YOKOHAMA MATHEMATICAL
JOURNAL VOL. 38, 1990

WEAK CONVERGENCE TO SOME PROCESSES DEFINED BY MULTIPLE WIENER INTEGRALS

By

KEN-ICHI YOSHIHARA

(Received December 4, 1989; Revised January 20, 1990)

Summary. We show that the random process $X_n=\{X_n(t): 0\leqq t\leqq 1\}$ defined by $X_n(t)=\Sigma Q(i_1/N, \cdots, i_m/N)\xi_{n,i_1}\cdots\xi_{n,i_m}$ converges weakly in $D[0,1]$ to some process defined by multiple Wiener integrals when $\{\xi_{n,i}\}$ is a martingale difference array or a strictly stationary sequence of random variables satisfying some mixing condition.

1. Introduction. Very recently, Teicher (1988) showed that the limit distribution of sums of product of martingale difference $\{\xi_{n,j};\ j\leqq N_n,\ n\geqq 1\}$, i.e.,

$$\sum_{1\leq i_1<\cdots<i_m\leq N_n}\xi_{n,i_1}\cdots\xi_{n,i_m}\quad(m\geqq 2) \tag{1.1}$$

coincides with the distribution of

$$\frac{1}{m!}H_m(Z) \tag{1.2}$$

where H_m is the Hermite polynomial of degree m and Z is a standard normal random variables. The problem concerning asymptotic distributions of the type (1.1) is closely related to the problem on asymptotic distributions of some symmetric statistics. (See, for example, Mandelbaum and Taqq (1984).) In this paper, we shall mainly consider the problem concerning weak convergence of random processes $X_n=\{X_n(t): 0\leqq t\leqq 1\}$ $(n\geqq 1)$ defined by

$$X_n(t)=\sum_{1\leq i_1<\cdots<i_m\leq[N_nt]}Q\Big(\frac{i_1}{N_n}, \cdots, \frac{i_m}{N_n}\Big)\xi_{n,i_1}\cdots\xi_{n,i_m}\quad(0\leqq t\leqq 1) \tag{1.3}$$

where the function $Q: [0, 1]^m\to R$ is continuous and has continuous first derivatives and $[s]$ denotes the largest integer m such that $s\geqq m$.

American Mathematical Society 1980 subject classifications. Primary 60F17; Secondary 60H05.

Key words and phrases: multiple Wiener integral, weak convergence, sum of products of random variables, martingale difference array, strictly stationary process, ϕ-mixing, strongly mixing.

2. Main results. (I) The triangular array case. Let $N=N(n)$ be a nondegreasing integer-valued function such that $N(n)\to\infty$ and $n\to\infty$. Let $\{\xi_{n,j};1\leqq j\leqq N,\ n\geqq 1\}$ be a triangular array of real-valued independent random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $D[0, L]$ be the space of functions x on $[0, L]$ that are right-continuous and have left-hand limits. We endow $D[0, 1]$ the J_1-topology. Let Q be a real-valued continuous function, defined on $[0, 1]^m$, which has continuous first partial derivatives.

Theorem 1. *Let $\{\xi_{n,i}:1\leqq i\leqq N,\ n\geqq 1\}$ be a triangular array of independent zero-mean random variables which satisfy the following conditions:*

(i) *For arbitrary numbers t_1 and t_2* $(0\leqq t_1<t_2\leqq 1)$

$$\sum_{i=[Nt_1]+1}^{[Nt_2]} E\xi_{n,i}^2 \longrightarrow t_2-t_1 \quad \textit{as } n\to\infty. \tag{2.1}$$

(ii) *For all* $\varepsilon>0$

$$\sum_{j=1}^{N} E\{\xi_{n,j}^2 I(|\xi_{n,j}>\varepsilon)\} \longrightarrow 0 \quad \textit{as } n\to\infty. \tag{2.2}$$

Then the stochastic process X_n, defined by (1.3) *converges weakly in* $D[0, 1]$ *to the process $X=\{X(t):0\leqq t\leqq 1\}$ defined by*

$$X(t)=\int_0^t\int_0^{u_{m-1}}\cdots\int_0^{u_1} Q(u_0, u_1, \cdots, u_{m-1})\prod_{j=0}^{m-1} W(du_j) \tag{2.3}$$

where the right hand side of (1.4) *is the m-ple Wiener integral with respect to the standard Wiener process $W=\{W(t):0\leqq t\leqq 1\}$.*

Remark. It is known that as $n\to\infty$

$$\left\{\sum_{j=1}^{[nt]}\xi_{n,j}:0\leqq t\leqq 1\right\} \xrightarrow{D} W \tag{2.4}$$

if conditions of Theorem 1 are satisfied.

(II) The mixing case. Let $\{\xi_j\}$ be a strictly stationary sequence of zero-mean random variables. We say that $\{\xi_j\}$ satisfies the ϕ-mixing condition if

$$\phi(n)=\sup_{A\in\mathcal{M}_{-\infty}^0,\ B\in\mathcal{M}_n^\infty}\left|\frac{P(AB)-P(A)P(B)}{P(A)}\right| \longrightarrow 0 \quad \text{as } n\to\infty \tag{2.5}$$

and that $\{\xi_j\}$ satisfies the strongly mixing condition if

$$\alpha(n)=\sup_{A\in\mathcal{M}_{-\infty}^0,\ B\in\mathcal{M}_n^\infty}|P(AB)-P(A)P(B)| \longrightarrow 0 \quad \text{as } n\to\infty, \tag{2.6}$$

where $\mathcal{M}_a^b$ denotes the σ-algebra generated by $\xi_a, \cdots, \xi_b$.

We put

(2.7) $$\sigma_0^2 = E\xi_0^2.$$

We define formally σ^2 by

(2.8) $$\sigma^2 = E\xi_0^2 + 2\sum_{j=1}^{\infty} E\xi_0\xi_j$$

and assume $\sigma^2 > 0$. It is known that if the conditions in Theorem 2 or 3 (below) are satisfied, then the series in (2.8) converges absolutely.

Theorem 2. *Let $\{\xi_j\}$ be a strictly stationary sequence of zero-mean random variables. Suppose one of the following two groups of conditions holds:*

(i) *$\{\xi_i\}$ is ϕ-mixing,*

(2.9) $$E|\xi_0|^8 < \infty$$

and

(2.10) $$\sum n\{\phi(n)\}^{1/8} < \infty;$$

(ii) *$\{\xi_i\}$ is strongly mixing and there exists a δ $(0<\delta<1)$ such that*

(2.11) $$E|\xi_0|^{8+2\delta} < \infty$$

and

(2.12) $$\sum n\{\alpha(n)\}^{\delta/(8+2\delta)} < \infty.$$

Let Q and σ^2 be as before. Then, the process $Y_n = \{Y_n(t): 0 \leqq t \leqq 1\}$, defined by

(2.13) $$Y_n(t) = n^{-1}\sigma^{-2}\sum_{1\leq i<j\leq[nt]} Q\left(\frac{i}{n}, \frac{j}{n}\right)\xi_i\xi_j \quad (0\leqq t\leqq 1)$$

converges weakly in $D[0, 1]$ to the process $Y = \{Y(t): 0\leqq t\leqq 1\}$ defined by

(2.14) $$Y(t) = \int\int_{0\leq t_1<t_2\leq t} Q(t_1, t_2)dW(t_1)dW(t_2) + \frac{\sigma^2-\sigma_0^2}{2\sigma^2}\int_0^t Q(s, s)ds \quad (0\leqq t\leqq 1).$$

Next, we consider the process $Z_n^{(m)} = \{Z_n^{(m)}(t): 0\leqq t\leqq 1\}$ defined by

(2.15) $$Z_n^{(m)}(t) = (n\sigma)^{-m/2}\sum_{1\leq i_1<\cdots<i_m\leq[nt]} \xi_{i_1}\cdots\xi_{i_m} \quad (0\leqq t\leqq 1).$$

Let

(2.16)
$$Z^{(1)}(t) = W(t)$$
$$Z^{(2)}(t) = \int_0^t W(s)dW(s) + \frac{\sigma^2-\sigma_0^2}{2\sigma^2}t$$
$$Z^{(k)}(t) = \int_0^t Z^{(k-1)}(s)dW(s) + \frac{\sigma^2-\sigma_0^2}{2\sigma^2}\int_0^t Z^{(k-2)}(s)ds \quad (k\geqq 3)$$
$$(0\leqq t\leqq 1).$$

Then

$$Z^{(2)}(t)=\frac{1}{2!}H_2(W(t))+\frac{\sigma^2-\sigma_0^2}{2\sigma^2}t$$

(2.17)
$$Z^{(3)}(t)=\frac{1}{3!}H_3(W(t))+\frac{\sigma^2-\sigma_0^2}{2\sigma^2}t\,W(t)$$

$$Z^{(4)}(t)=\frac{1}{4!}H_4(W(t))+\frac{\sigma^2-\sigma_0^2}{2\sigma^2}\frac{1}{2!}t\,H_2(W(t))+\frac{1}{2!}\left(\frac{\sigma^2-\sigma_0^2}{\sigma^2}\right)^2 t^2.$$

Let $Z^{(k)}=\{Z^{(k)}(t): 0\leqq t\leqq 1\}$.

Theorem 3. *Let $\{\xi_j\}$ be a strictly stationary sequence of zero-mean random variables. Suppose one of the following two groups of conditions hold:*

(i) *$\{\xi_j\}$ is ϕ-mixing and there exists a p $(p>0)$ such that*

(2.18)
$$E|\xi_0|^{2(m+p)}<\infty$$

and

(2.19)
$$\sum n^{m+p-1}\{\phi(n)\}^{1/2(m+p)}<\infty\,;$$

(ii) *$\{\xi_n\}$ is strongly mixing and there exist two positive numbers p and δ $(0<\delta<1)$ such that*

(2.20)
$$E|\xi_0|^{2(m+p+\delta)}<\infty$$

and

(2.21)
$$\sum n^{m+p-1}\{\alpha(n)\}^{\delta/2(m+p)+\delta}<\infty.$$

Let σ^2 and σ_0^2 be the ones defined above. Then

(2.22)
$$Z_n^{(m)}\xrightarrow{D}Z^{(m)}\quad in\ D[0,1]$$

as $n\to\infty$.

3. Proof of Theorem 1. To prove Theorem 1 we need a theorem which is applicable to many cases.

Let F_M be the space of functions defined on $[0,1]\times(-\infty,\infty)$ satisfying the following condition: there exists an absolute constant M such that if $f\in F_M$, then f and its derivatives satisfy inequalities of the form

(3.1)
$$|Df(s,x)|\leqq M(1+|x|^\alpha),$$

where D denotes either the identity operator or a first derivatives and α is some positive constant. Let $\{(\xi_{n,j},\mathcal{F}_{n,j});1\leqq j\leqq N, n\geqq 1\}$ be a martingale difference array, i.e., an array satisfying the following conditions:

(i) $E|\xi_{n,j}|<\infty$.

(ii) $\xi_{n,j}$ is measurable with respect to the σ-algebra $\mathcal{F}_{n,j}$ where $\mathcal{F}_{n,0}\subset\cdots\subset\cdots\subset\mathcal{F}_{n,N}\subset\mathcal{F}$ $(n\geqq 1)$.

(iii) $E(\xi_{n,j}|\mathcal{F}_{n,j-1})=0$ a.s.

Let $\{T_{n,j};\ 1\leqq j\leqq N,\ n\geqq 1\}$ be an array of random variables defined by

$$T_{n,j}=g_{n,j}(\xi_{n,1},\cdots,\xi_{n,j})\quad(1\leqq j\leqq N,\ n\geqq 1)\tag{3.2}$$

where $g_{n,j}: R^j\to R$ is Borel measurable for each n and j and

$$E|T_{n,j}-T_{n,k}|^2\leqq cN^{-1}|j-k|\quad(1\leqq j,\ k\leqq N)\tag{3.3}$$

for each n. Define a sequence $\{T_n\}=\{T_n(t): 0\leqq t\leqq 1\}$ of random elements in $D[0, 1]$ by

$$T_n(t)=T_{n,[Nt]}\quad(0\leqq t\leqq 1).$$

Theorem 4. *Let $\{(\xi_{n,j}, \mathcal{F}_{n,j}): 1\leqq j\leqq N,\ n\geqq 1\}$ be a square-integrable martingale difference array. Suppose that $f_n\in F_M$ $(n\geqq 1)$, $f\in F_M$ and for every $s\in[0, 1]$*

$$Df_n(s, x)\longrightarrow Df(s, x)\tag{3.4}$$

uniformly in x on every finite interval. Suppose there exists a double array $\{C_{n,j},\ 1\leqq j\leqq N,\ n\geqq 1\}$ of negative numbers such that

$$E\{\xi^2_{n,j+1}|\mathcal{F}_{n,j}\}\leqq C_{n,j}\quad(1\leqq j\leqq N)$$

and for arbitrary numbers t_1 and t_2 $(0\leqq t_1<t_2\leqq 1)$

$$\sum_{j=[Nt_1]+1}^{[Nt_2]} C_{n,j}\leqq C_0(t_2-t_1)$$

where C_0 is some positive numbers. Suppose further that as $n\to\infty$

$$\left\{\sum_{j=1}^{[Nt]}\xi_{n,j}: 0\leqq t\leqq 1\right\}\xrightarrow{D} Z=\{Z(t): 0\leqq t\leqq 1\}\quad in\ D[0, 1]\tag{3.5}$$

and

$$\{T_n(t): 0\leqq t\leqq 1\}\xrightarrow{D} T=\{T(t): 0\leqq t\leqq 1\}\quad in\ D[0, 1].\tag{3.6}$$

Then

$$\left\{\sum_{j=1}^{[Nt]} f_n\left(\frac{j}{n}, T_{n,j}\right)\xi_{n,j+1}: 0\leqq t\leqq 1\right\}\tag{3.7}$$

$$\xrightarrow{D}\left\{\int_0^t f(s, T(s))dZ(s): 0\leqq t\leqq 1\right\}\quad in\ D[0, 1]$$

where the stochastic integral in (3.7) is taken in the sense of convergence in probability.

Proof. The proof is carried out by the essentially same method as that of Theorem 1 in Szyszkowski (1988) and so is omitted. (cf. Yoshihara (1982) and Strasser (1986).)

From Theorem 4 we easily obtain the following corollary.

Corollay 1. *Let* $\{(\xi_{n,j}, \mathcal{F}_{n,j}): 1 \leqq j \leqq N, n \geqq 1\}$ *be a square-integrable martingale difference array such that*

(3.8) $$E\xi_n^* = E \max_{1 \leq j \leq N} |\xi_{n,j}| \longrightarrow 0 \quad \text{as } n \to \infty$$

and

(3.9) $$\sum_{j=1}^{N} \xi_{n,j}^2 \xrightarrow{P} 1 \quad \text{as } n \to \infty .$$

Let $\{f_n\}$ *and* f *be the ones in Theorem* 4. *Suppose* (3.4) *and* (3.6) *hold. Then*

(3.10) $$\left\{ \sum_{j=0}^{[Nt]-1} f_n\left(\frac{j}{n}, T_{n,j}\right)\xi_{n,j+1} : 0 \leqq t \leqq 1 \right\} \xrightarrow{D} \left\{ \int_0^t f(s, T(s))dW(s) : 0 \leqq t \leqq 1 \right\} \quad \text{in } D[0, 1],$$

where the stochastic integral in (3.9) *is taken in the* L^2*-sense.*

Using the above corollary repeatedly, we have the following corollary which was obtained by Teicher (1988) when $t=1$.

Corollary 2. *Suppose conditions of Corollary* 1 *to Theorem* 2 *are satisfied. Then*

(3.11) $$\{ \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq [Nt]} \xi_{n,i_1}\xi_{n,i_2} \cdots \xi_{n,i_m} : 0 \leqq t \leqq 1\} \xrightarrow{D} \left\{ \int_0^t \int_0^{u_{m-1}} \cdots \int_0^{u_1} \sum_{j=0}^{m-1} dW(u_{j-1}) : 0 \leqq t \leqq 1 \right\} \quad \text{in } D[0, 1].$$

Proof of Theorem 1. Firstly, we consider the case $m=2$. We note that by virtue of Theorem 4 for any t_1 $(0 < t_1 \leqq 1)$

(3.12) $$\left\{ \sum_{i=1}^{[Nt]-1} Q\left(\frac{i}{N}, t_1\right)\xi_{n,i} : 0 \leqq t \leqq t_1 \right\} \xrightarrow{D} \left\{ \int_0^t Q(s, t_1)dW(s) : 0 \leqq t \leqq t_1 \right\} \quad \text{in } D[0, t_1]$$

as $n \to \infty$.

Next, for any t $(0 < t \leqq 1)$ let

$$\Delta : 0=u_0<u_1< \cdots <u_b=t$$

be an arbitrary partition of the interval $[0, t]$ and put

$$\gamma=\gamma_\Delta=\max_{1\leq i\leq b}(u_i-u_{i-1}).$$

Further, let

$$\varepsilon_{i,j,l,n}(\Delta)=Q\Big(\frac{i}{N}, \frac{j}{N}\Big)-Q\Big(\frac{i}{N}, u_l\Big), \tag{3.13}$$

and

$$\varepsilon_n(\Delta)=\max_{1\leq i\leq N}\max_{1\leq l\leq b}\max_{[Nu_{l-1}]+1\leq j\leq [Nu_l]}|\varepsilon_{i,j,l,n}(\Delta)|. \tag{3.14}$$

Since by assumption Q has continuous partial derivatives on $[0, 1]^2$.

$$\lim_{\gamma\to 0}\varepsilon_n(\Delta)=0. \tag{3.15}$$

Now, put

$$Y_n(t:\Delta)=\sum_{l=1}^{b}\sum_{i=[Nu_{l-1}]}^{[Nu_l]-1}\Big\{\sum_{j=i+1}^{[Nu_l]}Q\Big(\frac{i}{N}, u_l\Big)\xi_{n,i}\xi_{n,j} + \sum_{k=l}^{b-1}\sum_{j=[Nu_k]+1}^{[Nu_{k+1}]}Q\Big(\frac{i}{N}, u_k\Big)\xi_{n,i}\xi_{n,j}\Big\}. \tag{3.16}$$

Then

$$\begin{aligned} X_n(t)-Y_n(t:\Delta) &= \sum_{l=1}^{b-1}\sum_{i=[Nu_{l-1}]}^{[Nu_l]-1}\Big\{\sum_{j=i+1}^{[Nu_l]}\varepsilon_{i,j,l,n}(\Delta)\xi_{n,i}\xi_{n,j} \\ &\quad + \sum_{k=l}^{b-1}\sum_{j=[Nu_k]+1}^{[Nu_{k+1}}\Big\}\varepsilon_{i,j,k,n}(\Delta)\xi_{n,i}\xi_{n,j}. \end{aligned} \tag{3.17}$$

and

$$E|X_n(t)-Y_n(t:\Delta)|^2\leqq \varepsilon_n^2(\Delta)\Big\{\sum_{i=1}^{[Nt]}E\,\xi_{n,i}^2\Big\}^2\leqq c\,\varepsilon_n^2(\Delta)t^2 \tag{3.18}$$

which implies that for any τ (>0)

$$\lim_{\gamma\to 0}\lim_{n\to\infty}P(|X_n(t)-Y_n(t:\Delta)|>\tau)=0. \tag{3.19}$$

On the other hand, by virtue of Theorem 4, we have that for each t $(0<t\leqq 1)$

$$Y_n(t:\Delta)\xrightarrow{D} Y(t:\Delta)=\sum_{k=0}^{b-1}\int_0^{u_k}Q(s, u_k)dW(s)\{W(u_{k+1})-W(u_k)\} \tag{3.20}$$

as $n\to\infty$.

Since it is easily shown that

$$\lim_{\gamma\to 0} E|Y(t:\Delta)-X(t)|^2=0, \tag{3.21}$$

so we have that for each t $(0<t\leqq 1)$

$$X_n(t)\xrightarrow{D} X(t) \quad \text{as } n\to\infty. \tag{3.22}$$

Clearly, using the Szyszkowski method (1988) and (2.1) we can prove that the finite dimensional distributions of $\{X_n, n\geqq 1\}$ converge weakly to those of X and that $\{X_n, n\geqq 1\}$ is tight. Hence, we obtain the desired conclusion when $m=2$.

Now, using this method repeatedly, we have the conclusion in the general case m $(\geqq 3)$. □

4. Proof of Theorem 2.

We only consider the strongly mixing case since the proof in the ϕ-mixing case is carried out analogously.

We often use the following inequalities:

(A) Let ζ_1 be an $\mathcal{M}_{-\infty}^{k}$-measurable random variable with $E|\zeta_1|^{2+\gamma}<\infty$ and let ζ_2 be an $\mathcal{M}_{k+n}^{\infty}$-measurable random variable with $E|\zeta_2|^{2+\gamma}<\infty$ where γ is a positive number. Then

$$|E\zeta_1\zeta_2-E\zeta_1E\zeta_2|\leqq 10\|\zeta_1\|_{2+\gamma}\|\zeta_2\|_{2+\gamma}\{\alpha(n)\}^{\gamma/(2+\gamma)} \tag{4.1}$$

where $\|\zeta\|_p=\{E|\zeta|^p\}^{1/p}$ $(p\geqq 1)$ (cf. Ibragimov and Linnik (1971)).

(B) Suppose condition (ii) in Theorem 2 is satisfied. Then

$$E\left|\sum_{i=1}^{n} a_i\xi_i\right|^{\gamma}\leqq c\, M_\delta\left(\sum_{i=1}^{n} a_i^2\right)^{\gamma/2} \quad (1\leqq\gamma\leqq 8+\delta) \tag{4.2}$$

where

$$M_\delta=\max_{1\leq i\leq n}\|\zeta_i\|_{\gamma+\delta}^{\gamma}.$$

Remark. If condition (i) in Theorem 2 is satisfied, then

$$E\left|\sum_{i=1}^{n} a_i\xi_i\right|^{\gamma}\leqq c\, M_0\left(\sum_{i=1}^{n} a_i^2\right)^{\gamma/2} \quad (1\leqq\gamma\leqq 8) \tag{4.3}$$

where

$$M_0=\max_{1\leq i\leq n}\|\xi_i\|_{\gamma}^{\gamma}.$$

(cf. Utev (1985)).

(C) Suppose conditions in Theorem 3 are satisfied. Then

(4.4) $$\left\| \sum_{i=a}^{b} \xi_i \right\|_{\gamma} \leqq c(b-a)^{1/2}$$

for $2 \leqq \gamma \leqq 2(m+p)$. (cf. Yokoyama (1980))

In what follows, we denote by c, with or without subscript, an absolute constant which does not depend on n, s and t.

(I) Weak convergence of the finite-dimensional distributions. For brevity, let

$$\eta_j = \xi_{n,j} = (n^{1/2}\sigma)^{-1}\xi_j \quad (1 \leqq j \leqq n)$$

and

$$Q_{n,i,j} = Q\left(\frac{i}{n}, \frac{j}{n}\right) \quad (1 \leqq i,\ j \leqq n).$$

Further, let $q = [n^{1/4}]$. Then, we have

(4.5) $$\sum_{1 \leq i < j \leq [nt]} Q\left(\frac{i}{n}, \frac{j}{n}\right)\xi_{n,i}\xi_{n,j}$$

$$= \sum_{\substack{1 \leq i < j \leq [nt] \\ j-i > q}} Q_{n,i,j}\eta_i\eta_j + \sum_{\substack{1 \leq i < j \leq [nt] \\ j-i \leq q}} Q_{n,i,j}(\eta_i\eta_j - E\,\eta_i\eta_j)$$

$$+ \sum_{\substack{1 \leq i < j \leq [nt] \\ j-i \leq q}} Q_{n,i,j} E\,\eta_i\eta_j$$

$$= U_n^{(1)}(t) + U_n^{(2)}(t) + U_n^{(3)}(t), \quad \text{(say)}.$$

By the method used in the proof of Theorem in Yoshihara (1980), analogously to (3.22) we can prove

(4.6) $$U_n^{(1)}(t) \xrightarrow{D} \int\!\!\int_{0 \leq t_1 < t_2 \leq t} Q(t_1, t_2)\,dW(t_1)\,dW(t_2)$$

(cf. Takahata (1987)).

Next, we note that by the continuity of Q

(4.7) $$K_0 = \sup_{0 \leq s,\, t \leq 1} |Q(s, t)| < \infty$$

and by the Schwarz inequality

(4.8) $$E\,\eta_i^2\eta_j^2 \leqq \|\eta_1\|_4^4 = \|(n^{1/2}\sigma)^{-1}\xi_1\|_4^4 \leqq c\,n^{-2}.$$

Therefore, by (4.1), (4.7) and (4.8)

$$\|U_n^{(2)}(t)\|_2^2 = \left\| \sum_{i=1}^{[nt]-q} \sum_{j=i+1}^{i+q} Q_{n,i,j}(\eta_i\eta_j - E\,\eta_i\eta_j) \right\|_2^2$$

$$= \sum_{i=1}^{[nt]-q} \left\| \sum_{j=i+1}^{i+q} Q_{n,i,j}(\eta_i\eta_j - E\,\eta_i\eta_j) \right\|_2^2$$

$$+2\Big\{\sum_{\substack{1\le i<i'\le[nt]-q\\ i'-i>2q}}+\sum_{\substack{1\le i<i'\le[nt]-q\\ i'-i\le 2q}}\Big\}$$

$$\Big|E\Big\{\sum_{j=i+1}^{i+q}Q_{n,i,j}(\eta_i\eta_j-E\,\eta_i\eta_j)\sum_{j'=i'+1}^{i'+q}Q_{n,i',j'}(\eta_{i'}\eta_{j'}-E\,\eta_{i'}\eta_{j'})\Big\}\Big|$$

$$\le c\,n\,q^2\cdot n^{-2}$$

$$+c\sum_{\substack{1\le i<i'\le[nt]-q\\ i'-i>2q}}\Big\|\sum_{j=i+1}^{i+q}Q_{n,i,j}(\eta_i\eta_j-E\,\eta_i\eta_j)\Big\|_{2+\delta}$$

$$\times\Big\|\sum_{j'=i'+1}^{i'+q}Q_{n,i',j'}(\eta_{i'}\eta_{j'}-E\,\eta_{i'}\eta_{j'})\Big\|_{2+q}\{\alpha(q)\}^{\delta/(2+q)}$$

$$+c\,n\,q^3n^{-2}$$

$$=o(1)$$

as $n\to\infty$. Hence, we have that for every ε (>0)

$$P(|U_n^{(2)}(t)|>\varepsilon)\longrightarrow 0\quad\text{as } n\to\infty. \tag{4.9}$$

Finally, let

$$0=s_0<s_1<\cdots<s_b=t$$

be arbitrarily and put $a_k=[ns_k]$ $(k=1,\cdots,b)$. Then, we have

$$U_n^{(3)}(t)=\sum_{k=0}^{b-1}\sum_{i=a_k+1}^{a_{k+1}}\sum_{j=i+1}^{i+q}\{Q(s_k,s_k)E\,\eta_i\eta_j+(Q_{n,i,j}-Q(s_k,s_k))E\,\eta_i\eta_j\} \tag{4.10}$$

$$=U_{n,1}^{(3)}(t)+U_{n,2}^{(3)}(t),\quad\text{(say)}.$$

Since

$$E\,\eta_i\eta_j=n^{-1}\sigma^{-2}E\,\xi_i\xi_j$$

and as $n\to\infty$

$$\sum_{j=i+1}^{i+q}E\,\xi_i\xi_j=\sum_{j=1}^{q}E\,\xi_0\xi_j\longrightarrow\frac{1}{2}(\sigma^2-\sigma_0^2),$$

so

$$\lim_{n\to\infty}U_{n,1}^{(3)}(t)=\frac{\sigma^2-\sigma_0^2}{2\sigma^2}\sum_{k=0}^{b-1}Q(s_k,s_k)(s_{k+1}-s_k).$$

Therefore, by the arbitrariness of $\{s_k\}$ we have

$$\lim_{n\to\infty}U_{n,1}^{(3)}(t)=\frac{\sigma^2-\sigma_0^2}{2\sigma^2}\int_0^tQ(s,s)ds. \tag{4.11}$$

On the other hand, by (4.1), (2.12) and (2.13) we have

$$|U_{n,2}^{(3)}(t)|\le c\sum_{k=0}^{b-1}\sum_{i=a_k+1}^{a_{k+1}}\max_{i<j\le i+q}|Q_{n,i,j}-Q(s_k,s_k)|$$

$$\times \sum_{j=i+1}^{i+q} \|\eta_0\|_{2+\delta}^2 \{\alpha(j-i)\}^{\delta/(2+\delta)}$$

$$\leqq c n^{-1} \sum_{k=0}^{b-1} \sum_{i=a_k+1}^{a_{k+1}} \max_{i<j\leq i+q} |Q_{n,i,j}-Q(s_k, s_k)|$$

$$\leqq c \max_{0\leq k<b} \sup_{a_k<i\leq a_{k+1}} \max_{i<j\leq i+q} |Q_{n,i,j}-Q(s_k, s_k)|$$

and so

$$\lim_{n\to\infty} |U_{n,2}^{(3)}(t)| \leqq c \max_{0\leq k<b} \sup_{s_k\leq s, t\leq s_{k+1}} |Q(s, t)-Q(s_k, s_k)|.$$

Therefore, by the continuity of Q and the arbitrariness of $\{s_k\}$ we have

(4.12)
$$\lim_{n\to\infty} |U_{n,2}^{(3)}(t)| = 0.$$

Combining (4.10)-(4.12) we have

(4.13)
$$\lim_{n\to\infty} U_n^{(3)}(t) = \frac{\sigma^2-\sigma_0^2}{2\sigma^2} \int_0^t Q(s, s) ds.$$

Now, from (4.5), (4.6), (4.9) and (4.13) we have

(4.14)
$$Y_n(t) \xrightarrow{D} Y(t)$$

for each t $(0<t\leqq 1)$.

It remains to prove that for each finite set $\{t_i; t_i\in[0, 1], i=1, \cdots, k\}$ the joint distribution of $(Y_n(t_1), \cdots, Y_n(t_k))$ converges weakly to the corresponding one of $(Y(t_1), \cdots, Y(t_k))$. But, this is easily shown by the functional central limit theorem for $\{\sum_{i=1}^k Q(i/n, \cdot)\xi_{n,i}\}$ (cf. Takahata (1987)). The proof is omitted.

(II) The tightness of $\{Y_n\}$. Since $\{Y_n(0)\}$ is tight, so by Theorem 8.3 in Billingsley (1968) it suffices to show that for each positive ε_1 and ε_2 there exist a ρ $(0<\rho<1)$ and an integer n_0 such that

(4.15)
$$\frac{1}{\rho} P(\sup_{s\leq t\leq s+\rho} |Y_n(t)-Y_n(s)| \geqq \varepsilon_1) \leqq \varepsilon_2$$

for $n\geqq n_0$ and $0\leqq s\leqq 1$ (with s in the supremum restricted to $s\leqq t\leqq 1$ in case $1-\rho<s\leqq 1$).

Let s $(0\leqq s<1)$ be fixed arbitrarily. Put $a=[ns]$. Let $q=[n^{1/3}]$. Then, we have

(4.16)
$$\sup_{s\leq t\leq s+\rho} |Y_n(t)-Y_n(s)|$$

$$= \sup_{s\leq t\leq s+\rho} \Big|\sum_{1\leq i<j\leq [nt]} Q_{n,i,j}\eta_i\eta_j - \sum_{1\leq i<j\leq a} Q_{n,i,j}\eta_i\eta_j\Big|$$

$$\leqq \sup_{s\leq t\leq s+\rho} \Big|\sum_{1\leq i<a} \sum_{a\leq j\leq [nt]} Q_{n,i,j}\eta_i\eta_j\Big|$$

$$+\sup_{s\le t\le t+\rho}\Big|\sum_{a\le i<j\le[nt]}Q_{n,i,j}\eta_i\eta_j\Big|$$

$$\le\max_{a\le l\le[n(s+\rho)]}\Big|\sum_{a+q\le j\le l}\Big(\sum_{1\le i<a}Q_{n,i,j}\eta_i\Big)\eta_j\Big|$$

$$+\max_{a\le l\le a+q}\Big|\sum_{1\le i<a-q}\sum_{a\le j\le l}Q_{n,i,j}\eta_i\eta_j\Big|$$

$$+\sum_{a-q\le i<a}\sum_{a\le j\le a+q}|Q_{n,i,j}|\,|\eta_i|\cdot|\eta_j|$$

$$+\max_{a\le l\le[n(s+\rho)]}\Big|\sum_{a\le i<j\le l}Q_{n,i,j}\eta_i\eta_j\Big|$$

$$=U_{n,1}+U_{n,2}+U_{n,3}+U_{n,4},\quad\text{(say)}.$$

Firstly, we prove that for all n sufficiently large

$$P\Big(U_{n,1}>\frac{\varepsilon_1}{4}\Big)\le c\,\rho^2. \tag{4.17}$$

Put

$$V_j=\sum_{1\le i<a}Q_{n,i,j}\eta_i.$$

Then, we have

$$E\Big|\sum_{a+q\le j\le l}\Big(\sum_{1\le i<a}Q_{n,i,j}\eta_i\Big)\eta_j\Big|^4 \tag{4.18}$$

$$\le\sum_j E|V_j\eta_j|^4+\sum_{j_1\ne j_2}E|V_{j_1}\eta_{j_1}|^2|V_{j_2}\eta_{j_2}|^2$$

$$+\sum_{j_1\ne j_2}|E(V_{j_1}\eta_{j_1})^3(V_{j_2}\eta_{j_2})|$$

$$+\sum_{j_1\ne j_2\ne j_3}|E(V_{j_1}\eta_{j_1})^2(V_{j_2}\eta_{j_2})(V_{j_3}\eta_{j_3})$$

$$+\sum_{j_1\ne j_2\ne j_3\ne j_4}|E(V_{j_1}\eta_{j_1})(V_{j_2}\eta_{j_2})(V_{j_3}\eta_{j_3})(V_{j_4}\eta_{j_4})|$$

$$=I_1+I_2+I_3+I_4+I_5,\quad\text{(say)}.$$

We note that by (4.1), (4.2) and (4.7)

$$K_1=\sup_{0<s<1}\max_{1\le j\le n}E|V_j|^4\le c\,K_0^4\sup_{0<s<1}\frac{[ns]^2}{n^2}\le c_1, \tag{4.19}$$

$$K_2=\sup_{0<s<1}\max_{1\le j\le n}E|V_j|^{8+\delta}\le c\,K_0^{8+\delta}\sup_{0<s<1}\frac{[ns]^{4+\delta/2}}{n^{4+\delta/2}}\le c_2 \tag{4.20}$$

and

$$E|\eta_1|^\gamma\le c\,n^{-\gamma/2}E|\xi_1|^\gamma\le c\,n^{-\gamma/2}\quad(1<\gamma\le 8+2\delta)\qquad(j=1,\cdots,n) \tag{4.21}$$

and that, by the definition of q

(4.22) $$\beta=\{\alpha(q)\}^{\delta/(8+2\delta)}=o(q^{-8})=o(n^{-8/3}).$$

Further, we note that for each $j(a+q\leqq j\leqq l)$ V_j is $\mathcal{M}_{-\infty}^{a-1}$-measurable and η_j is $\mathcal{M}_{a+q}^{\infty}$-measurable.

From (4.1) and (4.19)-(4.22) we obtain the following inequalities:

$$I_1\leqq\sum_j\{EV_j^4E\eta_j^4+c\|V_j^4\|_{2+\delta/2}\|\eta_j^4\|_{2+\delta/2}\beta\}$$

$$\leqq\sum_j\{cn^{-2}+cn^{-2}\cdot o(n^{-3/2})\}=o(n^{-1});$$

$$I_2\leqq\sum_{j_1\neq j_2}\{EV_{j_1}^2V_{j_2}^2\cdot E\eta_{j_1}^2\eta_{j_2}^2+c\|V_{j_1}^2V_{j_2}^2\|_{2+\delta/2}\|\eta_{j_1}^2\eta_{j_2}^2\|_{2+\delta/2}\beta\}$$

$$\leqq\sum_{j_1\neq j_2}[K_1\{E\eta_{j_1}^2E\eta_{j_2}^2+c\|\eta_{j_1}^2\|_{2+\delta/2}\|\eta_{j_2}^2\|_{2+\delta/2}(\alpha(|j_2-j_1|)^{\delta/(4+\delta)}\}$$

$$+cK_2^{4/(8+\delta)}n^{-2}\beta]$$

$$\leqq c(l-a)^2(1+o(1))\cdot n^{-2};$$

$$I_3\leqq\sum_{j_1\neq j_2}\{|EV_{j_1}^3V_{j_2}|\cdot|E\eta_{j_1}^3\eta_{j_2}|+c\|V_{j_1}^3V_{j_2}\|_{2+\delta/2}\|\eta_{j_1}^3\eta_{j_2}\|_{2+\delta/2}\beta\}$$

$$\leqq\sum_{j_1\neq j_2}\{K_1\|\eta_{j_1}^3\|_{(8+\delta)/3}\|\eta_{j_2}\|_{8+\delta}(\alpha(|j_2-j_1|)^{(4+\delta)/(8+\delta)}+cn^{-2}\beta\}$$

$$=o(n^{-1});$$

$$I_4\leqq\sum_{j_1\neq j_2\neq j_3}\{|EV_{j_1}^2V_{j_2}V_{j_3}|\cdot|E\eta_{j_1}^2\eta_{j_2}\eta_{j_3}|$$

$$+c\|V_{j_1}^2V_{j_2}V_{j_3}\|_{2+\delta/4}\|\eta_{j_1}^2\eta_{j_2}\eta_{j_3}\|_{2+\delta/4}\beta\}$$

$$\leqq K_1\sum_{j_1\neq j_2\neq j_3}|E\eta_{j_1}^2\eta_{j_2}\eta_{j_3}|+cK_2^{4/(8+\delta)}n\beta$$

$$\leqq c(l-a)^2(1+o(1))\cdot n^{-2};$$

$$I_5\leqq\sum_{j_1\neq j_2\neq j_3\neq j_4}\{|EV_{j_1}V_{j_2}V_{j_3}V_{j_4}|\cdot|E\eta_{j_1}\eta_{j_2}\eta_{j_3}\eta_4|$$

$$+c\|V_{j_1}V_{j_2}V_{j_3}V_{j_4}\|_{2+\delta/4}\|\eta_{j_1}\eta_{j_2}\eta_{j_3}\eta_{j_4}\|_{2+\delta/4}\beta\}$$

$$\leqq K_1\sum_{j_1\neq j_2\neq j_3\neq j_4}|E\eta_{j_1}\eta_{j_2}\eta_{j_3}\eta_{j_4}|+cK_2^{4/(8+\delta)}n^2\beta$$

$$\leqq c(l-a)^2(1+o(1))\cdot n^{-2}.$$

(The last two inequalities are proved by the same method in Yoshihara (1978) or Yokoyama (1980).)

Therefore, we have that for n sufficiently large

$$E|\sum_{a\leq j\leq l}(\sum_{1\leq i<a}Q_{n,i,j}\eta_i)\eta_j|^4\leqq c(l-a)^2(1+o(1))n^{-2},$$

which implies that for all n sufficiently large

$$P\left(U_{n,1}>\frac{\varepsilon_1}{4}\right)\leqq c\rho^2 \tag{4.23}$$

(cf. Theorem 12.2 in Billingsley (1968)).

By the same method, we can prove that as $n\to\infty$

$$P\left(U_{n,2}>\frac{\varepsilon_1}{4}\right)\longrightarrow 0 \tag{4.24}$$

and that for all n snfficiently large

$$P\left(U_{n,4}>\frac{\varepsilon_1}{4}\right)\leqq c\rho^4, \tag{4.25}$$

Finally, we note that by (4.6) and (4.21)

$$EU_{n,3}\leqq K_0 \sum_{a-q\leq i<a}\ \sum_{a\leq j\leq[nt]+q} E|\eta_i||\eta_j|\leqq cq^2n^{-1}E|\xi_0|^2=O(n^{-1/3}).$$

Therefore, we have that as $n\to\infty$

$$P\left(U_{n,3}>\frac{\varepsilon_1}{4}\right)\longrightarrow 0. \tag{4.26}$$

Now, (4.15) follows from (4.16) and (4.23)-(4.26). Thus, the proof of Theorem 2 is completed. □

5. Proof of Theorem 3.

Firstly, we show a lemma.

Lemma. *Let $\{\xi_i\}$ be a strictly stationary sequence of zero-mean random variables. Suppose conditions in Theorem 3 are satisfied. Then, for any γ $(1\leqq\gamma\leqq 2(m+p)/m)$*

$$E\Big|\sum_{a\leq i_1<\cdots<i_m\leq b}\xi_{i_1}\xi_{i_2}\cdots\xi_{i_m}\Big|^\gamma\leqq c(b-a)^{m\gamma/2}. \tag{5.1}$$

Proof. Let J be the collection of all set $(r)=(r_1,\cdots,r_l)$ of integers such that for some k $(\geqq 1)$, $r_1\geqq r_2\geqq\cdots\geqq r_k\geqq 1$ and $r_1+r_2+\cdots+r_k=m$.

We note that

$$\Big|\sum_{a\leq i_1<\cdots<i_m\leq b}\xi_{i_1}\xi_{i_2}\cdots\xi_{i_m}\Big|\leqq\Big|\sum_i\xi_i\Big|^m+c\sum_{(r)}\Big|\prod_{j=1}^{k}\Big(\sum_{i_j}\xi_{i_j}^{r_j}\Big)\Big| \tag{5.2}$$

where $\sum_{(r)}$ denotes the summation over all $(r)\in J$.

Let $(r)\in J$ be fixed arbitrarily and let d be the largest integer such that $r_d=2$. Put

$$T_j=\sum_{i_j}\xi_{i_j}^{r_j} \quad (j=1, \cdots, k).$$

Using the Minkowski inequality first and then the Hölder inequality we get

$$I_{(r)}=\left\|\prod_{j=1}^{k} T_j\right\|_{\gamma} \leq \left\|(T_1-ET_1)\prod_{j=2}^{k} T_j\right\|_{\gamma}+\left\|\left|\sum_{i_1} E\xi_{i_1}^{r_1}\right|\cdot\left|\prod_{j=2}^{k} T_j\right|\right\|_{\gamma}$$

$$\leq \|T_1-ET_1\|_{m\gamma/r_1}\left\|\prod_{j=2}^{k} T_j\right\|_{m\gamma/(m-r_1)}+\left(\sum_{i_1} E|\xi_{i_1}|^{r_1}\right)\left\|\prod_{j=1}^{k} T_j\right\|_{\gamma}.$$

Therefore, by (4.4)

$$I_{(r)}=c(b-a)^{1/2}\left\|\prod_{j=2}^{k} T_j\right\|_{m\gamma/(m-r_1)}+c(b-a)\left\|\prod_{j=1}^{k} T_j\right\|_{\gamma}$$

$$\leq c(b-a)\left\{\left\|\prod_{j=2}^{k} T_j\right\|_{m\gamma/(m-r_1)}+\left\|\prod_{j=2}^{k} T_j\right\|_{\gamma}\right\}.$$

Repeating this procedure d times, we obtain the inequality

$$I_{(r)} \leq c(b-a)^d\{v_1+v_2+\cdots+v_{2d}\}$$

where for each j $(1\leq j\leq 2^d)$ v_j is of the form

$$\left\|\prod_{j=d+1}^{k} T_j\right\|_{\nu}=\left\|\prod_{j=d+1}^{k}\left(\sum_{i_j}\xi_{i_j}\right)\right\|_{\nu} \quad (\nu>1).$$

By the Hölder inequality and (4.4)

$$\left\|\prod_{j=d+1}^{k} T_j\right\|_{\nu} \leq \prod_{j=d+1}^{k}\|T_j\|_{(k-d)\nu} \leq c(b-a)^{(k-d)/2}$$

for any ν (>1).

Hence, noting that $2d+(k-d)\leq m$ we have

(5.3) $$I_{(r)} \leq c(b-a)^d\cdot(b-a)^{(k-d)/2} \leq c(b-a)^{m/2}.$$

On the other hand, by (4.4)

(5.4) $$\left\|\left|\sum_i \xi_i\right|^m\right\|_{\nu} \leq c(b-a)^{m/2}.$$

Now, (5.1) follows from the Minkowski inequality and (5.2)-(5.4). □

(I) The case $m=2$. We consider the case $m=2$. By the proof of Theorem 2 or Theorem 2.1 in Takahata (1987) it is obvious that each finite-dimensional distribution of $Z_n^{(2)}$ converges weakly to the corresponding finite-distribution of $Z^{(2)}$. Hence, it is enough to show that $\{Z_n^{(2)}\}$ is tight.

Let $q=[n^{1/5}]$. Put $a=[ns]$ and $b=[n(s+\rho)]$. Then, corresponding to (4.16) we have

$$\sup_{s\le t\le s+\rho} |Z_n^{(2)}(t)-Z_n^{(2)}(s)| \tag{5.5}$$

$$= \sup_{s\le t\le s+\rho} |\sum_{1\le i<j\le b} \eta_i\eta_j - \sum_{1\le i<j\le a} \eta_i\eta_j|$$

$$\le \max_{a\le l\le b} |\sum_{1\le i<a} \eta_i| \cdot |\sum_{a+q\le j\le l} \eta_j|$$

$$+ \max_{a\le l\le b} |\sum_{1\le i<a-q} \eta_i||\sum_{a\le j\le a+q} \eta_j| + \sum_{a-q\le i\le a} |\eta_i| \cdot \sum_{a\le j\le a+q} |\eta_j|$$

$$+ \max_{a\le l\le b} \left| \sum_{j=2}^{l} \Big(\sum_{i=a}^{j-1} \eta_i \Big) \eta_j \right|$$

$$= U'_{n,1}+U'_{n,2}+U'_{n,3}+U'_{n,4}, \quad \text{(say)}.$$

Put $S_{a,l}=\sum_{j=a}^{l}\eta_j$. Firstly, by (4.1), (4.2) and (4.20) we have

$$E|S_{1,a}|^{2+p}|S_{a+q,l}|^{2+p}$$

$$\le E|S_{1,a}|^{2+p}\cdot E|S_{a+q,l}|^{2+p}$$

$$+c\| |S_{1,a}|^{2+p}\|_{(4+2p+\delta)/(2+p)}\| |S_{a+q,l}|^{2+p}\|_{(4+2p+\delta)/(2+p)}\{\alpha(q)\}^{\delta/(4+2p+\delta)}$$

$$\le c(l-a)^{1+p/2}n^{-(1+p/2)}(1+o(1))$$

for all n sufficiently large. Therefore, we have

$$P\Big(U'_{n,1}>\frac{\varepsilon_1}{4}\Big)\le c\,\rho^{1+p/2} \tag{5.6}$$

(cf. Theorem 12.2 in Billingsley (1968).)

Next, we note that by (4.1), (4.2), (4.4) and (4.21)

$$E|S_{1,a-q}|^{2+p}|S_{a,a+q}|^{2+p}$$

$$\le E|S_{1,a-q}|^{2+p}E|S_{a,a+q}|^{2+p}$$

$$+c\| |S_{1,a}|^{2+p}\|_{(4+2p+\delta)/(2+p)}\| |S_{a,a+q}|^{2+p}\|_{(4+2p+\delta)/(2+p)}\{\alpha(q)\}^{\delta/(4+2p+\delta)}$$

$$\le c\,q^{1+p/2}n^{-(1+p/2)}(1+o(1))$$

as $n\to\infty$. Therefore, we can conclude that as $n\to\infty$

$$P\Big(U'_{n,2}>\frac{\varepsilon_1}{4}\Big)\longrightarrow 0. \tag{5.7}$$

Further, by the inequality

$$E\{\sum_{a-q\le i\le a}|\eta_i|\sum_{a\le j\le a+q}|\eta_j\}\le c\,n^{-1}q^2=O(n^{-1/3})$$

we have that as $n\to\infty$

(5.8) $$P\left(U'_{n,3}>\frac{\varepsilon_1}{4}\right)\longrightarrow 0.$$

Finally, by Lemma

$$E\left|\sum_{j=2}^{l}\left(\sum_{i=a}^{j-1}\eta_i\right)\eta_j\right|^{2+p}\leqq c\{(l-a)n^{-1}\}^{2+p}.$$

Therefore, we have that for all n sufficiently large

(5.9) $$P\left(U'_{n,4}>\frac{\varepsilon_1}{4}\right)\leqq c\,\rho^{2+\rho}.$$

Now, since it is obvious that $\{Z_n(0)\}$ is tight, so it follows from (5.5)-(5.9) that $\{Z_n\}$ is tight. Hence, we obtain the desired conclusion when $m=2$.

(II) The case $m=3$. We note that

(5.10) $$\begin{aligned}Z_n^{(3)}(t)=&\sum_{\substack{1\leq i<j<k\leq[nt]\\ k-j>q}}\eta_i\eta_j\eta_k\\ &+\sum_{\substack{1\leq i<j<k\leq[nt]\\ k-j\leq q,\ j-i>q}}\{(E\,\eta_j\eta_k)\eta_i+\eta_i(\eta_j\eta_k-E\,\eta_j\eta_k)\}\\ &+\sum_{\substack{1\leq i<j<k\leq[nt]\\ k-j\leq q,\ j-i\leq q}}\eta_i\eta_j\eta_k\\ =&V_{n,1}(t)+V_{n,2}(t)+V_{n,3}(t)+V_{n,4}(t),\quad \text{(say)}.\end{aligned}$$

Firstly, by the proof in (I) and Theorem in Yoshihara (1980) we can conclude that for each t $(0\leqq t\leqq 1)$

(5.11) $$\begin{aligned}V_{n,1}(t)\xrightarrow{D}&\int_0^t\left\{\frac{1}{2!}H_2(W(s))+\frac{\sigma^2-\sigma_0^2}{2\sigma^2}s\right\}dW(s)\\ =&\frac{1}{3!}H_3(W(t))+\frac{\sigma^2-\sigma_0^2}{2\sigma^2}\int_0^t s\,dW(s)\\ =&\frac{1}{3!}H_3(W(t))+\frac{\sigma^2-\sigma_0^2}{2\sigma^2}\left\{tW(t)-\int_0^t W(s)ds\right\}.\end{aligned}$$

Next, we note that

$$V_{n,2}(t)=\sum_{j=q+1}^{[nt]-1}\left(\sum_{k=j+1}^{j+q}E\,\eta_j\eta_k\right)\left(\sum_{i=1}^{j-q}\eta_i\right),$$

$$n\sum_{k=j+1}^{j+q}E\,\eta_j\eta_k=\sigma^{-2}\sum_{l=1}^{q}E\,\xi_0\xi_l\longrightarrow\frac{\sigma^2-\sigma_0^2}{2\sigma_0^2}\quad\text{as } n\to\infty,$$

and for each t $(0<t\leqq 1)$

$$\left\{\sum_{i=1}^{[ns]}\eta_i:0\leqq s\leqq t\right\}\xrightarrow{D}\{W(s):0\leqq s\leqq t\}\quad\text{in } D[0,t].$$

Hence, using the method in (II) we have that as $n\to\infty$

(5.12) $$V_{n,2}(t)\xrightarrow{D}\frac{\sigma^2-\sigma_0^2}{2\sigma_0^2}\int_0^t W(s)ds.$$

Thirdly, to evaluate $E|V_{n,3}|^2$, we note that

$$V_{n,3}(t)=\sum_{j=q+1}^{[nt]-1}\Big(\sum_{i=1}^{j-q}\eta_i\Big)\Big\{\sum_{k=j+1}^{j+q}(\eta_j\eta_k-E\,\eta_j\eta_k)\Big\}.$$

For brevity, let $b=[nt]$, $\zeta_j=\sum_{i=1}^{j-q}\eta_i$ and

$$\theta_j=\sum_{k=j+1}^{j+q}(\eta_j\eta_k-E\,\eta_j\eta_k)(q+1\leqq j\leqq b-1).$$

Then, by (4.2)

$$E\,\zeta_j=E\,\theta_j=0,\ \|\zeta_j\|_\gamma\leqq c\,n^{-1/2}j^{1/2}\quad\text{and}\quad\|\theta_j\|_\gamma\leqq c\,n^{-1}q$$

($1\leqq\gamma\leqq 6+2p$). Further, ζ_j is $\mathcal{M}_{-\infty}^{j-q}$-measurable and θ_j is $\mathcal{M}_j^\infty$-measurable. Hence, by the Hölder inequality we have

$$\begin{aligned}E|V_{n,3}(t)|^2&=E\Big|\sum_{j=q+1}^{b-1}\zeta_j\theta_j\Big|^2\\&=\sum_{j=q+1}^{b-1}E\,\zeta_j^2\theta_j^2+2\sum_{q+1\leq j<j'\leq b-1}E\,\zeta_j\theta_j\zeta_{j'}\theta_{j'}\\&\leqq\sum_{j=q+1}^{b-1}[E\,\zeta_j^2E\,\theta_j^2+c\|\zeta_j^2\|_{2+\delta}\|\theta_j^2\|_{2+\delta}\{\alpha(q)\}^{\delta/(2+\delta)}]\\&\quad+c\sum_{q+1\leq j<j'\leq b-1}\|\zeta_j\theta_j\zeta_{j'}\|_{(4+\delta)/3}\|\theta_{j'}\|_{4+\delta}\{\alpha(j'-j)\}^{\delta/(4+\delta)}\\&\leqq c\,b\,n^{-2}q^2(1+o(1))\\&\quad+c\sum_{q+1\leq j<j'\leq b-1}n^{-1}q\|\zeta_j\|_{4+\delta}\|\theta_j\|_{4+\delta}\|\zeta_{j'}\|_{4+\delta}\{\alpha(j'-j)\}^{\delta/(4+\delta)}\\&=o(1).\end{aligned}$$

Therefore, we have

(5.13) $$V_{n,3}\xrightarrow{P}0\quad\text{as }n\to\infty.$$

Finally, by the Hölder inequality we have

$$E|V_{n,4}(t)|\leqq\sum_{\substack{1\leq i<j<k\leq[nt]\\k-j\leq q,\,j-i\leq q}}|E\,\eta_i\eta_j\eta_k|\leqq n\,q^2\|\eta_1\|_3^3\leqq c\,n\,q^2\cdot n^{-3/2}=o(1).$$

Thus, we have

(5.14) $$V_{n,4}\xrightarrow{P}0\quad\text{as }n\to\infty.$$

Now, it follows from (5.11)-(5.14) that

$$Z_n^{(3)}(t) \longrightarrow \frac{1}{3!}H_3(W(t))+\frac{\sigma^2-\sigma_0^2}{2\sigma^2}\int_0^t s\,dW(s)+\frac{\sigma^2-\sigma_0^2}{2\sigma^2}\int_0^t W(s)ds$$

$$=\frac{1}{3!}H_3(W(t))+\frac{\sigma^2-\sigma_0^2}{2\sigma^2}tW(t).$$

It is not hard that each finite-dimensional distribution of $Z_n^{(3)}$ converges weakly to the corresponding finite-dimensional distribution of $Z^{(3)}$.

Further, by the analogous method to the case $m=2$ and Lemma we can easily prove that $\{Z_n^{(3)}\}$ is tight. Hence, the proof in the case $m=3$ is completed.

(III) The general case. By induction we can prove the general case ($m>3$) using the method in (I) and (II) and Lemma. □

References

[1] Billingsley, P.: *Convergence of Probability Measures*, Wiley, New York (1968).

[2] Ibragimov, I. A. and Linnik, Y. V.: *Independent and Stationary Sequence of Random Variables.* Wolters-Noordhoff, Groningen (1971).

[3] Mandelbaum, A. and Taqqu, M. S.: Invariance principle for symmetric statistics. *Ann. Statist.* **12** (1984), 483-496.

[4] Strasser, H.: Martingale difference arrays and stochastic integrals, *Prob. Th. Rel. Fields* **72** (1986), 83-98.

[5] Szyszkowski, I.: The invariance principle for functionals of sums of martingale differences. *Acta Sci. Math.* **52** (1988), 157-63.

[6] Takahata, H.: An invariance principle for integral type functionals of weakly dependent random processes. *Yokohama Math. J.* **35** (1987), 81-93.

[7] Teicher, H.: Distribution and moment convergence of martingales. *Prob. Th. Rel. Fields* **78** (1988), 303-316.

[8] Utev, S. S.: Inequalities for sums of weakly dependent random variables and estimates of the convergence rate in the invariance principle. *Adv. Prob. Th. Limit theorems for sums of random variables* (Ed. by Borovkov, A. A.), 73-114, Optimization Software, Inc., New York.

[9] Yokoyama, R.: Moment bounds for stationary sequences, *Z. Wahrscheinlichkeitstheorie verw. Geb.* **52** (1980), 240-259.

[10] Yoshihara, K.: Moment inequalities for mixing sequences. *Kodai Math. J.* **1** (1978), 316-328.

[11] Yoshihara, K.: A weak convergence theorem for functionals of sums of strong mixing sequences. *Sci. Rep. Yokohama Natl. Univ. Sect.* 1 **27** (1980), 19-25.

[12] Yoshihara, K.: A weak convergence theorem for functionals of sums of independent random variables. *Stochastic Process. Appl.* **12** (1982), 293-299.

[13] Yoshihara, K.: Limit distributions of degenerate U-statistics of degree two for strongly mixing sequences (Preprint) (1989).

Yokohama National University
Department of Mathematics
Faculty of Engineering
156, Tokiwadai, Hodogaya,
Yokohama 240, Japan.