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FREIDLIN-WENTZELL TYPE ESTIMATES FOR A CLASS OF SELF-SIMILAR PROCESSES REPRESENTED BY MULTIPLE WIENER INTEGRALS

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Abstract. A large deviations result is obtained for a class of self-similar processes represented by multiple Wiener integrals, which includes the limit processes appearing in functional "non-central" limit theorems.

1. Introduction

Let $B=\{B(t), t\in R=(-\infty, \infty)\}$ be a standard Brownian motion with $B(0)=0$, and let H_T denote the reproducing kernel Hilbert space (RKHS) associated with $\{B(t), |t|\leqq T\}$ $(T<\infty)$, i.e., H_T is the Hilbert space consisting of absolutely continuous functions ϕ on $[-T, T]$ such that $\phi(0)=0$ and its Radon-Nikodym derivative $d\phi/dt$ is square integrable, with the norm $\|\phi\|_H=\left(\int_{-T}^{T}(d\phi/dt)^2dt\right)^{1/2}$. We then have the following asymptotic estimates.

Theorem A. (i) *Let* $\phi\in H_T$. *Then, for any* $\delta, \delta'>0$, *there is a number* $\alpha_1=\alpha_1(\delta, \delta', \|\phi\|_T)$ *such that*

$$P(\sup_{-T\leq t\leq T}|B(t)/\alpha-\phi(t)|<\delta)\geqq\exp[-(\alpha^2/2)(\|\phi\|_T^2+\delta')] \tag{1}$$

for all $\alpha\geqq\alpha_1$.

(ii) *Let* $K_{T,r}=\{\phi\in H_T: \|\phi\|_T\leqq r\}$ *for any* $r>0$. *Then, for any* $\delta, \delta'>0$, *there is a number* $\alpha_2=\alpha_2(\delta, \delta', r)$ *such that*

$$P(\inf_{\phi\in K_{T,r}}\sup_{-T\leq t\leq T}|B(t)/\alpha-\phi(t)|>\delta)\leqq\exp[-(\alpha^2/2)(r^2-\delta')] \tag{2}$$

for all $\alpha\geqq\alpha_2$.

The estimates of the type (1) and (2) were first obtained by Freidlin-Wentzell [4] for diffusion processes. Theorem A above is a particular case of

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general results for Gaussian processes (cf. e. g. [1]). The purpose of this note is to remark that analogous estimates can be derived for a class of self-similar processes represented by multiple Wiener integrals.

2. Result

Let $X=\{X(t), t\in R^{+}=[0, \infty)\}$ be a process defined by

$$X(t)=\int_{R^m}\cdots\int Q_t(u_1, \cdots, u_m)dB(u_1)\cdots dB(u_m), t\in R^{+}, \tag{3}$$

where the right hand side is a multiple Wiener integral with respect to standard Brownian motion $\{B(u), u\in R\}$ with $B(0)=0$. We assume that the kernel $Q_t(u_1, \cdots, u_m)$ is of the form

$$Q_t(u_1, \cdots, u_m)=\int_0^t q(v-u_1, \cdots, v-u_m)dv, t\in R^{+}, \tag{4}$$

where $q(u_1, \cdots, u_m)$ is symmetric, homogeneous with degree $-\lambda=H-1-m/2$ $(1/2<H<1)$, i. e.

$$q(cu_1, \cdots, cu_m)=c^{-\lambda}q(u_1, \cdots, u_m), \quad \text{for any } c>0,$$

and satisfies the condition

$$\int_{R^m}\cdots\int |q(u_1, \cdots, u_m)q(u_1+1, \cdots, u_m+1)|du_1\cdots du_m<\infty.$$

The process X is self-similar with parameter H, i. e., $\{X(ct)\}$ and $\{c^H X(t)\}$ have the same finite dimensional distributions for any $c>0$, and it has stationary increments. Furthermore X may be supposed to have continuous paths ([2]). In the case when

$$Q_t(u_1, \cdots, u_m)=\int_{0\vee\max u_j}^{t\vee\max u_j}\prod_{j=1}^{m}(v-u_j)^{-\alpha}dv, \tag{5}$$
$$\alpha=(1/2)+(1-H)/m, m=1, 2, \cdots,$$

the process X is a limit self-similar process appearing in the so-called non-central limit theorems ([3]). If $m=1$, it is a fractional Brownian motion, and if $m\geqq 2$, it is non-Gaussian.

Let H^Q denote a class of real valued continuous functions y on R^{+} which can be represented in the form

$$y(t)=\int_{R^m}\cdots\int Q_t(u_1, \cdots, u_m)\xi(u_1)\cdots\xi(u_m)du_1\cdots du_m,$$

with $\xi\in L^2(R)$. Write $y=Q[\xi]$, and define, for $y\in H^Q$,

$$\|y\|_Q=\inf\{\|\xi\|_2 : y=Q[\xi]\},$$

where $\|\xi\|_2=\left(\int_{-\infty}^{\infty}\xi^2(u)du\right)^{1/2}$, and put

$$K_r^Q=\{y=Q[\xi] : \|\xi\|_2\leqq r\}.$$

Denote by $\|\cdot\|_\infty$ the supremum norm and by d the metric defined by $\|\cdot\|_\infty$ in $C[0, 1]$. Then we have

Theorem (i) *Let $y\in H^Q$. Then for any δ, $\delta'>0$, there is a number $\alpha_1=\alpha_1(\delta, \delta', \|y\|_Q)$ such that*

$$P(\|X/\alpha-y\|_\infty<\delta)\geqq\exp[-(\alpha^{2/m}/2)(\|y\|_2^2+\delta')] \tag{6}$$

for all $\alpha\geqq\alpha_1$.

(ii) *For any δ, δ', $r>0$, there is a number $\alpha_2=\alpha_2(\delta, \delta', r)$ such that*

$$P(d(X/\alpha, K_r^Q)>\delta)\leqq\exp[-(\alpha^{2/m}/2)(r^2-\delta')] \tag{7}$$

for all $\alpha\geqq\alpha_2$.

The estimates of this type, which will be called Freidlin-Wentzell type estimates, are closely related to functional laws of the iterated logarithm, and the above theorem will be proved by arguments parallel to the ones used in [2].

3. Proof of the Theorem

Let $C_\gamma(R)$ denote the space of real continuous functions x on R such that $x(0)=0$ and

$$\lim_{t\to\pm\infty} x(t)/\gamma(t)=\lim_{t\to 0} x(t(/\gamma(t)=0,$$

where

$$\gamma(t)=\{|t|(1+|\log|t||)\}^{1/2}, \quad t\neq 0,$$
$$\gamma(0)=0.$$

Define a norm

$$\|x\|_\gamma=\sup_{t\neq 0}|x(t)|/\gamma(t), \qquad x\in C_\gamma(R).$$

Then $(C_\gamma(R), \|\cdot\|_\gamma)$ is a Banach space, and, by the iterated logarithm law, $B=\{B(u), u\in R\}$ with $B(0)=0$ may be considered as a $C_\gamma(R)$-valued random element. Let H denote the RKHS associated with B, i.e., H is the Hilbert space of all absolutely continuous functions x on R such that $x(0)=0$ and $\dot{x}=dx/dt\in L_2(R)$ with the norm

$$\|x\|_H=\|\dot{x}\|_2=\left(\int_{-\infty}^{\infty}(\dot{x})^2dt\right)^{1/2}.$$

H is a subspace of $C_\gamma(R)$ because

$$|x(t)| \leqq |t|^{1/2} \|\dot{x}\|_2, \quad \text{for } x \in H.$$

Let

$$K_r = \{x \in H: \|x\|_H \leqq r\}, \quad r > 0.$$

We need the following Freidlin-Wentzell type estimates for B.

Lemma 1. *For any δ, $\delta' > 0$, there are numbers $\alpha_1 = (\delta, \delta', \|x\|_H)$ and $\alpha_2 = \alpha_2(\delta, \delta', r)$ such that*

$$P(\|(B/\alpha) - x\|_\gamma < \delta) \geqq \exp[-(\alpha^2/2)(\|x\|_H^2 + \delta')] \tag{8}$$

for all $\alpha \geqq \alpha_1$, and

$$P(d_\gamma(B/\alpha, K_r) > \delta) \leqq \exp[-(\alpha^2/2)(r^2 - \delta')] \tag{9}$$

for all $\alpha \geqq \alpha_2$, where d_γ is the metric defined by $\|\cdot\|_\gamma$ in $C_\gamma(R)$.

Proof. For any $T > 0$,

$$\begin{aligned} P(\|(B/\alpha) - x\|_\gamma < \delta) \geqq & P(\sup_{1/T \leq |t| \leq T} (1/\gamma(t)) |B(t)/\alpha - x(t)| < \delta/3) \\ & - P(\sup_{0 < |t| < 1/T} (1/\gamma(t)) |B(t)/\alpha - x(t)| \geqq \delta/3) \\ & - P(\sup_{|t| \geq T} (1/\gamma(t)) |B(t)/\alpha - x(t)| \geqq \delta/3). \end{aligned}$$

Applying Theorem A, we have, for and $0 < \delta'' < \delta'$,

$$\begin{aligned} P(\sup_{1/T \leq t \leq T} (1/\gamma(t)) |B(t)/\alpha - x(t)| < \delta/3) & \geqq P(\sup_{1/T \leq |t| \leq T} |B(t)/\alpha - x(t)| < (\delta/3)\gamma(1/T)) \\ & \geqq P(\sup_{|t| \leq T} |B(t)/\alpha - x(t)| < (\delta/3)\gamma(1/T)) \\ & \geqq \exp[-(\alpha^2/2)(\|x\|_T^2 + \delta'')] \end{aligned}$$

for sufficiently large α, where $\|x\|_T^2 = \int_{-T}^{T} (\dot{x})^2 dt$. Since $\|x\|_T \leqq \|x\|_H$ for any $T > 0$, $x \in H$, we have

$$P(\sup_{1/T \leq |t| \leq T} (1/\gamma(t)) |B(t)/\alpha - x(t)| < \delta/3) \geqq \exp[-(\alpha^2/2)(\|x\|_H^2 + \delta'')]$$

for sufficiently large α.

Since, for $x \in H$,

$$\begin{aligned} \sup_{|t| \geq T} |x(t)|/\gamma(t) & \leqq \sup_{|t| \leq T} \|\dot{x}\|_2/(1 + |\log|t||)^{1/2} \\ & \leqq \|\dot{x}\|_2/(1 + \log T)^{1/2} \\ & \longrightarrow 0, \quad \text{as } T \to \infty, \end{aligned}$$

and similarly

$$\sup_{0<|t|\leq 1/T} |x(t)|/\gamma(t) \longrightarrow 0, \quad \text{as } T\to\infty,$$

we may assume, choosing T large enough, that

$$\sup_{|t|\geq T} |x(t)|/\gamma(t)\leqq\delta/6 \quad \text{and} \quad \sup_{0<|t|\leq 1/T} |x(t)|/\gamma(t)\leqq\delta/6.$$

Then

$$P(\sup_{|t|\geq T}(1/\gamma(t))|B(t)/\alpha-x(t)|\leqq\delta/3)\leqq P((1/\alpha)\sup_{|t|\geq T}|B(t)|/\gamma(t)\geqq\delta/6)$$

$$\leqq 2P((1/\alpha)\sup_{|t|\geq T}|B(t)|/\gamma(t)\geqq\delta/6),$$

and similarly

$$P(\sup_{0<|t|\leq 1/T}(1/\gamma(t))|B(t)/\alpha-x(t)|\geqq\delta/3)$$

$$\leqq 2P((1/\alpha)\sup_{0<t\leq 1/T}|B(t)|/\gamma)t)\geqq\delta/6).$$

Now we may assume that $T=2^K$. Then

$$P(\sup_{t\geq 2^K}|B(t)|/\gamma(t)\geqq\alpha\delta/6)\leqq \sum_{k=K}^{\infty} P(\sup_{2^k\leq t\leq 2^{k+1}}|B(t)|/\gamma(t)\geqq\alpha\delta/6),$$

and

$$P(\sup_{2^k\leq t\leq 2^{k+1}}|B(t)|/\gamma(t)\geqq\alpha\delta/6)\leqq P(\sup_{2^k\leq t\leq 2^{k+1}}|B(t)|\geqq\gamma(2^k)\alpha\delta/6)$$

$$\leqq P(\sup_{0\leq t\leq 2^{k+1}}|B(t)|\geqq\gamma(2^k)\alpha\delta/6)$$

$$\leqq 2P(B(2^{k+1})\geqq\gamma(2^k)\alpha\delta/6)$$

$$\leqq 2P(B(1)\geqq\varepsilon'\alpha k^{1/2})$$

$$\text{with } \varepsilon'=(\log 2/2)^{1/2}(\delta/6)$$

$$\leqq \text{constant}\cdot\exp[-(\varepsilon'\alpha)^2 k/2].$$

Hence

$$P(\sup_{t\geq 2^K}|B(t)|/\gamma(t)\geqq\alpha\delta/6)\leqq \text{constant}\cdot\sum_{k=K}^{\infty}\exp[-(\varepsilon'\alpha)^2 k/2]$$

$$\leqq \text{constant}\cdot\exp[-(\varepsilon'^2 K/2)\alpha^2].$$

Using a well known fact that $\{tB(1/t),\ t>0\}$ is also a Brownian motion, we have

$$P(\sup_{0<t<1/2^K}|B(t)|/\gamma(t)\geqq\alpha\delta/6)\leqq \text{constant}\cdot\exp[-\varepsilon'^2 K)(\alpha^2/2)].$$

Thus we have, taking $T=2^K$ large enough,

$$P(\|B/\alpha-x\|_\gamma<\delta)\geqq\exp[-(\alpha^2/2)(\|x\|_H^2+\delta'')]-\text{constant}\cdot\exp[-(\varepsilon'^2 K)(\alpha^2/2)]$$

$$\geqq\exp(-(\alpha^2/2)(\|x\|_H^2+\delta')]$$

for sufficiently large α. This proves (8).

Now, for any $T>0$,

$$P(d_\gamma(B/\alpha, K_r)>\delta)\leqq P(\inf_{x\in K_r}\sup_{1/T\leqq|t|\leqq T}(1/\gamma(t))|B(t)/\alpha-x(t)|>\delta/3)$$
$$+P(\inf_{x\in K_r}\sup_{0<|t|\leqq 1/T}(1/\gamma(t))|B(t)/\alpha-x(t)|>\delta/3)$$
$$+P(\inf_{x\in K_r}\sup_{|t|\geqq T}(1/\gamma(t))|B(t)/\alpha-x(t)|>\delta/3).$$

It follows from Theorem A that

$$P(\inf_{x\in K_r}\sup_{1/T\leqq|t|\leqq T}(1/\gamma(t))|B(t)/\alpha-x(t)|>\delta/3)$$
$$\leqq P(\inf_{x\in K_r}\sup_{|t|\leqq T}|B(t)/\alpha-x(t)|>\gamma(1/T)\delta/3)$$
$$\leqq \exp[-(\alpha^2/2)(r^2-\delta'')],$$

for any $0<\delta''<\delta'$, if α is sufficiently large.

Next, choose T large enough that

$$\sup_{x\in K_r}\sup_{|t|\geqq T}|x(t)|/\gamma(t)\leqq\delta/6, \quad \text{and} \quad \sup_{x\in K_r}\sup_{|t|\leqq 1/T}|(x(t))|/\gamma(t)\leqq\delta/6.$$

Then

$$P(\inf_{x\in K_r}\sup_{|t|\geqq T}(1/\gamma(t))|B(t)/\alpha-x(t)|>\delta/3)$$
$$\leqq P((1/\alpha)\sup_{|t|\geqq T}|B(t)|/\gamma(t)>\delta/6)$$

and

$$P(\inf_{x\in K_r}\sup_{0<|t|\leqq 1/T}(1/\gamma(t))|B(t)/\alpha-x(t)|>\delta/3)$$
$$\leqq P((1/\alpha)\sup_{0<|t|\leqq 1/T}|B(t)|/\gamma(t)>\delta/6).$$

Hence, just as in the proof of (8), we have, taking $T=2^K$ large enough,

$$P(d_\gamma(B/\alpha, K_r)>\delta)\leqq\exp[-(\alpha^2/2)(r^2-\delta'')]+\text{constant}\cdot\exp[-(\alpha^2/2)\varepsilon'^2K]$$
$$\leqq\exp[-(\alpha^2/2)(r^2-\delta')]$$

for sufficiently large α. The proof is complete.

Let $F(\cdot)$ be a continuous mapping from $(C_\gamma(R), \|\cdot\|_\gamma)$ to $(C[0, 1], \|\cdot\|_\infty)$. Assume further that F is is homogeneous with degree $p>0$, i.e., $F(c\cdot)=c^pF(\cdot)$ for any $c>0$. Then, from Lemma 1, we immediately obtain

Lemma 2. *For any $x\in H$, δ, $\delta'>0$, there is a number α_1, such that*

$$P(\|F(B)/\alpha-F(x)\|_\infty<\delta)\geqq\exp(-(\alpha^{2/p}/2)(\|x\|_H^2+\delta')]$$

for all $\alpha\geqq\alpha_1$.

(ii) *For any* δ, δ', $r>0$, *there is a number* α_2 *such that*

$$P(d(F(B)/\alpha, F(K_r))>\delta)\leqq\exp(-(\alpha^{2/p}/2)(r^2-\delta')]$$

for all $\alpha\geqq\alpha_2$.

Furthermore, by the reasoning used in [1] (cf. Theorem 8), we have

Lemma 3.

$$\lim_{\alpha\to\infty}(1/\alpha^{2/p})\log P(\|F(B)\|_\infty>\alpha)=-b^2/2,$$

where

$$b^2=\inf\{\|x\|_H^2 : \|F(x)\|_\infty>1\}$$
$$=\sup\{r^2 : \sup(\|F(x)\|_\infty : x\in K_r)<1\}.$$

Let $\mathcal{G}_\gamma(R^m)$ denote the space of all symmetric functions f which are defined by

$$f(u_1, \cdots, u_m)=(-1)^l \int\cdots\int_{E(u_1,\cdots,u_m)} \phi(v_1, \cdots, v_m)/(\gamma(v_1)\cdots\gamma(v_m))dv_1\cdots dv_m,$$

with $\phi\in L^1(R^m)$, where $E(u_1, \cdots u_m)=E(u_1)\times\cdots\times E(u_m)$ for each $(u_1, \cdots, u_m)\in R^m$ $(u_j\neq 0, 1\leqq j\leqq m)$ with $E(u)=(-\infty, u)$ or (u, ∞) according as $u<0$ or $u>0$, and $l=l(u_1, \cdots, u_m)$ is the number of positive u_j, $1\leqq j\leqq m$. For $f\in\mathcal{G}_\gamma(R^m)$, put

$$f_t(u_1, \cdots, u_m)=t^{H-m/2}f(u_1/t, \cdots, u_m/t), \quad t>0,$$

and

$$f_0(u_1, \cdots, u_m)=0.$$

Then (cf. Lemma 7.2 [2]), for any $\varepsilon>0$, there exists a function $f^\varepsilon\in\mathcal{G}_\gamma(R^m)$ such that

(10) $$\int_{R^m}\cdots\int|Q_1-f_1^\varepsilon|^2du_1\cdots du_m<\varepsilon^2/m!$$

and

$$\int_{R^m}\cdots\int|f_{1+h}^\varepsilon-f_1^\varepsilon|^2du_1\cdots du_m<Ah^{2H}, \quad 0<h<1,$$

where A is a constant independent of ε.

Define, for each $\varepsilon>0$, a process X^ε by

$$X^\varepsilon(t)=\int_{R^m}\cdots\int f_t^\varepsilon(u_1, \cdots, u_m)dB(u_1)\cdots dB(u_m), \quad t\geqq 0$$

Then X^ε is self-similar with parameter H, and we have

Lemma 4. *For any* η, $M>0$,

$$P(\|X-X^{\varepsilon}\|_{\infty}>\eta)\leqq \text{constant}\cdot\exp(-M\eta^{2/m})$$

for all sufficiently small $\varepsilon>0$.

Proof. Put $Z^{\varepsilon}(t)=X(t)-X^{\varepsilon}(t)$. Then (cf. [2], p. 387) $\{E|Z^{\varepsilon}(t+h)-Z^{\varepsilon})t)|^{2}\}^{1/2}\leqq\text{constant}\cdot h^{H}$, for $t\geqq 0$, $0\leqq h\leqq 1$, and $\|E|Z^{\varepsilon}(t)|^{2}\|_{\infty}$ can be made arbitrarily small by choosing $\varepsilon>0$ small enough. Thus the lemma follows from Lemma 6.3 (ii) of [2].

Let

$$(f_{t}^{\varepsilon})^{[r]}(u_1,\cdots,u_{m-2r})=\int_{R^r}\cdots\int f_{t}^{\varepsilon}(u_1,\cdots,u_{m-2r},v_1,v_1,\cdots,v_r,v_r)dv_1\cdots dv_r$$

for $r=1,2,\cdots,[m/2]$, and $(f_{t}^{\varepsilon})^{[0]}=f_{t}^{\varepsilon}$. Define mappings $F_{\varepsilon}^{[r]}$, $r=0,1,\cdots,[m/2]$, from $C_{\gamma}(R)$ to $C[0,1]$ by

$$F_{\varepsilon}^{[r]}(x)(t)=(-1)^{m-2r}\int_{R^{m-2r}}\cdots\int D_1\cdots D_{m-2r}(f_{t}^{\varepsilon})^{[r]}(u_1,\cdots,u_{m-2r})$$
$$\cdot x(u_1)\cdots x(u_{m-2r})du_1\cdots du_{m-2r},\quad t>0,\ x\in C_{\gamma}(R),$$

where $D_j=\partial/\partial u_j$. Then $F_{\varepsilon}^{[r]}$ are continuous and homogeneous with degree $m-2r$ (cf. [2], pp. 378-379). Applying Lemma 5.3 of [2], we can write X^{ε} in the form

$$X^{\varepsilon}=F_{\varepsilon}(B)+\sum_{r=1}^{[m/2]}(-1)^{r}\frac{m!}{2^{r}r!(m-2r)!}F_{\varepsilon}^{[r]}(B),$$

where $F_{\varepsilon}=F_{\varepsilon}^{[0]}$ and

$$F_{\varepsilon}^{[r]}(B)=\int_{R^m}\cdots\int f_{t}^{\varepsilon}(v_1,v_1,\cdots,v_r,v_r)dv_1,\cdots,dv_r\quad\text{if } m=2r.$$

Lemma 5. *For any* η, $M'>0$,

$$P(\|X^{\varepsilon}-F_{\varepsilon}(B)\|_{\infty}>\alpha\eta)\leqq\text{constant}\cdot\exp(-M'\alpha^{2/m})$$

for all sufficiently large α.

Proof. Since

$$\|X^{\varepsilon}-F_{\varepsilon}(B)\|_{\infty}\leqq\sum_{r=1}^{[m/2]}\frac{m!}{2^{r}r!(m-2r)!}\|F_{\varepsilon}^{[r]}(B)\|_{\infty},$$

it suffices to show that, for and η', $M'>0$.

$$P(\|F_{\varepsilon}^{[r]}(B)\|_{\infty}>\alpha\eta')\leqq\text{constant}\cdot\exp(-M'\alpha^{2/m})$$

for all sufficiently large α, for $r=1,2,\cdots,[m/2]$. But by Lemma 3,

$$P(\|F_\varepsilon^{[r]}(B)\|^\infty > \alpha\eta') \leqq \exp(-C\alpha^{2/(m-2r)})$$

for all sufficiently large α, with some constant $C>0$, for $r=1, 2, \cdots, [m/2]$, and the lemma follows.

Note that, if $x \in H$, then $F_\varepsilon(x) = F_\varepsilon^{[0]}(x)$ can be written as

$$F_\varepsilon(x) = \int_{R^m} \cdots \int f_t^\varepsilon(u_1, \cdots, u_m)\dot{x}(u_1) \cdots \dot{x}(u_m) du_1 \cdots du_m.$$

Lemma 6. *For any element $y \in H^Q$ and for any $\eta>0$, there is an element $F_\varepsilon(x) \in F_\varepsilon(H)$ such that $\|y - F_\varepsilon(x)\|_\infty < \eta$ if $\varepsilon>0$ is small enough. Also, if $(K_\gamma^Q)^\eta$ denotes the η-neighborhood of K_γ^Q, then $F_\varepsilon(K_\gamma) \subset (K_\gamma^Q)^\eta$ for all sufficiently small $\varepsilon>0$.*

Proof. Let $y \in H^Q$ be of the form

$$y(t) = Q[\xi](t) = \int_{R^m} \cdots \int Q_t(u_1, \cdots, u_m)\xi(u_1) \cdots \xi(u_m) du_1 \cdots du_m, \ \xi \in L^2(R).$$

Let $x \in H$ be such that $\dot{x} = \xi$. Then, by Schwarz's inequality and (10),

$$\begin{aligned}\|y - F_\varepsilon(x)\|_\infty &= \sup_{0\le t\le 1}\left\{\int_{R^m} \cdots \int |Q_t - f_t^\varepsilon|^2 du_1 \cdots du_m\right\}^{1/2} \|\xi\|_2^m \\ &= \sup_{0\le t\le 1} t^H \left\{\int_{R^m} \cdots \int |Q_1 - f_1^\varepsilon|^2 du_1 \cdots du_m\right\}^{1/2} \|\xi\|_2^m \\ &< (\varepsilon/(m')^{1/2}) \cdot \|\xi\|_2^m.\end{aligned}$$

The lemma follows immediately from the above.

Now, let $y \in H^Q$ and $\delta, \delta'>0$ be given. Assume that y is of the form $y = Q[\xi]$, $\xi \in L^2(R)$. Let $x \in H$ be an element such that $\dot{x} = \xi$. By Lemmas 4, 5, and 6 we can choose $f^\varepsilon \in \mathcal{F}_\gamma(R^m)$ such that

$$\|y - F_\varepsilon(x)\|_\infty < \delta/4,$$

and

$$P(\|X - X^\varepsilon\|_\infty > \alpha\delta/4) \leqq \text{constant} \cdot \exp[-M(\delta/4)^{2/m}\alpha^{2/m}],$$

$$P(\|X_\varepsilon - F_\varepsilon(B)\|_\infty > \alpha\delta/4) \leqq \text{constant} \cdot \exp(-M'\alpha^{2/m}),$$

for all sufficiently large α, where M and M' are some constants such that $M(\delta/4)^{2/m}$, $M' > \|x\|_H^2 + \delta'$. Note that $F_\varepsilon(\cdot)$ is continuous and homogeneous with degree m. Hence, by Lemma 2 (i), for any $0<\delta''<\delta'/2$,

$$P(\|F_\varepsilon(B)/\alpha - F_\varepsilon(x)\|_\infty < \delta/4) \geqq \exp[-(\alpha^{2/m}/2)(\|x\|_H^2 + \delta'')],$$

if α is large enough. Thus

$$P(\|X/\alpha-y\|_\infty<\delta)\geqq P(\|F_\varepsilon(B)/\alpha-F_\varepsilon(x)\|_\infty<\delta/4)$$
$$-P(\|X-X^\varepsilon\|_\infty>\alpha\delta/4)$$
$$-P(\|X^\varepsilon-F_\varepsilon(B)\|_\infty>\alpha\delta/4)$$
$$\geqq\exp[-(\alpha^{2/m}/2)(\|x\|_H^2+\delta'')]$$
$$-\text{constant}\cdot\exp[-M(\delta/4)^{2/m}\alpha^{2/m}]-\text{constant}\cdot\exp(-M'\alpha^{2/m})$$
$$\geqq\exp[-(\alpha^{2/m}/2)(\|x\|_H^2+\delta''')],$$

for all sufficiently large α, where $\delta''<\delta'''<\delta'/2$. Since $\|y\|_Q=\inf\{\|x\|_H : y=Q(\dot{x})\}$, we obtain

$$P(\|X/\alpha-y\|_\infty<\delta)\geqq\exp[-(\alpha^{2/m}/2)(\|y\|_Q^2+\delta')]$$

for sufficiently large α, which is (6).

To prove (7), we choose $f^\varepsilon\in\mathfrak{F}_\gamma(R^m)$ such that

$$F_\varepsilon(K_r)\subset(K_r^Q)^{\delta/2}$$

and

$$P(\|X-X^\varepsilon\|_\infty>\alpha\delta/6)\leqq\text{constant}\cdot\exp[-M(\delta/6)^{2/m}\alpha^{2/m}],$$
$$P(\|X^\varepsilon-F_\varepsilon(B)\|_\infty>\alpha\delta/6)\leqq\text{constant}\cdot\exp(-M'\alpha^{2/m})$$

for all sufficiently large α, with sufficiently large constants M and M'. By Lemma 2 (ii), for $0<\delta''<\delta'/2$,

$$P(d(F_\varepsilon(B)/\alpha,\ F_\varepsilon(K_r))>\delta/6)\leqq\exp[-(\alpha^{2/m}/2)(r^2-\gamma'')],$$

if α is sufficiently large. Hence

$$P(d(X/\alpha,\ K_r^Q)>\delta)\leqq P(d(X/\alpha,\ F_\varepsilon(K_r))>\delta/2)$$
$$\leqq P(d(F_\varepsilon(B)/\alpha,\ F_\varepsilon(K_r))>\delta/6)$$
$$+P(\|X-X^\varepsilon\|_\infty>\alpha\delta/6)+P(\|X^\varepsilon-F_\varepsilon(B)\|_\infty>\alpha\delta/6)$$
$$\leqq\exp[-(\alpha^{2/m}/2)(r^2-\delta'')]$$
$$+\text{constant}\cdot\exp[-M(\delta/6)^{2/m}\alpha^{2/m}]$$
$$+\text{constant}\cdot\exp[-M'\alpha^{2/m}]$$
$$\leqq\exp[-(\alpha^{2/m}/2)(r^2-\delta')]$$

for all sufficiently large α. This completes the proof of the theorem.

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