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NON-REGULAR LOCAL RINGS OF FINITE BUCHSBAUM–REPRESENTATION TYPE

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1. Introduction.

Let R be a commutative Noetherian local ring of $\dim R=d$. Then we say that R has finite Buchsbaum-representation type, if there exist only finitely many isomorphism classes of indecomposable Buchsbaum R-modules M which are maximal, i.e. $\dim_R M=d$. D. Eisenbud and S. Goto [1, 2] showed that any regular local ring R has finite Buchsbaum-representation type. The converse is also true, if R is a Cohen-Macaulay complete local ring of $\dim R\geqq 2$ and if R contains an algebraically closed coefficient field k of $\mathrm{ch}\, k\neq 2$ (cf. [4, 5]. See [3, 7] for the case where $\dim R=1$.)[2)].

The Buchsbaum-representation theory was just started and according to a conjecture in [4, Introduction] it seemes to be expected, first of all, to find higher-dimensional non-regular local rings of finite Buchsbaum-representation type.

In this paper we shall construct, for each integer $d\geqq 1$, a (non-Cohen-Macaulay) local ring R of $\dim R=d$ that possesses exactly $d+1$ isomorphism classes of indecomposable maximal Buchsbaum R-modules. The representatives M_n will be arranged so that $\mathrm{depth}_R M_n=n$ for each $0\leqq n\leqq d$.

2. The construction of non-regular local rings of finite Buchsbaum-representation type.

In this section we shall prove the following theorem, which is already known in the case where $d=2$ ([4, Theorem (3.1)]):

Theorem 1. *Let P be a $(d+1)$-dimensional $(d>0)$ regular local ring with the maximal ideal $\mathfrak{n}$. Let $X\in\mathfrak{n}\setminus\mathfrak{n}^2$ and let I be a proper ideal of P with $\mathrm{ht}_P I\geqq 2$. We put $R=P/XI$, $\mathfrak{p}=XR$ and $\mathfrak{m}=\mathfrak{n}R$. Then R has finite Buchsbaum-representation*

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2) This is true for the Cohen-Macaulay complete local rings of dimension two whose residue class fields are algebraically closed, cf. [4].

type and

$$E_n=\mathrm{Syz}^n_{R/\mathfrak{p}}(R/\mathfrak{m})\ (1\leqq n\leqq d),\quad R/\mathfrak{m}\mathfrak{p}$$

are the representatives of indecomposable maximal Buchsbaum R-modules, where $\mathrm{Syz}^n_{R/\mathfrak{p}}(R/\mathfrak{m})$ *denotes the n-th syzygy module of the residue class field* $R/\mathfrak{m}$ *of* $R/\mathfrak{p}$.

Before the proof of Theorem 1, let us recall the foundamental theorem of Buchsbaum-representation theory which plays a key role in our argument (cf. [1] and [2]):

Let A be a regular local ring. Then the syzygy modules of the residue class field of A are the reprentatives of indecomposable maximal Buchsbaum A-modules and so A has finite Buchsbaum-representation type.

We divide the proof of Theorem 1 into several steps. First we show the following

Proposition 2. E_n $(1\leqq n\leqq d)$ *and* $R/\mathfrak{m}\mathfrak{p}$ *are indecomposable maximal Buchsbaum R-modules which are not isomorphic to each other.*

Proof. Since $R/\mathfrak{p}=P/XP$ is a d-dimensional regular local ring with residue class field $R/\mathfrak{m}$, so E_n $(1\leqq n\leqq d)$ are the representatives of indecomposable maximal Buchsbaum $R/\mathfrak{p}$-modules by the foundamental theorem. We regard E_n as R-modules via the canonical map $R\to R/\mathfrak{p}$. Then E_n are still indecomposable maximal Buchsbaum R-modules and $\mathrm{depth}_R E_n=n$. Let us apply the local cohomology functor $H^i_\mathfrak{m}(\cdot)$ to the sequence

$$0\longrightarrow \mathfrak{p}/\mathfrak{m}\mathfrak{p}\longrightarrow R/\mathfrak{m}\mathfrak{p}\longrightarrow R/\mathfrak{p}\longrightarrow 0.$$

Then we have

$$\begin{aligned}H^i_\mathfrak{m}(R/\mathfrak{m}\mathfrak{p})&=\mathfrak{p}/\mathfrak{m}\mathfrak{p}\ (i=0),\\ &=0\quad (i\neq 0,\ d).\end{aligned}$$

Hence $R/\mathfrak{m}\mathfrak{p}$ is also an indecomposable maximal Buchsbaum R-module of $\mathrm{depth}_R(R/\mathfrak{m}\mathfrak{p})=0$ (cf. [8, Ch I, (2.12)]). Because the depths are different, the R-modules $E_1, E_2, \cdots, E_d$ and $R/\mathfrak{m}\mathfrak{p}$ are not isomorphic to each other.

Conversely let M be an indecomposable maximal Buchsbaum R-module. We put $\overline{M}=M/H^0_\mathfrak{m}(M)$.

Lemma 3. $\overline{M}$ *is a maximal Buchsbaum* $R/\mathfrak{p}$*-module which has a decomposition*

$$\overline{M}\cong\bigoplus_{n=1}^{d}(E_n)^{\alpha_n}$$

with integers $\alpha_n\geqq 0$, *where* $(E_n)^{\alpha_n}$ *denotes the direct sum of* α_n *copies of* E_n.

Proof. Since $\bar{M}$ is again a maximal Buchsbaum R-module of $\operatorname{depth}_R \bar{M}>0$ (cf. [8, Ch. I, (2.22)]), we get $\operatorname{Ass}_R \bar{M}=\operatorname{Assh} R=\{p\in \operatorname{Ass} R \mid \dim R/p=d\}$ by [8, Ch. I, (1.10)]. On the other hand $\operatorname{Assh} R=\{\mathfrak{p}\}$, since XP is the unique ideal of P such that $XI\subset XP$ and $\dim P/XP=d$. Thus we find any element of $R\setminus\mathfrak{p}$ is a non-zero divisor on $\bar{M}$ and there is an embedding $\bar{M}\subset(\bar{M})_{\mathfrak{p}}$. Since $\mathfrak{p}R_{\mathfrak{p}}=0$, $\mathfrak{p}\bar{M}=0$. Hence $\bar{M}$ is a maximal Buchsbaum $R/\mathfrak{p}$-module. Therefore because $R/\mathfrak{p}$ is a d-dimensional regular local ring, we get the required decomposition by the fundamental theorem.

We choose elements $Y_1, Y_2, \cdots, Y_d$ of P so that $\mathfrak{n}=(X, Y_1, \cdots, Y_d)P$ and $Y_1, Y_2, \cdots, Y_d$ form a system of parameters for R. Let $x=X \bmod XI$, $y_i=Y_i \bmod XI$ for $1\leqq i\leqq d$ and let $\bar{\ }$ denote the reduction mod $\mathfrak{p}$. Then $\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_d$ is a regular system parameters of $R/\mathfrak{p}$ and so the Koszul complex $K.(\bar{y}_1, \cdots, \bar{y}_d; R/\mathfrak{p})$ generated by the sequence $\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_d$ is a minimal free resolution of the $R/\mathfrak{p}$-module $R/\mathfrak{m}$. Hence E_n is the $(n-1)$-th boundary module of $K.(\bar{y}_1, \cdots, \bar{y}_d; R/\mathfrak{p})$. Let $K_n=K_n(y_1, \cdots, y_d; R)$ and let $p_n: K_n\to E_n$ be the epimorphism induced from the canonical map $K_n\to K_n(\bar{y}_1, \cdots, \bar{y}_d; R/\mathfrak{p})$. We put $L_n=\operatorname{Ker} p_n$. Then we easily see the following

Lemma 4. *$L_n=B_n+xK_n$, where B_n is the n-th boundary module of $K.(y_1, \cdots, y_d; R)$.*

Since $H^0_{\mathfrak{m}}(M)\subset\mathfrak{m}M$ (cf. [5, Claim in the proof of Theorem (5.3)]), we get by Lemma 3 a commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \uparrow & & \\
0 & \longrightarrow & \bigoplus_{n=1}^{d}(L_n)^{\alpha_n} & \longrightarrow & \bigoplus_{n=1}^{d}(K_n)^{\alpha_n} & \longrightarrow & \bigoplus_{n=1}^{d}(E_n)^{\alpha_n} & \longrightarrow & 0\\
 & & \uparrow & & \| & & \uparrow & & \\
0 & \longrightarrow & N & \longrightarrow & \bigoplus_{n=1}^{d}(K_n)^{\alpha_n} & \longrightarrow & M & \longrightarrow & 0\\
 & & \uparrow & & & & \uparrow & & \\
 & & 0 & & & & H^0_{\mathfrak{m}}(M) & & \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}
$$

with exact rows and columns. Let $F=\bigoplus_{n=1}^{d}(K_n)^{\alpha_n}$ and $W=\bigoplus_{n=1}^{d}(L_n)^{\alpha_n}$. Then $W/N\cong H^0_{\mathfrak{m}}(M)$ and hence $\mathfrak{m}W\subset N$ by [8, Ch. I, (2.4)]. Furthermore by [4, Claim 2 in the proof of Theorem (3.1)]* we have the next

*) The proof of [4, Claim 2 in the proof of Theorem (3.1)] works in this case.

Lemma 5. *$W\cap\mathfrak{q}F\subset N$ for any parameter ideal $\mathfrak{q}$ of R.*

Because $y_1, y_2, \cdots, y_d$ form, by the choice, a system of parameters for R, we see $\dim R/(y_1, \cdots, \hat{y}_i, \cdots, y_d)R=1$ for any $1\leqq i\leqq d$. Hence we may choose elements $\{a_i\}_{1\leq i\leq d}$ of R so that $y_1, y_2, \cdots, x+a_iy_i, \cdots, y_d$ form a system of parameters for R (cf. [6], Theorem 124). Let $\mathfrak{q}_0=(y_1, y_2, \cdots, y_d)R$ and $\mathfrak{q}_i=(y_1, \cdots, x+a_iy_i, \cdots, y_d)R$ for $1\leqq i\leqq d$.

Lemma 6. *$L_n\subset \sum_{i=0}^{d}(L_n\cap\mathfrak{q}_iK_n)$ for $1\leqq n<d$.*

Proof. Let $T_1, T_2, \cdots, T_d$ be a free basis of $K.(y_1, \cdots, y_d; R)$. We put for each subset I of $\{1, 2, \cdots, d\}$ with $\#I=m$

$$T_I=T_{i_1}\wedge T_{i_2}\wedge\cdots\wedge T_{i_m},$$

where $I=\{i_1, i_2, \cdots, i_m\}$ with $i_1<i_2<\cdots<i_m$.

Let ∂ denote the boundary operator of $K.(y_1, \cdots, y_d; R)$ and let $1\leqq n<d$. Then B_n is generated by

$$\{\partial T_J \mid J\subset\{1, 2, \cdots, d\},\quad \#J=n+1\}$$

and if $J=\{j_1, j_2, \cdots, j_{n+1}\}$.

$$\partial T_J=\sum_{r=1}^{n+1}(-1)^{r+1}y_{j_r}T_{J\setminus\{j_r\}}.$$

So $B_n\subset L_n\cap\mathfrak{q}_0K_n$. Hence by Lemma 4 it is sufficient to prove that $xK_n\subset \sum_{i=0}^{d}(L_n\cap\mathfrak{q}_iK_n)$. Let $I=\{i_1, i_2, \cdots, i_n\}$ be a subset of $\{1, 2, \cdots, d\}$ with $i_1<i_2<\cdots<i_n$. Since $n<d$, we can choose $j\in\{1, 2, \cdots, d\}\setminus I$. Then

$$\begin{aligned}\partial(T_j\wedge T_I)&=(\partial T_j)\wedge T_I-T_j\wedge(\partial T_I)\\&=y_jT_I-\sum_{r=1}^{n}(-1)^{r+1}y_{i_r}T_j\wedge T_{I\setminus\{i_r\}}\end{aligned}$$

and so we have

$$\begin{aligned}xT_I&=-a_j\partial(T_j\wedge T_I)+(x+a_jy_j)T_I-\sum_{r=1}^{n}(-1)^{r+1}a_jy_{i_r}T_j\wedge T_{I\setminus\{i_r\}}\\&\in B_n+\mathfrak{q}_jK_n.\end{aligned}$$

We are now ready to prove that $M\cong E_n$ for some $1\leqq n\leqq d$ or $M\cong R/\mathfrak{m}\mathfrak{p}$. By Lemma 5 and Lemma 6 it follows that $\left[\left(\bigoplus_{n=1}^{d-1}(L_n)^{\alpha_n}\right)\right]\oplus(0)\subset N$ and so there is an R-submodule N' of $(L_d)^{\alpha_d}(=\mathfrak{p}^{\alpha_d})$ such that $N=\left[\bigoplus_{n=1}^{d-1}(L_n)^{\alpha_n}\right]\oplus N'$ Consequently

$$M\cong\left[\bigoplus_{n=1}^{d-1}(K_n/L_n)^{\alpha_n}\right]\oplus(R^{\alpha_d}/N').$$

Therefore if $\alpha_n\geqq 1$ for some $1\leqq n\leqq d-1$, we have $M\cong E_n$ since M is indecomposable. And if $\alpha_1=\alpha_2=\cdots=\alpha_{d-1}=0$, we have $M\cong R/\mathfrak{m}\mathfrak{p}$ or $M\cong R/\mathfrak{p}=E_d$ by the same argument as the last part of the proof of [4, Theorem (3.1)]. This completes the proof of Theorem 1.

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