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WAVE MOVES ON CRYSTALLIZATIONS*

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Abstract. In this paper, the relations between the notions of "wave move" [HO] and "frame" [T] are investigated.

A genus three frame of S^3 is produced, giving a counterexample to a conjecture of [T]; on the contrary, the conjecture is proved to be true in genus two.

1. Introduction and notations.

In [T] Tsukui introduces a special class of edge-coloured graphs, called frames, which represent, via a standard construction, closed 3-manifolds and conjectures the non-existence of frames for the 3-sphere S^3.

In the present paper, we relate the notion of frame with wave theory on the bridge-presentation of links. This kind of relations appear as a useful tool for the study of Tsukui's conjecture. In particular, we prove that there are no genus $g\leqq 2$ frame for S^3, while we produce counterexamples to the conjecture for the genus three and four.

Throughout this paper, all spaces and maps are piecewise-linear (P. L.) in the sense of [Gl] or [RS]. Manifolds are always assumed to be closed, conected and *orientable*.

For basic graph theory, we refer to [H].

An *edge-coloration* on a muligraph $\Gamma=(V(\Gamma), E(\Gamma))$ is a map $\gamma: E(\Gamma)\rightarrow \Delta_n=\{0, 1, \cdots, n\}$ such that $\gamma(e)\neq\gamma(f)$ for each pair e, f of adjacent edges. The pair (Γ, γ), where Γ denotes a regular multigraph of degree $n+1$ and $\gamma: E(\Gamma)\rightarrow\Delta_n$ is an edge-coloration, is said to be an $(n+1)$-*coloured graph*. If Γ has no multiple edge (i.e. if Γ is a graph), (Γ, γ) is said to be *simple*. Given two $(n+1)$-coloured graphs (Γ, γ) and (Γ', γ') with colour set C and C' respectively, an isomorphism $\Psi: \Gamma\rightarrow\Gamma'$ is called a *colour-isomorphism* iff there exists a bijection $\Phi: C\rightarrow C'$ such that $\gamma'\circ\Psi=\Phi\circ\gamma$. For each $\mathcal{I}\subseteqq\Delta_n$, we set $\Gamma_{\mathcal{I}}=(V(\Gamma), \gamma^{-1}(\mathcal{I}))$; each connected component of $\Gamma_{\mathcal{I}}$ is often called an $\mathcal{I}$-*residue*. An m-*residue* is an $\mathcal{I}$-residue such that the cardinality of $\mathcal{I}$ is m. For each

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colour $i\in\Delta_n$, we set $\hat{i}=\Delta_n-\{i\}$.

A c-coloured edge of an $(n+1)$-coloured graph (Γ, γ) is said to be $\{i, j\}$-*diagonal* $(i, j\neq c)$ if its end-points belong to the same $\{i, j\}$-residue; an edge is called *diagonal* if it is $\{i, j\}$-diagonal for some i, j.

Two ball complexes B_1 and B_2 are said to be *isomorphic* if there is a bijection $f: B_1\rightarrow B_2$ preserving the face-incidence relation. A *pseudocomplex* [HW] is a ball complex in which each h-ball, considered with all its faces, is isomorphic with the complex underlying an h-simplex. As shown in [FGG], every $(n+1)$-coloured graph (Γ, γ) represents an n-dimensional pseudocomplex $K(\Gamma)$; moreover, $K(\Gamma)$ is a pseudomanifold [ST], which is orientable iff Γ is bipartite.

An $(n+1)$-coloured graph (Γ, γ) is *contracted* if $\Gamma_{\hat{c}}$ is connected, for each $c\in\Delta_n$; the geometrical interpretation of this property is that the associated pseudocomplex $K(\Gamma)$ has exactly $n+1$ vertices. A *crystallization* of an n-manifold M is any contracted $(n+1)$-coloured graph representing M; every n-manifold admits a crystallization [P].

A 2-*cell embedding* [W] $\alpha: \Gamma\rightarrow F$ of an $(n+1)$-coloured graph (Γ, γ) into a closed surface F is said to be *regular* iff there is a cyclic permutation $\varepsilon=(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$ of Δ_n such that each region of α is bounded by the image of a cycle whose edges are alternatively coloured by ε_i, ε_{i+1} (i being an integer mod.$(n+1)$). Given a bipartite $(n+1)$-coloured graph (Γ, γ), its (*regular*) *genus* $g(\Gamma)$ is the smallest integer k such that (Γ, γ) regularly embeds into the orientable closed surface of genus k. The *regular genus* of an n-manifold M is the non-negative integer $g(M)=\min\{g(\Gamma)/(\Gamma, \gamma)$ is a crystallization of $M\}$. If $h(M)$ denotes the Heegaard genus of a 3-manifold, then $h(M)=g(M)$ ([G]). For a survey on crystallization theory, see [FGG].

For each natural number r, let us call the r-bridge presentation of the trivial knot the $2r$-*gonal* if its projection has no crossing.

Let $L=(B_1, \cdots, B_r; b_1, \cdots, b_r)$ be an r-bridge presentation of a link $\mathcal{L}$, where B_i are the bridges and b_i are the arcs (or underbridges). Let π be the plane containing all arcs b_i and let P be the projection of L on π. The projection of the bridge B_i is denoted by P_i.

The projection P of a link $\mathcal{L}$ is said to be *reduced* if it has no cancelling region (that is, a region bounded by only two edges) and each bridge (resp. arc) of L has at least one undercrossing (resp. overcrossing).

A *wave* in L is a path ω in π such that:

—$\omega\cap P=\partial\omega\cap A_i$, A_i being a suitable P_i or a suitable b_i;

—if β_i denotes the subpath of A_i bounded by $\partial\omega$, the interior of β_i must contain at least one crossing of P.

The corresponding *wave move* is the replacement of A_i with $(A_i-\beta_i)\cup\omega$ which leads to the projection P' of a new r-bridge presentation L' of $\mathcal{L}$ having a fewer number of crossings than P.

Proposition 1 [HO]. *Every 3-bridge presentation of the trivial knot can be transformed into the hexagonal one by a finite sequence of wave moves.* ■

Since the presence of a wave ω can be established by checking if the same P_i (or the same b_i) appears twice in the boundary of each region of P and since a wave move strictly decreases the number of crossings, Homma-Ochiai result leads to an algorithm for recognizing the triviality of a knot from the projection of a 3-bridge presentation of it.

Wave moves are also defined for Heegaard diagrams; the relations between the two wave theories are investigated in [NO].

2. Frames and wave moves.

If i, j, k are distinct colours in Δ_3, the pair $(i, \{j, k\})$ is briefly denoted by $(i; j, k)$; let Ω be the set of all these pairs.

Definition 1. A 4-coloured graph (Γ, γ) is said to be $(i; j, k)$-*irreducible*, $(i; j, k) \in \Omega$, if any $\{i, j\}$-residue and $\{i, k\}$-residue meet at most in one i-coloured edge.

Definition 2. A simple crystallization (Γ, γ) of a 3-manifold M is said to be a *frame* of M if it is $(i; j, k)$-irreducible, for each $(i; j, k) \in \Omega$.

Note that the only non-simple crystallization which is $(i; j, k)$-irreducible for each $(i; j, k) \in \Omega$ is the standard genus zero crystallization of S^3 with two vertices and four multiple edges.

The idea of wave move on the projection of a link $\mathcal{L}$ is strictly related to the notion of frame for the 3-manifold M which is the 2-fold covering space of S^3 branched over $\mathcal{L}$, via the following construction [F].

If $L = (B_1, \cdots B_r; b_1, \cdots, b_r)$ is an r-bridge presentation of a link $\mathcal{L}$, let π denote the plane containing all arcs b_i and let P be the projection of L on π. P can always be assumed to be connected; this is immediate if L is not splittable. If L splits, we can isotope arcs of L on π to pass "in and out" under bridges of different components. Let E_i $(i \in N_r = \{1, \cdots, r\})$ be an ellipse in π whose principal axis is the projection P_i of the bridge B_i and such that E_i intersects each arc of P at most in one point; set $V = \bigcup_{i \in N_r} (E_i \cap L)$.

V subdivides $L \cap \pi$ into edges: let C (resp. D) be the set of these edges which are internal (resp. external) to the ellipsis. Let α be the involution on V which exchanges the end-points of the edges in C and fixes the points of

$\bigcup_{i\in N_r}(E_i\cap B_i)$; let δ be the involution on V which exchanges the end-points of the edges in D. Note that V subdivides the ellipsis into a set F consisting of an even number of edges. Colour all the edges in D by 2 and colour all the edges in E_1 alternatively by 0 and 1, starting from an arbitrary vertex. Complete the coloration of F by 0 and 1 so that each region of the planar 2-cell embedding of $F\cup D$ is bounded by edges alternatively coloured by two colours (note that the edges of the regions not bounded by $E_1, \cdots, E_r$ alternatively belong to F and D). Join by a 3-coloured edge any pair of vertices in V exchanged by the involution $\alpha\delta\alpha$ and let D' be the set of these edges.

If Γ is the graph defined by $V(\Gamma)=V$, $E(\Gamma)=D\cup D'\cup F$ and γ is the edge-coloration defined above, then $F(L)=(\Gamma, \gamma)$ is a crystallization of the 2-fold covering space of S^3 branched over $\mathcal{L}$.

Since the involution α may be thought of as the symmetry in π whose axis contains the projections P_i and which exchanges colour 0 (resp. 2) with colour 1 (resp. 3) in $F(L)$, it seems natural to call $F(L)$ 2-*symmetric*.

Definition 3. If L' is obtained from L by a wave move, the crystallization $F(L')$ is also said to be obtained from $F(L)$ by a *wave move*.

Note that $F(L')$ has strictly fewer vertices than $F(L)$.

Proposition 2. *Let P be the projection of a bridge-presentation L of a link.*

(a) *Each cancelling region in P gives rise to an $\{i_0, i_2\}$-residue and an $\{i_1, i_3\}$--residue in $F(L)$, both of length two, where (i_0, i_1) (resp. (i_2, i_3)) is a suitable permutation on $\{0, 1\}$ (resp. $\{2, 3\}$), and viceversa.*

(b) *Each bridge (resp. arc) with no under-crossing (resp. overcrossing) gives rise to a $\{0, 1\}$-residue (resp. $\{2, 3\}$-residue) of length two in $F(L)$, and viceversa.*

(c) *If P is reduced, then P admits no wave iff $F(L)$ is $(0; 1, 2)$-, $(1; 0, 2)$-, $(2; 0, 3)$- and $(2; 1, 3)$-irreducible.*

In order to prove the previous result, we need the following notations:

Given a bridge-presentation L of a link and the projection P of L on a plane π, let π_+ (resp. π_-) be the upper (resp. lower) half-space of π whose generatrix is the symmetry axis containing all bridge-projections of L.

If B is a region of P, let S be a connected component of the intersection (supposed non-void) of B with a bridge-projection P_i. Let x be any end-point of the edge S and $U(x, \rho)$ the disk centered at x of radius ρ.

We call *intersection index* between B and S the symbol $\varepsilon(B, S)\in\{+, -\}$ defined in the following way:

$$\varepsilon(B, S)=\begin{cases} + & \text{if there exists } \rho>0 \text{ such that } B\cap U(x, \rho)\subset\pi_+; \\ - & \qquad\qquad \text{"} \qquad\qquad B\cap U(x, \rho)\subset\pi_-. \end{cases}$$

Proof of Proposition 2. (a) and (b) are direct consequences of Ferri's construction.

(c1) Suppose that P admits a wave.

First, assume that there is a wave on a bridge, that is a bridge projection P_i intersects a region B of P at least in two connected components S and T. Let $S_+\subset\pi_+$, $S_-\subset\pi_-$ (resp. $T_+\subset\pi_+$, $T_-\subset\pi_-$) be the edges belonging to the $\{0,, 1\}$-residue C_i of $F(L)$ arising from S (resp. T) by means of Ferri's construction.

Let ψ be the one-to-one correspondence between the set of all regions of P and the set of all $\{0, 2\}$- and $\{1, 2\}$-residues in $F(L)_{\hat{3}}$. Then $\psi(B)$ contains $S_{\varepsilon(B,S)}$, $T_{\varepsilon(B,T)}$. Note that the edges $S_{\varepsilon(B,S)}$, $T_{\varepsilon(B,T)}$ have the same colour c ($c\in\{0, 1\}$); thus, the $\{0, 1\}$-residue C_i and the $\{c, 2\}$-residue $\psi(B)$ meet at least the two c-coloured edges $S_{\varepsilon(B,S)}$, $T_{\varepsilon(B,T)}$ and $F(L)$ is $(c\,;c', 2)$-reducible, with $\{c, c'\}=\{0, 1\}$.

Assume now that there is a wave on an arc, that is, an arc b_i intersects a region B of P at least in two connected components S, T. Let ϕ be the one-to-one correspondence between the arcs of P and the $\{2, 3\}$-residues of $F(L)$. The edge S (resp. T) gives rise to the 2-coloured edge S_2 (resp. T_2) belonging to the $\{2, 3\}$-residue $\phi(b_i)$ in $F(L)$. Moreover, $\psi(B)$ is a $\{c, 2\}$-residue ($c\in\{0, 1\}$) containing S_2 and T_2; thus, the $\{2, 3\}$-residue $\phi(b_i)$ and the $\{c, 2\}$-residue $\psi(B)$ meet at least the two 2-coloured edges S_2, T_2 and $F(L)$ is $(2\,;c, 3)$-reducible, with $c\in\{0, 1\}$.

(c2) Suppose now P reduced.

Assume that there is a $\{0, 1\}$-residue C_i and a $\{c, 2\}$-residue R ($c\in\{0, 1\}$) in $F(L)$ meeting at least two c-coloured edges σ, τ. In this situation, the region $\psi^{-1}(R)$ intersects the projection P_i at least in two (disjoint) edges obtained by orthogonally projecting σ and τ over P_i in π.

Finally, the existence of a $\{2, 3\}$-residue Q and a $\{c, 2\}$-residue R ($c\in\{0, 1\}$) meeting at least two 2-coloured edges σ and τ implies that the arc $\phi^{-1}(Q)$ intersects the region $\psi^{-1}(R)$ at least in two (disjoint) edges of P respectively containing σ and τ. ∎

Remark. If $\tilde{\alpha}$ denotes the permutation of Δ_3 exchanging colour 0 with 1 and colour 2 with 3, then the $(i\,;j, k)$-irreducibility is equivalent to the $(\tilde{\alpha}(i)\,;\tilde{\alpha}(j), \tilde{\alpha}(k))$-irreducibility. ∎

This immediately follows from the 2-symmetry of $F(L)$ induced by the involution α.

Note that, if a crystallization (Γ, γ) contains a 2-residue of length two, then (Γ, γ) is not simple; hence, it can not be a frame. Thus, Prop. 2 yields a necessary condition on L for $F(L)$ to be a frame.

Corollary 3. *If a 2-symmetric crystallization $F(L)$ is a frame, then the projection P of L is reduced and admits no wave.* ■

Actually, Prop. 2 and the related remark prove that, if the projection P is reduced and admits no wave, it suffices to test the $(2;0,1)$- (or its symmetric $(3;1,0)$-) irreducibility and the $(0;2,3)$- (or its symmetric $(1;3,2)$-) irreducibility for proving that $F(L)$ is a frame. Nevertheless, the converse of Corollary 3 is false since it is possible to find (simple) 2-symmetric crystallizations $F(L)$ which are $(2;0,1)$-reducible and/or $(0;2,3)$-reducible and such that the projection P of L is reduced and admits no wave (see, for example, Fig. 1a, 1b).

In [T], the following conjecture is stated:

Conjecture [T]. S^3 admits no frame.

Corollary 3 implies that possible counterexamples to Tsukui's conjecture may be found among 2-symmetric crystallizations $F(L)$, for some bridge-presentation L of the trivial knot whose projection P is reduced and admits no wave. For such an $F(L)$ it suffices to check $(2;0,1)$-irreducibility and $(0;2,3)$-irreducibility.

Fig. 2a represents the projection P of a 4-bridge presentation $\bar{L}$ of the trivial knot; this projection, firstly given in [M], is reduced and admits no wave. If $F(\bar{L})$ is the 2-symmetric crystallization obtained as the result of Ferri's construction applied to $\bar{L}$ (Fig. 2b), it is easy to prove that $F(\bar{L})$ is both $(2;0,1)$-irreducible and $(0;2,3)$--irreducible. Hence, we have:

Proposition 4. *$F(\bar{L})$ is a genus three frame of S^3.* ■

Remark. If $\bar{\bar{L}}$ is the 5-bridge-presentation of the trivial knot given in [O], the genus four crystallization $F(\bar{\bar{L}})$ is a counterexample to Tsukui's conjecture, too.

Nevertheless, Tsukui's conjecture is true in genus $g \leqq 2$, as we will prove in the next section.

3. The genus two case.

From now on, we restrict our attention to genus two 3-manifolds.

First, we recall the possibility of representing these 3-manifolds by means of 2-symmetric crystallizations.

Proposition 5 [CG]. *If (Γ, γ) is a simple genus two crystallization of a 3-manifold with no diagonal edge, then there exists a 3-bridge presentation L of a* *link such that the* 2-*symmetric crystallization* $F(L)$ *is colour-isomorphic to* (Γ, γ). ■

In order to prove that Tsukui's conjecture is true in genus $g \leqq 2$, we need the following straightforward lemma:

Lemma 6. *If* (Γ, γ) *is a simple crystallization which contains an* (i, j)-*diagonal* k-*coloured edge, then* (Γ, γ) *is* $(i; j, k)$-*reducible.* ■

Proposition 7. S^3 *admits no genus two frame.*

Proof. It is well known that there are no simple genus zero crystallization (of S^3), while the only simple genus one crystallizations are the "normal" crystallizations of the lens spaces $L(p, q)$, $p>1$ [DG]. Thus, if $g=0$ or $g=1$, the result is straightforward.

Let (Γ, γ) be a genus two frame of S^3; Lemma 6 ensures that (Γ, γ) has no diagonal edge. Thus, Prop. 5 gives $(\Gamma, \gamma)=F(L)$ for a suitable 3-bridge presentation L of the trivial knot.

Corollary 3 implies that the projection P of L is reduced and admits no wave; this contradicts the fact that the only 3-bridge projection of the trivial knot without waves is the hexagonal one (which is not reduced). ■

We point out that Homma-Ochiai algorithm leads to an algorithm for recognizing if a given genus two crystallization (Γ, γ) represents S^3 or not.

We now show that, for r-bridge presentations, $r \leqq 3$, the converse of Corollary 3 holds, too.

The following lemmas are straightforward.

Lemma 8. *If the projection* P *of a* 3-*bridge presentation* L *of a link is reduced and admits no wave, then* P *is of "triangular type"* [N]. ■

Lemma 9. *Let* L *be a* 3-*bridge presentation of a link such that its projection is of triangular type. If* $c \in \{0, 1\}$ *and* $c' \in \{2, 3\}$, *then every* $\{c, c'\}$-*residue of the* 2-*symmetric crystallization* $F(L)$ *has length* $\leqq 6$. ■

Proposition 10. *If the projection* P *of an* r-*bridge presentation* L *of a link* $(r \leqq 3)$ *is reduced and admits no wave, then* $F(L)$ *is a frame.*

Proof. If $r \leqq 2$, the only r-bridge presentations L whose projections P are reduced and admit no wave, are the Schubert's normal form $K(p, q)$ of a 2-bridge knot; since $F(K(p, q))$ is the "normal" crystallization of $L(p, q)$, the result is true for $r \leqq 2$.

Assume $r=3$; since P is reduced, Prop. 2 (cases (a), (b)) proves that $F(L)$ is simple. By Prop. 2 (case (c)), we only have to prove the $(2; 0, 1)$- and $(0; 2, 3)$-irreducibility of $F(L)$. Lemmas 8 and 9 prove that, if $c \in \{0, 1\}$ and $c' \in \{2, 3\}$, every $\{c, c'\}$- residue of $F(L)$ has length $\leqq 6$.

Suppose $F(L)$ is $(2; 0, 1)$-reducible: let σ, τ be two 2-coloured edges belonging both to the same $\{0, 2\}$-residue R and to the same $\{1, 2\}$-residue S. Since R has length $\leqq 6$, σ and τ respectively have at least one end-point ($\sigma(0)$ and $\tau(0)$, say) which are adjacent on the same $\{0, 1\}$-residue C; if σ' (resp. τ') denotes the 1-coloured edge adjacent to σ (resp. τ) in $\sigma(0)$ (resp. $\tau(0)$), σ' and τ' must be distinct edges, otherwise C would be a $\{0, 1\}$-residue of length two (and $F(L)$ would not be simple). Moreover, σ' and τ' belong both to C and to S. Hence, $F(L)$ is also $(1; 0, 2)$-reducible, against the hypothesis that P admits no wave (Prop. 2, case (c)).

Suppose $F(L)$ is $(0; 2, 3)$-reducible: let σ, τ be two 0-coloured edges belonging both to the same $\{0, 3\}$-residue R and to the same $\{0, 2\}$-residue S. Since R has length $\leqq 6$, σ has at least one end-point ($\sigma(0)$, say) which is adjacent to one end-point of τ ($\tau(0)$, say); if σ' (resp. τ') denotes the 2-coloured edge adjacent to σ (resp. τ) in $\sigma(0)$ (resp. $\tau(0)$), σ' and τ' must be distinct edges, otherwise $F(L)$ would contain a $\{2, 3\}$-residue of length two (and $F(L)$ would not be simple). Moreover, σ' and τ' belong both to S and to the $\{2, 3\}$-residue containing $\sigma(0)$ and $\tau(0)$. Hence, $F(L)$ is also $(2; 0, 3)$-reducible, against the hypothesis that P admits no wave (Prop. 2, case (c)).

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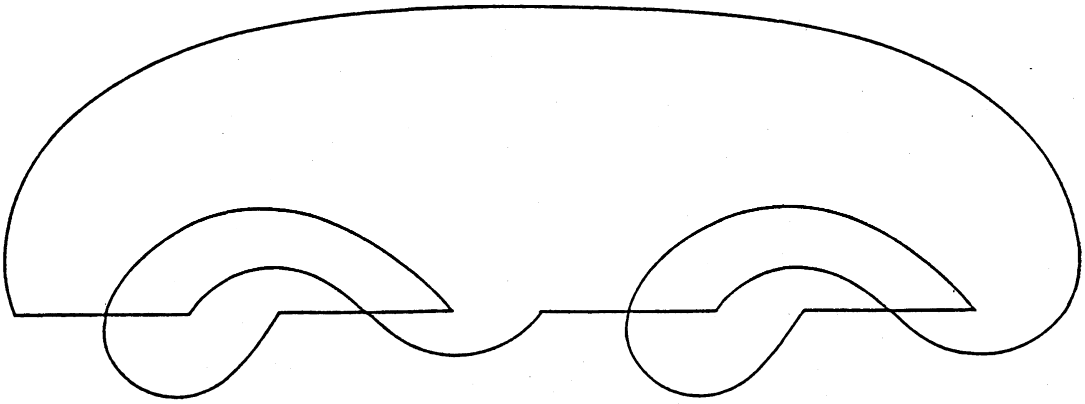

Fig. 1a.

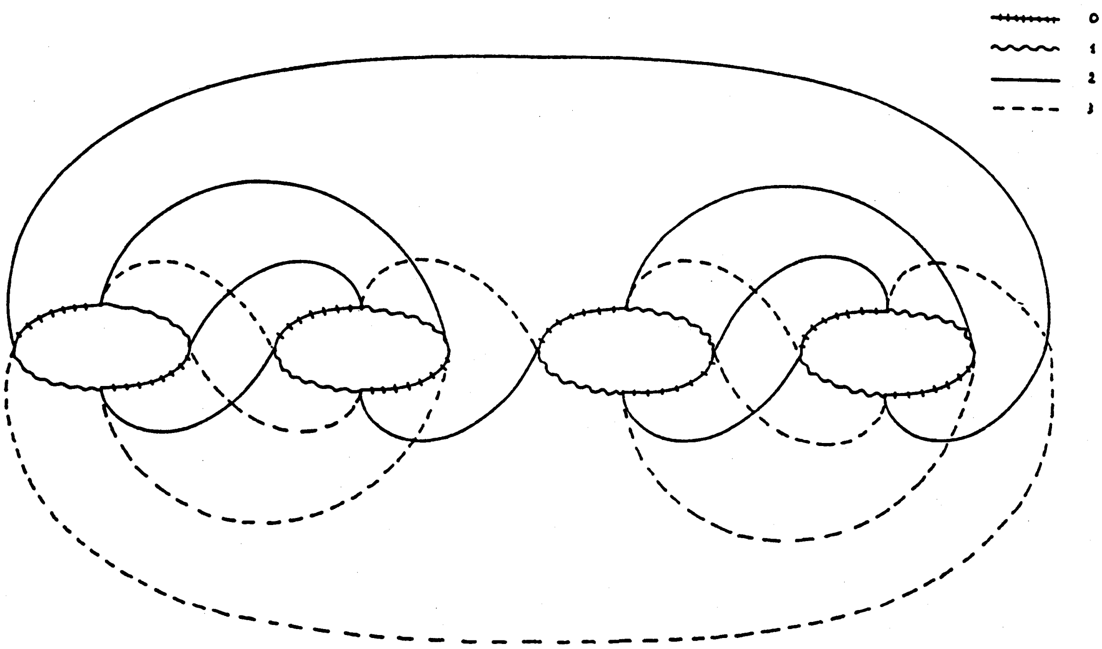

Fig. 1b.

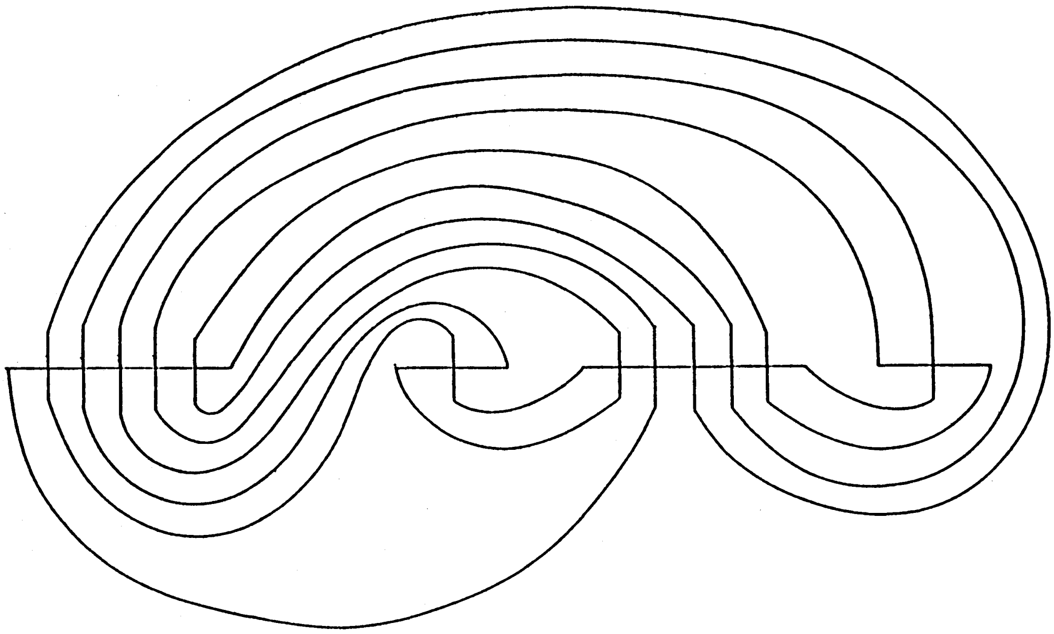

Fig. 2a.

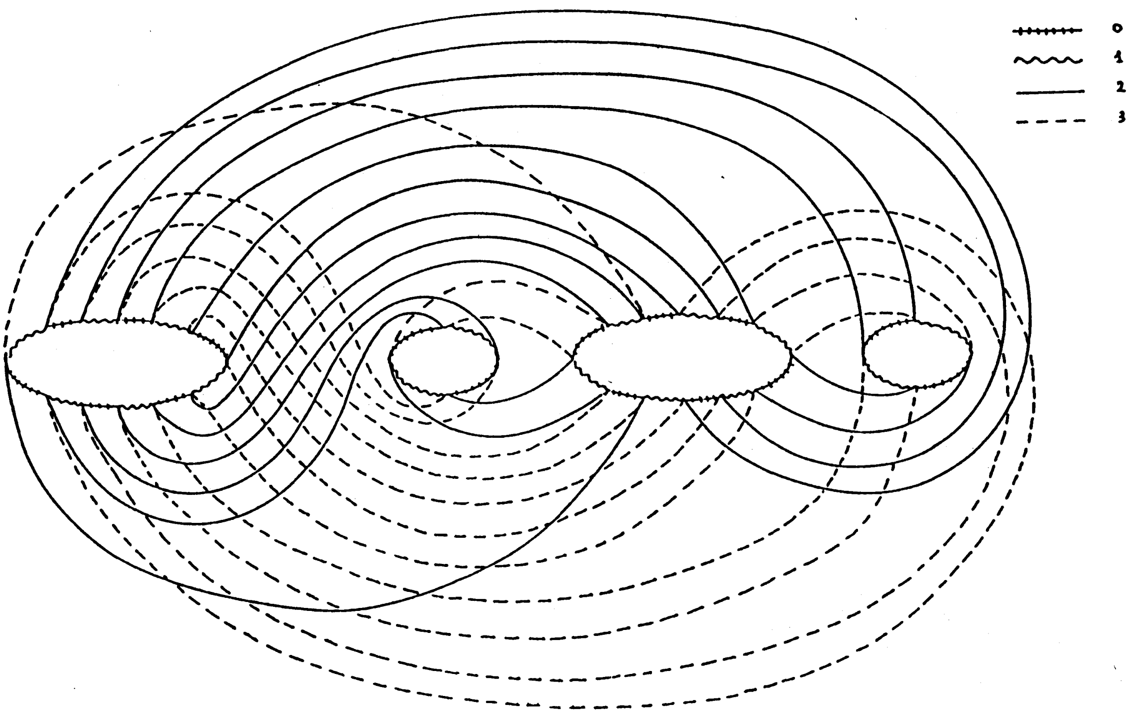

Fig. 2b.