $\|\phi_i\|\leqq 1+\varepsilon_0$ and $\|\phi_i^{-1}\|\leqq 1+\varepsilon_0$). So by Proposition 2.4,

$$\phi=(\phi_i)_{\mathcal{U}_0}: (M_i)_{\mathcal{U}_0} \longrightarrow (N_i)_{\mathcal{U}_0}$$

and

$$\phi^{-1}=(\phi_i^{-1})_{\mathcal{U}_0}: (N_i)_{\mathcal{U}_0} \longrightarrow (M_i)_{\mathcal{U}_0}$$

are completely bounded with $\|\phi\|_{cb}$ and $\|\phi^{-1}\|_{cb}\leqq 1+\varepsilon_0$ (respectively completely positive).

Now M_0 can be considered as a subspace of $(M_i)_{\mathcal{U}_0}$ and $\iota=\phi|_{M_0}$ and $\iota^{-1}=\phi^{-1}|_{\iota(M_0)}$ satisfy

$$\begin{aligned}\|\iota_n(A)\| = \|\phi_n(A)\| &\leqq \|\phi_n\| \|A\| \leqq (1+\varepsilon_0)\|A\| \\ &\leqq (1+\varepsilon_0)\|\phi_n^{-1}\phi_n(A)\| \\ &\leqq (1+\varepsilon_0)\|\phi_n^{-1}\| \|\iota_n(A)\| \\ &\leqq (1+\varepsilon_0)^2 \|\iota_n(A)\|\end{aligned}$$

(respectively, $\iota_n(A)$ is positive if and only if A is positive). Since $\varepsilon_0>0$ is arbitrary, we have that $\|\iota_n(A)\|=\|A\|$, further since A is arbitrary we have that ι_n is an isometry and since n is arbitrary it follows that ι is a complete isometry (respectively, ι_n and ι_n^{-1} are both positive and since n is arbitrary it follows that both ι and ι ^{1} are completely positive).

Remark 3.4. In the second part of the above theorem in case X is an operator system, we may replace I and $\mathcal{U}$ by $I_1=\{(M, \varepsilon): 1_F \in M\}$ and $\mathcal{U}_1:=\{J \cap I_1 : J \in \mathcal{U}\}$ respectively and as explained in Remark 2.5(b), ι becomes a complete contraction.

Remark 3.5. If $\mathscr{B}$ in the above theorem is taken to be a collection of C^*-subalgebras of a C^*-algebra $Y \subset L(E)$ then we may term conditions in the theorems as X is completely finitely (respectively completely order finitely) representable in Y. Then Theorem 3.3 gives us that if X is completely (respectively completely order) finitely reprsentable in Y then there is an ultrafilter $\mathcal{U}$ such that X is completely isometrically (respectively completely order isometrically) isomorphic to a subspace of $(Y)_{\mathcal{U}}$.

Unlike in Banach spaces only partial converses of these results are true which we now discuss. We have the following partial converse to Theorem 3.3 under the inaccessible condition of countable completeness of $\mathcal{U}$.

Theorem 3.6. *Let $(E_i)_{i\in I}$ be a family of C^*-algebras and let M be a finite dimensional *-subspace of $(E_i)_{\mathcal{U}}$. Let $\varepsilon>0$. If $\mathcal{U}$ is countably complete then there exists an $i \in I$, a *-subspace $M_i \subset E_i$ such that M is completely $(1+\varepsilon)$-isomorphic to M_i.*

Proof. *Step I*: We first proceed as in the proof of the corresponding result in Banach spaces (cf. [15], [11]). Let $\{x_1, x_2, \cdots, x_n\}$ be a basis of M with x_j represented by $(x_{ji})_i$. Since all norms on $\boldsymbol{C}^n$ are equivalent, there exists $K_n>0$ such that for each $(\lambda_j)_{1\leq j\leq n} \in \boldsymbol{C}^n$,

$$K_n \left\| \sum_{j=1}^{n} \lambda_j x_j \right\| \geqq \sup_{j=1}^{n} |\lambda_j| \, .$$

Put $\varepsilon_n = \dfrac{\varepsilon}{(1+\varepsilon)K_n}$. So there exists a member I_0 of $\mathcal{U}$ such that for each i

in I_0,

$$\left|\left(\left\|\sum_{j=1}^{n}\lambda_j x_j\right\|-\left\|\sum_{j=1}^{n}\lambda_j x_{ji}\right\|\right)\right|<\varepsilon_n K_n\left\|\sum_{j=1}^{n}\lambda_j x_j\right\|.$$

So for each $i\in I_0$, ϕ_i defined on M onto M_i, the space spanned by $\{x_{ji}, 1\leqq j\leqq n\}$ in E_i as $\phi_i\left(\sum_{j=1}^{n}\lambda_j x_j\right)=\sum_{j=1}^{n}\lambda_j x_{ji}$ satisfies

$$\|\phi_i\|\leqq 1+\varepsilon \quad \text{and} \quad \|\phi_i^{-1}\|\leqq 1+\varepsilon .$$

Step II: Let us assume that $\mathcal{U}$ is countably complete and $M\subset(E_i)_{\mathcal{U}}$ be a finite dimensional subspace of dimension n as above. Let $\{x_1, x_2, \cdots, x_n\}$ be a basis of M and x_j be represented by $(x_{ji})_i$. As proved in Step I, there exists $J_1\in\mathcal{U}$ such that for each i in J_1, $\|\phi_i\|\leqq 1+\varepsilon$ and $\|\phi_i^{-1}\|\leqq 1+\varepsilon$.

Consider $M_2(M)$. $(x_j E_{pq})_{1\leq p,q\leq 2,\ 1\leq j\leq n}$ is a basis of $M_2(M)$ and dimension of $M_2(M)$ $=2^2 n$. There exists a set $J_2'\in\mathcal{U}$ such that for each i in J_2', $\|\phi_{i,2}\|\leqq 1+\varepsilon$ and $\|\phi_{i,2}^{-1}\|\leqq 1+\varepsilon$. Put $J_2=J_1\cap J_2'$. Then for each i in J_2,

$$\|\phi_i\|\leqq 1+\varepsilon,\ \|\phi_i^{-1}\|\leqq 1+\varepsilon,\ \|\phi_{i,2}\|\leqq 1+\varepsilon,\ \|\phi_{i,2}^{-1}\|\leqq 1+\varepsilon .$$

Proceeding like this, we get a countable family $J_1\supset J_2\supset\cdots\supset J_n\cdots$ in $\mathcal{U}$, such that for each i in J_n, $\|\phi_{i,n}\|\leqq 1+\varepsilon$ and $\|\phi_{i,n}^{-1}\|\leqq 1+\varepsilon$. Since $\mathcal{U}$ is countably complete, $\bigcap_{i=1}^{\infty} J_i$ is nonempty. Let $J=\bigcap_{i=1}^{\infty} J_i$. Then for each i in J, ϕ_i is a complete $(1+\varepsilon)$-isomorphism.

Remark 3.7. Taking each E_i to be a fixed C^*-subalgebra Y of $L(E)$, we may say that if $\mathcal{U}$ is countably complete, then $(Y)_{\mathcal{U}}$ is completely finitely representable in Y.

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