YOKOHAMA MATHEMATICAL
JOURNAL VOL. 38, 1990

FINITE ABELIAN GROUP ACTIONS ON SURFACES

By

BRUNO ZIMMERMANN

(Received February 28, 1989)

We classify free actions of a finite abelian group on orientable and nonorientable surfaces of high genus.

Let $\tilde{M}$ be a closed surface of genus g (orientable or nonorientable), and G a finite abelian group. We consider fixed-point free actions of G on $\tilde{M}$. To each such action corresponds a finite covering $\tilde{M}\to M:=\tilde{M}/G$ determined by a surjection $s: F:=\pi_1 M\twoheadrightarrow G$ with kernel $\pi_1\tilde{M}$. We call two such actions of G on $\tilde{M}$ *equivalent* if they are conjugate by a homeomorphism of $\tilde{M}$ (orientation preserving if $\tilde{M}$ is orientable), inducing the identity on G. Two such actions are equivalent iff for the corresponding surjections $s_1, s_2: F\twoheadrightarrow G$ there exists a commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{\;s_1\;} & \\ a\downarrow\cong & & M \\ F & \xrightarrow{\;s_2\;} & \end{array}$$

where a is an automorphism of the fundamental group F of the quotient surface M which is the same for both actions by the Riemann-Hurwitz formula; one direction of this is clear, for the other one uses that every automorphism of a surface group is induced by a homeomorphism of the surface. In the above situation, we also call s_1 and s_2 *equivalent*.

Theorem. *Let $\tilde{M}$, M and G be as above, $\tilde{g}$=genus $\tilde{M}$, g=genus M. Let n be the minimal number of generators of G.*

a) *Suppose $\tilde{M}$ is orientable, $2(\tilde{g}-1)=|G|2(g-1)$ and $g\geqq n$. Then the equivalence classes of orientation preserving actions of G on $\tilde{M}$ correspond bijectively to the elements of the second homology group $H_2(G, \boldsymbol{Z})$ of G.*

b) *Suppose $\tilde{M}$ is nonorientable, $\tilde{g}-2=|G|(g-2)$ and $g\geqq 2n+4$. Then the equivalence classes of actions of G on M correspond to the elements of $H_2(G, \boldsymbol{Z}_2)$.*

c) *Suppose $\tilde{M}$ is orientable, $2(\tilde{g}-1)=|G|(g-2)$, $g\geqq 2n+4$ and the action of G on $\tilde{M}$ has orientation reversing elements; let $G_0\subset G$ be the (fixed) subgroup of index 2 of orientation preserving elements. We have two cases:*

c1) *There exists an element of order* 2 *in* $G-G_0$, *i.e.* $G\cong G_0\oplus \boldsymbol{Z}_2$. *Then the number of equivalence classes of such actions is half the order of* $H_2(G, \tilde{\boldsymbol{Z}})$, *where* $\tilde{\boldsymbol{Z}}$ *is the G-module* $\boldsymbol{Z}$ *with* G_0 *operating trivially and the elements in* $G-G_0$ *by (-identity) (half of the elements in* $H_2(G, \tilde{\boldsymbol{Z}})$ *correspond to actions with g odd, the other half to actions with g even).*

c2) *There exists no element of order* 2 *in* $G-G_0$, *i.e.* $G\not\cong G_0\oplus \boldsymbol{Z}_2$. *Then there is no action for g odd, and for g even the equivalence classes of such actions correspond to the elements of* $H_2(G, \tilde{\boldsymbol{Z}})$.

Note that the equalities in the theorem between $\tilde{g}$, g and $|G|$ are forced by the Riemann-Hurwitz formula.

In the proof of the theorem, we also give normal forms for the equivalence classes of surjections $F\twoheadrightarrow G$, and an algorithm to bring such a surjection into normal form, hence we get:

Corollary. *There exists an algorithm to decide whether two surjections* $F\twoheadrightarrow G$ *(resp. the corresponding G-actions on* $\tilde{M}$*) are equivalent or not, for* $g=$*genus F as in the theorem.*

The main invariant associated to an action of G on $\tilde{M}$ resp. to the corresponding surjection $s: F\twoheadrightarrow G$ is $\Omega(s)$ defined as follows:

a) $\tilde{M}$ orientable, G orientation preserving:

$$s_*: H_2(F, \boldsymbol{Z})\cong \boldsymbol{Z} \longrightarrow H_2(G, \boldsymbol{Z}), \qquad \Omega(s):=s_*(1),\ 1\in \boldsymbol{Z}.$$

b) $\tilde{M}$ nonorientable:

$$s_*: H_2(F, \boldsymbol{Z}_2)\cong \boldsymbol{Z}_2 \longrightarrow H_2(G, \boldsymbol{Z}_2), \qquad \Omega(s):=s_*(-1),\ -1\in \boldsymbol{Z}_2=\{\pm 1\}.$$

c) $\tilde{M}$ orientable, G with orientation reversing elements:

$$s_*: H_2(F, \tilde{\boldsymbol{Z}})\cong \boldsymbol{Z} \longrightarrow H_2(G, \tilde{\boldsymbol{Z}}), \qquad \Omega(s):=s_*(1),\ 1\in \boldsymbol{Z},$$

where $\tilde{\boldsymbol{Z}}$ is the twisted F- resp. G-module $\boldsymbol{Z}$, with the orientation reversing elements in F resp. G operating by (-identity), the orientation preserving ones trivially.

It is clear that for equivalent surjections $s_1, s_2: F\twoheadrightarrow G$ we have $\Omega(s_1)=\Omega(s_2)$.

Remarks. 1) Part a) of the theorem has also been proved by Edmonds ([E1]), by different methods. For a classification of actions of certain nonabelian finite groups on orientable surfaces, see [E2]. It follows from [L] that for finite nonabelian groups actions are not classified by H_2, in general (at least for small g).

2) In [Z1] we classified orientable 4-dimensional Seifert fiber spaces over

orientable base-surfaces. For this we had to classify surjections $F \twoheadrightarrow K \subset SL_2(\boldsymbol{Z})$ (the "structure maps" of the fibrations) from the fundamental group F of an orientable surface onto subgroups of the special linear group $SL_2(\boldsymbol{Z})$, up to equivalence. The methods of the present paper allow such a classification also for nonorientable surface groups F, leading to a classification of nonorientable 4-dimensional Seifert fiber spaces (see [Z2]).

1. Proof of the theorem, part a)

Let $G=\boldsymbol{Z}_{p_1}\oplus\cdots\oplus\boldsymbol{Z}_{p_n}$, $p_i>1$, $p_n\,|\,p_{n-1}\cdots|\,p_1$; let x_i be a generator of $\boldsymbol{Z}_{p_i}$, $i=1,\cdots,n$.

Let $F=\langle a_1, b_1, \cdots, a_g, b_g\,|\,\prod_i[a_i, b_i]=1\rangle$.

1.1 Lemma. *For $g\geqq n$, each surjective homomorphism $s: F\twoheadrightarrow G$ is equivalent to one of the following:*

$$\begin{aligned}
&(s(a_1),\ s(b_1))=(x_1,\ x_2^{m_{12}}\cdots x_n^{m_{1n}}),\\
&(s(a_2),\ s(b_2))=(x_2,\ x_3^{m_{23}}\cdots x_n^{m_{2n}}),\\
&\qquad\vdots\\
&(s(a_n),\ s(b_n))=(x_n,\ 1),\\
&(s(a_j),\ s(b_j))=(1,\ 1)\qquad \textit{for } j>n.
\end{aligned}$$

Proof. We need the following automorphisms of F; here $\bar{a}_i$ resp. $\bar{b}_i$ denotes the image of a_i resp. b_i, generators which don't occur remain fixed.

1.2a) $[\bar{a}_1, \bar{b}_1]=[a_1, b_1]$, with

$\bar{a}_1=a_1b_1^k$, $\bar{b}_1=b_1$ or

$\bar{a}_1=b_1$, $\bar{b}_1=b_1a_1^k$, $k\in\boldsymbol{Z}$.

b) $[\bar{a}_1, \bar{b}_1][\bar{a}_2, \bar{b}_2]=[a_1, b_1][a_2, b_2]$, with

$\bar{a}_1=a_1a_2b_2^{-1}$, $\bar{b}_1=b_2a_2^{-1}b_1a_2b_2^{-1}$,

$\bar{a}_2=b_2a_2^{-1}b_1a_2b_2^{-1}b_1^{-1}a_2b_2a_2^{-1}b_1^{-1}a_2b_2^{-1}$,

$\bar{b}_2=b_2b_2a_2^{-1}b_1^{-1}a_2b_2^{-1}$;

$s(\bar{a}_1)=s(a_1a_2b_2^{-1})$, $s(\bar{b}_1)=s(b_1)$,

$s(\bar{a}_2)=s(a_2b_1^{-1})$, $s(\bar{b}_2)=s(b_2b_1^{-1})$.

c) automorphism inverse to 1.2b):

$\bar{a}_1=a_1b_1^{-1}b_2a_2^{-1}b_1$, $\bar{b}_1=b_1^{-1}a_2b_2^{-1}b_1b_2a_2^{-1}b_1$,

$\bar{a}_2=b_1^{-1}a_2b_2^{-1}b_1b_2b_1$, $\bar{b}_2=b_2b_1$, with

$s(\bar{a}_1)=s(a_1a_2^{-1}b_2)$, $s(\bar{b}_1)=s(b_1)$,

$s(\bar{a}_2)=s(a_2b_1), \quad s(\bar{b}_2)=s(b_2b_1).$

d) permutations of the commutators $[a_i, b_i]$, e.g. for $i=1, 2$:

$\bar{a}_1=a_2, \bar{b}_1=b_2,$

$\bar{a}_2=[a_2, b_2]^{-1}a_1[a_2, b_2], \quad \bar{b}_2=[a_2, b_2]^{-1}b_1[a_2, b_2].$

We prove lemma 1.1 by induction on n. We compose s with automorphisms 1.2 and call the new surjection again s.

The case $n=1$ is easy. Suppose the lemma is true for $n-1$. We divide out $\boldsymbol{Z}_{p_n}$ and apply the induction hypothesis. Therefore we can assume that s has the following form:

$$\begin{aligned}
&(s(a_1), s(b_1))=(x_1x_n^{m_1}, x_2^{m_{12}}\cdots x_n^{m_{1n}}),\\
&(s(a_2), s(b_2))=(x_2x_n^{m_2}, x_3^{m_{23}}\cdots x_n^{m_{2n}}),\\
&\vdots \qquad\qquad \vdots\\
&(s(a_n), s(b_n))=(x_n^{m_n}, 1),\\
&(s(a_j), s(b_j))=(1, 1) \qquad \text{for} \quad j>n,\ m_i\in \boldsymbol{Z},\ i=1, \cdots, n.
\end{aligned}$$

By applying all automorphisms 1.2, we achieve $m_n=1$ (because s is surjective), without changing all other values $(s(a_i), s(b_i))$ (see also [Z3, Lemma 2.6]).

After a permutation of commutators, we apply the following process to $(s(a_1), s(b_1))(s(a_n), s(b_n))$:

$$(x_1x_n^{m_1}, x_2^{m_{12}}\cdots x_n^{m_{1n}})(x_n, 1)\xrightarrow[\text{iteration of 1.2b)}]{}$$

$$(x_1, x_2^{m_{12}}\cdots x_n^{m_{1n}})(x_nx_2^{m'_{12}}\cdots x_n^{m'_{1n}}, x_2^{m'_{12}}\cdots x_n^{m'_{1n}}).$$

Now the pair $(s(a_1), s(b_1))$ is in normal form 1.1. We consider the "restriction" of s to the generators $a_i, b_i, i\geqq 2$, with image $\boldsymbol{Z}_{p_2}\oplus\cdots\oplus\boldsymbol{Z}_{p_n}$. By the induction hypothesis we bring this restriction into normal form. Then also s is in normal form. This finishes the proof of the lemma.

Proof of part a) of the theorem.

We have $H_2(G, \boldsymbol{Z})=H_2(\boldsymbol{Z}_{p_1}\oplus\cdots\oplus\boldsymbol{Z}_{p_n}, \boldsymbol{Z})=\boldsymbol{Z}_{p_2}\oplus\boldsymbol{Z}_{p_3}^2\oplus\cdots\oplus\boldsymbol{Z}_{p_{n-1}}^{n-2}\oplus\boldsymbol{Z}_{p_n}^{n-1}$ (apply the Künneth-formula, see e.g. [B], [HS] or [McL]), so there are as many different normal forms 1.1 as elements in $H_2(G, \boldsymbol{Z})$. Now for big enough g every element in $H_2(G, \boldsymbol{Z})$ is the image $\Omega(s)=s_*(1)$ of some surjection $s: F\twoheadrightarrow G$ (represent the element by a 2-cycle in a $K(G, 1)$ which is the image of some closed orientable surface; by adding homologically trivial handles to the surface we can make s surjective. For a purely algebraic proof see [Z3]). It follows that $\Omega(s_1)\neq\Omega(s_2)$ for any two different normal forms s_1, s_2 in 1.1. Therefore different normal forms are not equivalent and the normal forms correspond bijectively via Ω to the elements in $H_2(G, \boldsymbol{Z})$.

2. Proof of the theorem, part b) and c)

Let $F=\langle v, a_i, b_i \mid v^2\prod_i[a_i, b_i]=1\rangle$ if g is odd,

$F=\langle c, d, a_i, b_i \mid \{c, d\}\prod_i[a_i, b_i]=1\rangle$ if g is even,

where $\{c, d\}:=cdc^{-1}d$.

For a surjection $s: F\twoheadrightarrow G$, let $G_0\subset G$ be the image of the orientation preserving elements in F (a subroup of index 1 or 2; note that the generators v and c in the presentations above are orientation reversing, all other generators are orientation preserving). Let $G_0=\boldsymbol{Z}_{p_1}\oplus\cdots\oplus\boldsymbol{Z}_{p_n}$, $p_i>1$, $p_n\mid p_{n-1}\mid\cdots\mid p_1$ (so G_0 is in the role of G now). If $G_0=G$, let $r=1$, otherwise let $r\in G$ be a fixed representative of the nontrivial coset of G_0 in G.

2.1 Lemma. *For $g\geqq 2n+4$, each surjective homomorphism $s: F\twoheadrightarrow G$ is equivalent to one of the following:*

$$s(v) \textit{ resp. } s(d)=\textit{arbitrary element of order } \leqq 2 \textit{ in } G \textit{ resp. } G_0,$$

$$s(c)=r,\ (s(a_i), s(b_i))=\textit{as in } 1.1,$$

with $m_{ij}\in\{0, 1\}$, $m_{ij}=0$ if 2 does not divide p_j, $i+1\leqq j\leqq n$.

Proof. We need the following changes of generators for F:

2.2a) $v^2[a, b]=v_1^2v_2^2v_3^2$, with

$$v_1:=v^2abv^{-1},\quad v_2:=vb^{-1}a^{-1}v^{-1}a^{-1}v^{-1},\quad v_3:=va;$$

2.2b) $v^2[a, b]=\bar{v}^2[\bar{a}, \bar{b}]$, with

$$\bar{v}:=a^{-1}va,\quad \bar{a}:=a^{-1}v^{-2}av^2a,\quad \bar{b}:=ba^{-1}v^2a;$$

2.2c) $\{c, d\}=\{\bar{c}, \bar{d}\}$, with $\bar{c}:=cd^k$, $\bar{d}:=d$, $k\in\boldsymbol{Z}$;

2.2d) $\{c, d\}[a, b]=\{\bar{c}, \bar{d}\}[\bar{a}, \bar{b}]$, with

$$\bar{c}:=cab^{-1},\quad \bar{d}:=ba^{-1}dab^{-1},$$

$$\bar{a}:=ba^{-1}d^{-1}ab^{-1}daba^{-1}dab^{-1},\quad \bar{b}:=b^2a^{-1}dab^{-1},$$

$$s(\bar{c})=s(cab^{-1}),\quad s(\bar{d})=s(d),\quad s(\bar{a})=s(ad),\quad s(\bar{b})=s(bd);$$

2.2e) $\{c, d\}=v_1^2v_2^2$, with $v_1:=c$, $v_2:=c^{-1}d$.

i) Suppose first that g is odd.

We prove 2.1 by induction on n. The case $n=1$ is easy. Suppose 2.1 is true for $n-1$. We divide out $\boldsymbol{Z}_{p_n}$. By the induction hypothesis we can assume

$$s(v)=w\in G,\quad w^2=1,$$

$$(s(a_i), s(b_i))=(x_ix_n^{m_i}, x_{i+1}^{m_{ii+1}}\cdots x_n^{m_{in}}),\quad i<n,$$

$$(s(a_n), s(b_n))=(x_n^{m_n}, 1),$$

$$(s(a_i),\ s(b_i))=(1,\ 1),\quad i>n,$$

where $m_{ij}\in\{0, 1\}$, $m_{ij}=0$ if $(2, p_j)=1$, $i+1\leqq j<n$ (but not for $j=n$ for the moment).

As in the proof of 1.1 we achieve $m_n=1$.

We use the following process to normalize m_{1n} and $(s(a_1), s(b_1))$ (together with some permutations of commutators):

$$v^2[a_n,\ b_n]\xrightarrow[\text{2.2a)}]{} v_1^2v_2^2v_3^2,\ \text{with } s(v_3)=wx_n;$$

$$v_3^2[a_1,\ b_1]\xrightarrow[\text{2.2b)}]{} \bar{v}_3^2[\bar{a}_1,\ \bar{b}_1],\ \text{with } s(\bar{v}_3)=s(v_3),\quad s(\bar{a}_1)=s(a_1),$$

$$s(\bar{b}_1)=s(b_1v_3^2)=x_2^{m_{12}}\cdots x_n^{m_{1n}}x_n^2;$$

$$v_1^2v_2^2\bar{v}_3^2\xrightarrow[\text{inverse of 2.2a)}]{} \bar{v}^2[\bar{a}_n,\ \bar{b}_n],\ \text{with } s(\bar{v})=w,\quad (s(\bar{a}_n),\ s(\bar{b}_n))=(x_n,\ 1);$$

Using this we can achieve $m_{1n}\in\{0, 1\}$, $m_{1n}=0$ if $(2, p_n)=1$. Now, using an iteration of 1.2b) as in the proof of 1.1, we achieve $m_1=0$. Then $(s(a_1), s(b_1))$ is in normal form 2.1.

Now we bring the commutator $[a_1, b_1]$ to the end of the defining relation of F and consider the restriction of s to the subgroup generated by v and a_i, b_i, $i>2$, with $\boldsymbol{Z}_{p_2}\oplus\cdots\oplus\boldsymbol{Z}_{p_n}$ as the new G_0. By the induction hypothesis, we can bring this restriction into normal form 2.1. Then also s is in normal form.

ii) Now suppose that g is even.

By processes 2.2c), d) and 1.2a), d) we achieve $s(c)=r$. If $s(a_i)$, $s(b_i)$, $i\geqq1$, generate a proper subgroup of G_0, we bring $(s(a_i), s(b_i))$, $i\geqq1$, into orientable normal form to get $s(a_{n+1})=s(b_{n+1})=1$. Then we apply 2.2d) to $\{c, d\}[a_{n+1}, b_{n+1}]$ and get $s(a_{n+1})=s(b_{n+1})=s(d)$. To get also $s(c^2)$, we apply 2.2e) to $\{c, d\}=v_1^2v_2^2$, with $s(v_1)=s(c)$, $s(v_2)=s(c^{-1}d)$, then 2.2b) to $v_2^2[a_{n+1}, b_{n+1}]$, getting $s(v_2)=s(c^{-1}d)$, $s(a_{n+1})=s(d)$, $s(b_{n+1})=s(dc^{-2})$ (note that $s(d^2)=1$) and then the inverse of 2.2e) to get back $\{c, d\}$. Therefore we can assume that $s(a_i)$, $s(b_i)$, $i>1$, generate G_0. Now lemma 2.1 follows from the following

2.3 Lemma. *Let* $s: F\twoheadrightarrow G$ *and* U *the subgroup of* G *generated by* $s(a_i)$, $s(b_i)$, $i>1$. *Suppose* $U=\boldsymbol{Z}_{p_1}\oplus\cdots\oplus\boldsymbol{Z}_{p_n}$, $p_n\,|\,p_{n-1}|\cdots|\,p_1$. *Then* s *can be brought into normal form* 2.1 *by automorphisms of* F *which don't change the values* $s(c)$, $s(d)$ *and the subgroup generated by* $s(a_i)$, $s(b_i)$, $i\geqq1$.

Proof. We proceed as in case i) by induction on n, the case $n=1$ being clear. In the induction step, we again divide out $\boldsymbol{Z}_{p_n}$ and bring s into the form

$$s(c)=\text{some } r,\ s(d)=\text{some } w \text{ with } w^2=1,$$

$(s(a_i), s(b_i))$: as in the proof of part i).

We normalize $(s(a_1), s(b_1))$ by the following process:

$$\{c, d\} \xrightarrow[2.2e)]{} \bar{v}^2 v^2, \text{ with } s(\bar{v})=r, \quad s(v)=r^{-1}w;$$

$$v^2[a_n, b_n] \xrightarrow[2.2a)]{} v_1^2 v_2^2 v_3^2, \text{ with } s(v_3)=r^{-1}wx_n;$$

$$v_3^2[a_1, b_1] \xrightarrow[2.2b)]{} \bar{v}_3^2[\bar{a}_1, \bar{b}_1], \text{ with } s(\bar{v}_3)=s(v_3), \quad s(\bar{a}_1)=s(a_1),$$

$$s(\bar{b}_1)=x_2^{m_{12}} \cdots x_n^{m_{1n}} r^{-2} x_n^2;$$

$$\bar{v}^2[\bar{a}_1, \bar{b}_1] \xrightarrow[2.2b)]{} \bar{\bar{v}}^2[\bar{\bar{a}}_1, \bar{\bar{b}}_1], \text{ with } s(\bar{\bar{v}})=s(\bar{v}), \quad s(\bar{\bar{a}}_1)=s(a_1),$$

$$s(\bar{\bar{b}}_1)=x_2^{m_{12}} \cdots x_n^{m_{1n}} x_n^2;$$

Use this to normalize m_{1n}, then apply the inverse of 2.2a) and 2.2e) to get back $\{c, d\}$. Then apply 1.2b) to get $m_1=1$. Note that we also used some permutations between the parts of the defining relation of F, and that the values $s(c)$, $s(d)$ and the subgroup U are not changed by the above processes. Now we bring the commutator $[a_1, b_1]$ to the end of the defining relation and consider the restriction of s to the generators $c, d, a_i, b_i, i \geqq 2$, with $s(\langle a_i, b_i, i \geqq 2 \rangle) = \boldsymbol{Z}_{p_2} \oplus \cdots \oplus \boldsymbol{Z}_{p_n}$. By the induction hypothesis, we can bring s into normal form. This finishes the proof of lemma 2.3 and also of lemma 2.1.

Proof of part b) of the theorem.

Note that here we are interested in actions on nonorientable surfaces, so the kernel of the corresponding $s: F \twoheadrightarrow G$ contains orientation reversing elements. Therefore we have $G=G_0=\boldsymbol{Z}_{p_1} \oplus \cdots \oplus \boldsymbol{Z}_{p_n}$. Now $H_2(G, \boldsymbol{Z}_2) \cong H_2(G, \boldsymbol{Z}) \otimes \boldsymbol{Z}_2 \oplus \mathrm{Tor}(H_1(G, \boldsymbol{Z}), \boldsymbol{Z}_2) \cong (\boldsymbol{Z}_{p_2} \oplus \boldsymbol{Z}_{p_3}^2 \oplus \cdots \oplus \boldsymbol{Z}_{p_n}^{n-1}) \otimes \boldsymbol{Z}_2 \oplus \mathrm{Tor}(G, \boldsymbol{Z}_2)$. The elements of $\mathrm{Tor}(G, \boldsymbol{Z}_2)$ correspond bijectively to the elements of order $\leqq 2$ in G. It follows that the different normal forms 2.1 are in bijective correspondence to the elements in $H_2(G, \boldsymbol{Z}_2)$, which proves part b) of the theorem (note that each element in $H_2(G, \boldsymbol{Z}_2)$ is the image in a $K(G, 1)$ of a nonorientable surface of odd and also of even genus because we can always add a projective plane which is mapped trivially).

Proof of part c) of the theorem.

Now the kernel of $s: F \twoheadrightarrow G$ contains no orientation reversing element, so G_0 is a subgroup of index 2 in G. For g even, the possible s(d)'s in the normal form 2.1 are in bijective correspondence to the elements of order $\leqq 2$ in G_0. For g odd, $s(v)$ has order 2 and $s(v) \notin G_0$, so there is no surjection $s: F \twoheadrightarrow G$ if $G \not\cong G_0 \oplus \boldsymbol{Z}_2$; if $G \cong G_0 \oplus \boldsymbol{Z}_2$, the possible $s(v)$'s are in bijective correspondence to the elements of order $\leqq 2$ in G_0. For a moment we discuss the case $G=\boldsymbol{Z}_2=$

$\langle x \mid x^2=1\rangle$, $G_0=1$. For g odd we have only one normal form 2.1, namely $s(v)=x$, $s(a_i)=s(b_i)=1$, and for this $\Omega(s)\in H_2(\boldsymbol{Z}_2, \tilde{\boldsymbol{Z}})\cong \boldsymbol{Z}_2$ is nontrivial because a nonorientable surface of odd genus does not bound. For g even we have also one normal form $s(c)=x$, $s(d)=1$, $s(a_i)=s(b_i)=1$, with $\Omega(s)$ representing the trivial element in $H_2(\boldsymbol{Z}_2, \tilde{\boldsymbol{Z}})$. It follows similarly that $\Omega(s_1)\neq\Omega(s_2)$, where $s_1\colon F\twoheadrightarrow G$ is a surjection from a surface group of odd genus, s_2 a surjection from a surface group of even genus (in the situation of part c) of the theorem), e.g. by dividing out G_0. Now we compute $H_2(G, \tilde{\boldsymbol{Z}})$ for a general G, using the Lyndon-Hochschild-Serre spectral sequence for the group extension $1\to G_0\hookrightarrow G\to Z_2\to 1$ (see [B], [HS] or [McL]). The E^2-terms of the spectral sequence are as follows:

$$E^2_{0,2}\cong H_0(\boldsymbol{Z}_2, H_2(G_0, \tilde{\boldsymbol{Z}}))=H_0(\boldsymbol{Z}_2, \boldsymbol{Z}_{p_2}\oplus \boldsymbol{Z}^2_{p_3}\oplus\cdots\oplus \boldsymbol{Z}^{n-1}_{p_n})$$
$$\cong(\boldsymbol{Z}_{p_2}\oplus\cdots\oplus \boldsymbol{Z}^{n-1}_{p_n})\otimes \boldsymbol{Z}_2,$$

because the nontrivial element of $\boldsymbol{Z}_2$ operates by (-identity). Note that the elements of $E^2_{0,2}$ are in bijective correspondence to the possible values $m_{i,j}$ in the normal forms 2.1.

$$E^2_{1,1}\cong H_1(\boldsymbol{Z}_2, H_1(G_0, \tilde{\boldsymbol{Z}}))\cong H_1(\boldsymbol{Z}_2, G_0)\cong G_0\otimes \boldsymbol{Z}_2,$$

because the nontrivial element in $\boldsymbol{Z}_2$ operates by (-identity); the elements of $E^2_{1,1}$ are in bijective correspondence to the elements of order$\leqq 2$ in G_0.

$$E^2_{2,0}\cong H_2(\boldsymbol{Z}_2, H_0(G, \tilde{\boldsymbol{Z}}))\cong H_2(\boldsymbol{Z}_2, \tilde{\boldsymbol{Z}})\cong \boldsymbol{Z}_2.$$

It is easy to check (using induction on n for example), that all differentials entering or leaving $E^2_{0,2}$ and $E^2_{1,1}$ are trivial. If $G\not\cong G_0\oplus \boldsymbol{Z}_2$, the elements of $E^2_{0,2}\cup E^2_{1,1}$ are in bijective correspondence to the different normal forms 2.1 (for g even, no normal form for g odd). As we cannot have more elements in $H_2(G, \tilde{\boldsymbol{Z}})$ than normal forms, the normal forms correspond bijectively to the elements in $H_2(G, \tilde{\boldsymbol{Z}})$ (and $E^2_{0,2}$ gives no contribution to $H_2(G, \tilde{\boldsymbol{Z}})$). If $G\cong G_0\oplus \boldsymbol{Z}_2$, we have the same number of normal forms in the cases g odd and g even, corresponding to the elements in $E^2_{0,2}\cup E^2_{1,1}$ in each case, so the normal forms in each case are all nonequivalent (or divide out $\boldsymbol{Z}_2\subset G$ and use the classification in case b) of the theorem). As the cases g odd and g even also give different values in $H_2(G, \tilde{\boldsymbol{Z}})$, the number of different normal forms in each case is half the order of $H_2(G, \tilde{\boldsymbol{Z}})$ (and the differential leaving $E^2_{2,0}$ is also trivial, so $E^2_{2,0}$ gives a contribution to $H_2(G, \tilde{\boldsymbol{Z}})$; for another computation of $H_2(G, \tilde{\boldsymbol{Z}})$ see the following remark).

This finishes the proof of part c) of the theorem.

Remark. To compute $H_2(G, \tilde{\boldsymbol{Z}})$, one can also use the exact coefficient sequence $1\to\tilde{\boldsymbol{Z}}\xrightarrow{2\cdot}\tilde{\boldsymbol{Z}}\to \boldsymbol{Z}_2\to 1$, from which one gets the long exact sequence

$$\begin{array}{ccccccccc} H_2(G, \tilde{\tilde{\boldsymbol{Z}}}) & \xrightarrow{\cdot 2} & H_2(G, \tilde{\boldsymbol{Z}}) & \longrightarrow & H_2(G, \boldsymbol{Z}_2) & \longrightarrow & H_1(G, \tilde{\tilde{\boldsymbol{Z}}}) & \xrightarrow{\cdot 2} & H_1(G, \tilde{\boldsymbol{Z}}) \longrightarrow \\ & \searrow \; 0 \; \nearrow & & & & & & \searrow \; 0 \; \nearrow & \end{array}$$

$$\begin{array}{ccccccccc} H_1(G, \boldsymbol{Z}_2) & \longrightarrow & H_0(G, \tilde{\tilde{\boldsymbol{Z}}}) & \xrightarrow{\cdot 2} & H_0(G, \tilde{\boldsymbol{Z}}) & \longrightarrow & H_0(G, \boldsymbol{Z}_2) & \longrightarrow & 0 . \\ & & \wr\Vert & \searrow \; 0 \; \nearrow & \wr\Vert & & \wr\Vert & & \\ & & \boldsymbol{Z}_2 & & \boldsymbol{Z}_2 & & \boldsymbol{Z}_2 & & \end{array}$$

It follows $|H_2(G, \boldsymbol{Z}_2)| = |H_2(G, \tilde{\boldsymbol{Z}})| \cdot |H_1(G, \tilde{\tilde{\boldsymbol{Z}}})|$,

$$|H_1(G, \boldsymbol{Z}_2)| = 2|H_1(G, \tilde{\boldsymbol{Z}})| = 2|H_1(G, \tilde{\tilde{\boldsymbol{Z}}})|, \text{ so}$$

$$|H_2(G, \tilde{\boldsymbol{Z}})| = 2|H_2(G, \boldsymbol{Z}_2)| / |H_1(G, \boldsymbol{Z}_2)| .$$

Problem. 1) Show that for finite nonabelian groups G actions of G on surfaces of high genus (i.e. in a "stable" range) are not classified by H_2, in general; for low genus this follows from [L] (see also [E2]).

2) Is it true that a surjection $F \twoheadrightarrow G$, G a fixed finite group, has always a nontrivial simple closed curve in its kernel, for high genus of F?

References

[B] Brown, K.S.: *Cohomology of Groups.* Graduate Texts in Mathematics 87, Springer 1982.

[E1, 2] Edmonds, A.L.: Surface symmetry I, II. *Michigan Math. J.* **29** (1982), 171-183 and **30** (1983), 143-154.

[HS] Hilton, P.J., Stammbach, U.: *A Course in Homological Algebra.* Graduate Texts in Mathematics 4, Springer 1971.

[L] Livingston, C.: Inequivalent, bordant group actions on a surface. *Math. Proc. Camb. Philos. Soc.* **99** (1986), 233-238.

[McL] MacLane, S.: *Homology.* Springer 1963.

[Z1] Zimmermann, B.: Zur Klassifikation höherdimensionaler Seifertscher Faserräume. In: *Low dimensional Topology*, London Math. Soc. Lecture Notes **95** (1985), 214-255.

[Z2] Zimmermann, B.: Über Homomorphismen von Flächengruppen in abelsche Gruppen und $SL_2(\boldsymbol{Z})$. Preprint.

[Z3] Zimmermann, B.: Surfaces and the second homology of a group. *Mh. Math.* **104** (1987), 247-253.

Dipartimento di Scienze Matematiche
Università degli Studi di Trieste
34100 Trieste, Italia.