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ANALYSIS OF SCHMITT TRIGGERS

By

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Abstract. In this note we will show that Schmitt triggers, one consisting of an operational amplifier and one consiting of two transisitors, undergo the same type of bifurcation when inputs are considered as distinguished parameters and outputs as state variables. These are very simple and illustrative examples of the singularity theory applied for bifurcations developed by Golubitsky and Schaeffer ([1]). It is interesting that two apparently different circuits have the same mathematical principle corresponding to the function of the circuits.

Furthermore, we can identify the unfolding parameters so that the possible local behaviors of the perturbed circuits are determined.

1. Preliminaries

We recall what we need from Golubitsky-Schaeffer's theory ([1]). Let $f: R\times R\rightarrow R$ be a smooth map, precisely defined only on a neighborhood of the origin (0, 0).

Consider the nonlinear equation;

$$f(x, \lambda)=0,$$

where the variable x represents the state of the system and the variable λ the parameter of the system. We will treat the problem that how the state x changes when λ is changed slightly. Now we can assume $f(0,0)=0$ without loss of generality by parallel translations. Two equations $f=0$ and $g=0$, or simply f and g, are strongly equivalent if and only if there exist two smooth functions X, S defined on some neighborhood of the origin, satisfying the following conditions;

(1) $g(x, \lambda)=S(x, \lambda)f(X(x, \lambda), \lambda)$,

(2) $X(0,0)=0$

(3) $S(x, \lambda)>0$, $X_x(x, \lambda)>0$.

This means that the appropriate coordinate changes transform the function f into g and hence the local behaviors of the variable x satisfying $f(x, \lambda)=0$ and $g(x, \lambda)=0$ are qualitatively the same. We will use the following result of [1].

Proposition A. *If a function f satisfies the following conditions:*

$$f(0,0)=D_x f(x,\lambda)=D_x^2 f(x,\lambda)=\cdots=D_x^{k-1}f(0,0)=0,$$

$$D_x^k f(0,0)\neq 0,\quad D_\lambda f(0,0)\neq 0,$$

then f is strongly equivalent to the function

$$\varepsilon x^k+\delta\lambda,$$

where ε is the sign of $D_x^k f(0,0)$ and δ is the sign of $D_\lambda f(0,0)$.

(By D_x and D_λ we denote the partial derivatives with respect to x and λ, respectively.)

Next we recall the notion of universal unfoldings. The smooth function $F: R\times R\times R^k\rightarrow R$ defined on some neighborhood of the origin is called k-parameter unfolding of f iff $F(x,\lambda,0)=f(x,\lambda)$. Let F be a k-parameter unfolding of f and G a l-parameter unfolding of f. We say the unfolding G factors through the unfolding F iff there exist smooth mappings

$$S, X: R\times R\times R^l\longrightarrow R,\quad \Lambda: R\times R^l\longrightarrow R$$

and

$$A: R^l\longrightarrow R^k$$

satisfying

$$G(x,\lambda,\beta)=S(x,\lambda,\beta)F(X(x,\lambda,\beta),\Lambda(x,\beta),A(\beta))$$

and

$$S(x,\lambda,0)=1,\quad X(x,\lambda,0)=x,\quad \Lambda(x,0)=x\quad \text{and}\quad A(0)=0.$$

This means that all the behaviors of x satisfying $G(x,\lambda,\beta)=0$ are included in that of x satisfying $F(x,\lambda,\alpha)=0$.

An unfolding F of f is versal if every unfolding of f factors through F. A versal unfolding of f with the minimum number of parameters is called universal. If unfolding F of f is versal, then all the behaviors of x satisfying $\hat{f}(x,\lambda)=0$ where $\hat{f}$ is any perturabation of f are realized in $F(x,\lambda,\alpha)=0$. The following result characterized the universal unfolding of $x^3\pm\lambda$ ([1]).

Proposition B. *Let $g(x,\lambda)$ be a smooth function strongly equivalent to $x^3\pm\lambda$. An unfolding G of g is universal if and only if*

$$\det\begin{vmatrix} G_\lambda(0,0,0) & G_{\lambda x}(0,0,0)\\ G_\alpha(0,0,0) & G_{\alpha x}(0,0,0)\end{vmatrix}\neq 0.$$

2. Analysis of Schmitt trigger consisting of an operation amplifier.

We consider the circuit shown in Fig. 1. We assume the input-output voltage relation of the operational amplifier is represented as follows;

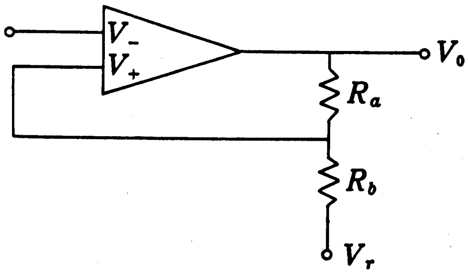

Fig. 1 A Schmitt trigger circuit consisting of an operational amplifier.

$$V_0=f(V_+-V_-)$$

where the function f satisfies the following;

(1) $f(0)=0$, (2) $f'(0)>0$, (3) $f''(0)=0$, (4) $f'''(0)<0$.

The conditions (3), (4) mean that voltage amplication factor $f'(x)$ attains its local maximum at $x=0$. For simplicity assuming the input impedance is high enough, the output impedance low enough and the values of R_a, R_b large enough, we consider only the voltage relations. We get the following;

(5) $V_0=f(V_+-V_-)$, (6) $V_+=(R_b/(R_a+R_b))V_0+(R_a/(R_a+R_b))V_r$.

Putting $x=V_0$, $\lambda=V_-$, $m=(R_b/(R_a+R_b))$ and $\gamma=V_r$, we obtain the equation,

(7) $x=f(mx+(1-m)\gamma-\lambda)$.

Put

(8) $g(x, \lambda)=x-f(m_0x+(1-m_0)\gamma-(\lambda+\lambda_0))$, where

$m_0=1/f'(0)$, $\lambda_0=(1-m_0)\gamma$,

then the solution of the equation $g(x, \lambda)=0$ for given λ represents the output of the circuit when the input is $\lambda-\lambda_0$ in the case of $m=m_0$.

Theorem 1. *The above g is strongly equivalent to* $x^3+\lambda$.

Proof. By the assumption (1)-(4) we can easily obtain the following;

$$g(0, 0)=g_x(0, 0)=g_{xx}(0, 0)=0,$$

$$g_{xxx}(0, 0)=-m_0^3f'''(0)>0, \quad g_\lambda(0, 0)=f'(0)>0.$$

Then by Proposition A we get the result.

Next we investigate the behaviors of the system when the system parameter m is perturbed around m_0. Put

$$G_0(x, \lambda, m)=x-f(mx+(1-m)\gamma-\lambda),$$

$$G(x, \lambda, m)=G_0(x, \lambda+\lambda_0, m+m_0)$$

and we obtain the following.

Theorem 2. *The unfolding G is universal.*

Proof. We can easily obatain the following;

$$\det\begin{vmatrix} G_\lambda(0,0,0) & G_{\lambda x}(0,0,0) \\ G_a(0,0,0) & G_{ax}(0,0,0) \end{vmatrix}=\det\begin{vmatrix} f'(0) & 0 \\ \gamma f'(0) & -f'(0) \end{vmatrix}=-f'(0)^2<0.$$

Applying Proposition B, we get the result.

This theorem shows that even if the system parameters are perturbed its behaviors are qualitatively one of the cases shown in Fig. 2.

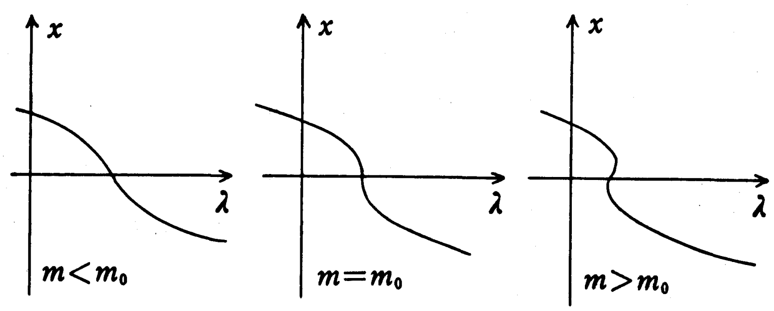

Fig. 2 Bifurcation diagram of Schmitt triggers.

3. Analysis of Schmitt trigger consisting of two transistors.

Consider the circuit shown in Fig. 3. Following Ridders ([2]), we assume the following equations hold.

$$I_1=I_s\exp(V_{BE_1}/V_T)$$

$$I_2=I_s\exp(V_{BE_2}/V_T)$$

$$I_0=I_1+I_2$$

$$V_r=V_1(R_b/(R_b+R_a))$$

$$V_1=E_c-I_1R_1$$

where the transisitors are described by the one-sided Ebers-Moll's model and the currents of the base and R_a, R_b are neglected.

Putting $\alpha=I_0$, $\beta=E_c/V_T$, $\gamma=R_1/V_T$, $m=R_a/(R_a+R_b)$, $\lambda=V_i/V_T$ and $x=I_1$, noting $V_{BE_2}-V_{BE_1}=V_r-V_i$, by simple calculations we can obtain the following

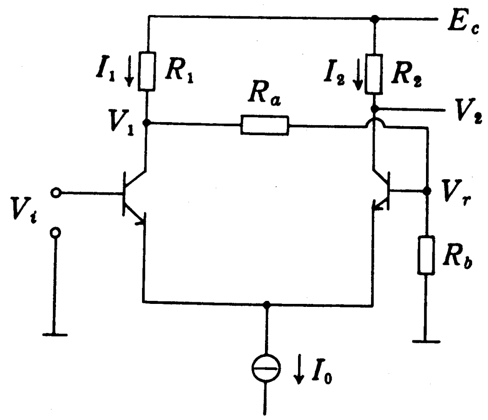

Fig. 3 A Schmitt trigger circuit consisting of two transistors.

equation;

$$x-(\alpha-x)\exp(\lambda-m\beta+m\gamma x)=0$$

We denote the left hand side of the above equation as $g_0(x, \lambda)$ when α is fixed as $\alpha=\alpha_0=4/(m\gamma)$. Put

$$g(x, \lambda)=g_0(x+\alpha_0/2, \lambda+m(\beta-\alpha_0\gamma/2)),$$

then we get the following;

Theorem 3. *The above g is strongly equivalent to the function $x^3-\lambda$.*

Proof. It is easy to verify the followings;

$$g(0, 0)=g_x(0, 0)=g_{xx}(0, 0)=0$$

$$g_{xxx}(0, 0)=m^2\gamma^2>0, \quad g_\lambda(0, 0)=-\alpha_0/2<0.$$

Then by Proposition A, the result follows.

Next we put

$$G_0(x, \lambda, \alpha)=x-(\alpha-x)\exp(\lambda-m\beta+m\gamma x),$$

$$G(x, \lambda, \alpha)=G_0(x+\alpha_0/2, \lambda+m(\beta-\alpha_0\gamma/2), \alpha+\alpha_0).$$

Then the following theorem holds.

Theorem 4. *The function G is a universal unfolding of g.*

Proof. We can easily show the following;

$$\det\begin{vmatrix} G_\lambda(0, 0, 0) & G_{\lambda x}(0, 0, 0) \\ G_\alpha(0, 0, 0) & G_{\alpha x}(0, 0, 0)\end{vmatrix}=\det\begin{vmatrix} -\alpha_0 & -1 \\ -1 & -m\gamma\end{vmatrix}=m\gamma(\alpha_0/2)-1=2-1=1>0.$$

Apllying the Proposition B, we get the result.

The above theorem shows that also in this case when the system is perturbed we have no other type of behavior other than those in Fig. 2, with λ-axis reversed.

References

[1] M. Golubitsky and D.G. Schaeffer: "*Singularity and groups in bifurcation theory*", Vol. 1, Springer-Verlag, Appl. Math. Sci. Vol. 51, 1985.

[2] C.J.F. Ridders: "Accurate determination of threshold voltage levels of a Schmitt trigger", *IEEE Trans. on Circuits and Systems*, Vol. CAS-32, No. 9 (1985), 969-970.

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