YOKOHAMA MATHEMATICAL
JOURNAL VOL. 37, 1989

SOME REMARKS ON LOCAL SOLVABILITY OF FUCHSIAN PARTIAL DIFFERENTIAL EQUATIONS FOR HYPERFUNCTIONS

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(Received September 4, 1989)

Abstract. Local solvability in the space of hyperfunctions is proved for Fuchsian elliptic or hyperbolic partial differential equations without any additional conditions.

Let P be a Fuchsian partial differential operator of weight $m-k$ in the sense of Baouendi-Goulaouic [1]. The local solvability of the equation $Pu=f$ for hyperfunctions (f given, and u unknown) has been proved by Tahara [7] in hyperbolic case and by Ôaku [6] in elliptic case. However, both authors assume some genericness conditions on characteristic exponents of P. The aim of this article is to remove these conditions.

First, let us recall the definition of Fuchsian partial differential operators (cf. [1]). Put $N=\{x=(x_1, x_2, \cdots, x_n)\in \boldsymbol{R}^n ; x_1=0\}$ with an integer $n\geqq 2$. We use the notation $x'=(x_2, \cdots, x_n)$, $D_j=\partial/\partial x_j$, $D'=(D_2, \cdots, D_n)$. Then P is said to be a *Fuchsian partial differential operator of weight* $m-k$ *with respect to* N (around $x^0\in N$) if $0\leqq k\leqq m$ and if, on a neighborhood of x^0, P is written in the form

$$P=a(x)\Big(x_1^k D_1^m+\sum_{j=1}^{m} A_j(x, D')x_1^{\max(0, k-j)}D_1^{m-j}\Big),$$

where $A_j(x, D')$ is a linear partial differential operator of order at most j for $j=1, \cdots, m$; $A_j(0, x', D')$ equals a function $a_j(x')$ for $j=1, \cdots, k$; $a(x')$ is a real analytic function with $a(x^0)\neq 0$. Then the *indicial equation* of P at x^0 is the polynomial

$$e(P, \lambda, x^0)=\prod_{\nu=0}^{m-1}(\lambda-\nu)+\sum_{j=1}^{k} a_j(x^0)\prod_{\nu=0}^{m-j-1}(\lambda-\nu)$$

in λ and the *non-trivial characteristic exponents* $\lambda_j=\lambda_j(x^0)$ $(j=1, \cdots, k)$ of P at x^0 are defined as the roots of the equation

$$\frac{e(P, \lambda, x^0)}{\prod_{\nu=0}^{m-k-1}(\lambda-\nu)}=0$$

in λ.

In this article we assume that the principal symbol $\sigma_m(P)(x, \xi)$ of P is written in the form

$$\sigma_m(P)(x, \xi)=x_1^k p_m(x, \xi)$$

with a (complex valued) real analytic function $p_m(x, \xi)$ defined on $U\times \boldsymbol{R}^n$ where U is an open neighborhood of x^0.

Then P is said to be *Fuchsian elliptic* (at x^0) if $p_m(x^0, \xi)\neq 0$ for any $\xi\in \boldsymbol{R}^n\setminus\{0\}$. Note that Fuchsian elliptic equations appear, e.g., in the representation theory of semi-simple Lie groups.

On the other hand, P is said to be *Fuchsian hyperbolic* if p_m is hyperbolic in the x_1 direction, i.e., the equation $p_m(x, \zeta, \xi')=0$ in ζ has only real roots for any $x\in U$ and $\xi'\in \boldsymbol{R}^{n-1}$. One of the most typical Fuchsian hyperbolic operators is

$$x_1\Big(D_1^2-\sum_{j=2}^{n} D_j^2\Big)+\alpha D_1$$

with $\alpha\in \boldsymbol{C}$, which is called the Euler-Poisson-Darboux operator.

We denote by $\mathcal{B}_{x^0}$ the stalk at x^0 of the sheaf $\mathcal{B}$ of hyperfunctions. Hence, $f\in\mathcal{B}_{x^0}$ means that f is a hyperfunction defined on a neighborhood (in $\boldsymbol{R}^n$) of x^0, and $f=0$ holds if and only if its restriction to some smaller neighborhood of x^0 vanishes.

Theorem. *Let P be a Fuchsian elliptic or hyperbolic operator of weight $m-k$ with $0\leqq k\leqq m$ with respect to a hypersurface N defined on a neighborhood of $x^0\in N$. Then*

$$P:\ \mathcal{B}_{x^0}\longrightarrow \mathcal{B}_{x^0}$$

is surjective; i.e, the equation $Pu=f$ is locally solvable at x^0 for any hyperfunction f.

Proof. (i) First we assume $\lambda_j\notin\{\nu\in \boldsymbol{Z};\ \nu\geqq m-k\}$ for any $j=1, \cdots, k$. If P is Fuchsian elliptic, the local solvability has been proved in [6]. If P is Fuchsian hyperbolic, the local solvability was proved essentially by H. Tahara. However, in Theorem 2.3.6 of [7] an additional condition that $\lambda_i-\lambda_j\notin \boldsymbol{Z}$ if $i\neq j$ is imposed. Here we give a proof of local solvability without this additional assumption.

Hence now we assume that P is Fuchsian hyperbolic and that $\lambda_j\notin\{\nu\in \boldsymbol{Z};\ \nu\geqq m-k\}$ for any $j=1, \cdots, k$. Let $\sqrt{-1}S^*\boldsymbol{R}^n=\boldsymbol{R}^n\times\sqrt{-1}S^{n-1}\ni(x, \sqrt{-1}\xi\infty)$ be the purely imaginary cosphere bundle of $\boldsymbol{R}^n$ with $\xi=(\xi_1, \cdots, \xi_n)=(\xi_1, \xi')\in \boldsymbol{R}^n\setminus\{0\}$ ($\xi\infty$ is the projection of ξ to the $(n-1)$-sphere S^{n-1}). Put

$$Z=(\sqrt{-1}S^*\boldsymbol{R}^n|_N)\setminus\sqrt{-1}S_N^*\boldsymbol{R}^n=\{(0, x', \sqrt{-1}\xi\infty);\ x'\in \boldsymbol{R}^{n-1}, \xi\in \boldsymbol{R}^n, \xi'\neq 0\}$$

and let

$$\rho : Z \longrightarrow \sqrt{-1}S^*N = \sqrt{-1}S^*\boldsymbol{R}^{n-1}$$

be the map defined by $\rho(0, x', \sqrt{-1}\xi\infty) = (x', \sqrt{-1}\xi'\infty)$. We denote by $\mathcal{C}$ the sheaf of microfunctions on $\sqrt{-1}S^*\boldsymbol{R}^n$ and by $\rho_!$ the functor of direct image with proper support with respect to ρ.

By Theorem 2 of [3] (see also [4] for details of the proof), for any $g \in \rho_!(\mathcal{C}|_Z)_{x^*}$ with $x^* \in \sqrt{-1}S^*N$, there exists a $v \in \rho_!(\mathcal{C}|_Z)_{x^*}$ such that $Pv = g$ and that $D_1^\nu v(0, x') = 0$ for any $\nu = 0, \cdots, m-k-1$. Moreover, in view of Theorem 2.3 of [5], such v is unique (Note that sections of $\rho_!(\mathcal{C}|_Z)$ are naturally regarded as F-mild microfunctions in the sense of [5]).

Now let $f \in \mathcal{B}_{x^0}$ and denote by $\mathrm{sp}(f)$ the spectrum of f (i.e. the microfunction defined by f). By the flabbiness of the sheaf of microfunctions, we can take a microfunction g defined on $V \times \sqrt{-1}S^{n-1}$ with an open neighborhood V of x^0 such that $g = \mathrm{sp}(f)$ on

$$\{(x, \sqrt{-1}\xi\infty) \in \sqrt{-1}S^*\boldsymbol{R}^n ; x \in V, |\xi_1| < |\xi'|\}$$

and its support $\mathrm{supp}(g)$ is contained in

$$\{(x, \sqrt{-1}\xi\infty) \in \sqrt{-1}S^*\boldsymbol{R}^n ; x \in V, |\xi_1| \leqq |\xi'|\}.$$

By the above argument, we can take, for any $\eta\infty \in S^{n-2}$, a microfunction v_η on

$$\Omega(\eta) = \{(x, \sqrt{-1}\xi\infty) ; |x - x^0| < \varepsilon(\eta), \xi \in \boldsymbol{R}^n, |\xi' - \eta| < \varepsilon(\eta)\}$$

such that

$$\begin{gathered} Pv_\eta = g \quad \text{on } \Omega(\eta), \\ D_1^\nu v_\eta(0, x') = 0 \quad \text{for } \nu = 0, \cdots, m-k-1, \\ \mathrm{supp}(v_\eta) \subset \left\{(x, \sqrt{-1}\xi\infty) \in \Omega(\eta) ; |\xi_1| \leqq \frac{|\xi'|}{\varepsilon(\eta)}\right\} \end{gathered}$$

with some $\varepsilon(\eta) > 0$.

We can take finite number of $\eta^{(1)}, \cdots, \eta^{(J)} \in S^{n-2}$ such that

$$\bigcup_{j=1}^{J} \Omega(\eta^{(j)}) \supset Z_0 = Z \cap (\{x^0\} \times \sqrt{-1}S^{n-1}).$$

By the uniqueness we have $v_{\eta^{(i)}} = v_{\eta^{(j)}}$ on a neighborhood of $\Omega(\eta^{(i)}) \cap \Omega(\eta^{(j)}) \cap Z$. Hence these $v_{\eta^{(j)}}$'s define a microfunction v on

$$\Omega = \{(x, \sqrt{-1}\xi\infty) ; |x - x^0| < \varepsilon, |\xi_1| < |\xi'|\}$$

with some $\varepsilon > 0$ such that $Pv = \mathrm{sp}(f)$ on Ω. Hence by the same argument as the proof of Theorem 1 of [6], we get the surjectivity of

$$P:(\mathcal{B}/\mathcal{A})_{x^0} \longrightarrow (\mathcal{B}/\mathcal{A})_{x^0},$$

where $\mathcal{A}$ denotes the sheaf of real analytic functions on $\boldsymbol{R}^n$. Since $P: \mathcal{A}_{x^0} \to \mathcal{A}_{x^0}$ is surjective, $P: \mathcal{B}_{x^0} \to \mathcal{B}_{x^0}$ is also surjective.

(ii) Now we make no assumptions on characteristic exponents. But we assume $k=m$ for the moment. Let $\lambda_1, \cdots, \lambda_m$ be the characteristic exponents of P at x_0. We can take a nonnegative integer ν such that $\lambda_j - \nu \notin \{0, 1, 2, \cdots\}$ for any $j=1, \cdots, m$.

Note that P can be rewritten in the form

$$P=(x_1D_1)^m + \sum_{j=1}^{m} B_j(x, D')(x_1D_1)^{m-j}$$

with linear partial differential operators $B_j(x, D')$ of order$\leqq j$ such that $B_j(0, x', D')$ equals a function $b_j(x')$. Then it is easy to see that

$$e(P, \lambda, x^0)=\lambda^m + \sum_{j=1}^{m} b_j(x^0)\lambda^{m-j}.$$

Using the formula

$$x_1D_1x_1^\nu = x_1^\nu(x_1D_1+\nu)$$

(here x_1 is considered as a differential operator of order 0) we have

$$Px_1^\nu = x_1^\nu Q$$

with

$$Q=(x_1D_1+\nu)^m + \sum_{j=1}^{m} B_j(x, D')(x_1D_1+\nu)^{m-j}.$$

Then Q is also a Fuchsian elliptic or hyperbolic operator (since $\sigma_m(P)=\sigma_m(Q)$) and we have

$$e(Q, \lambda, x^0)=e(P, \lambda+\nu, x^0).$$

Since Q satisfies the assumptions in (i) with $k=m$,

$$Q: \mathcal{B}_{x^0} \longrightarrow \mathcal{B}_{x^0}$$

is surjective. Since $x_1: \mathcal{B}_{x^0} \to \mathcal{B}_{x^0}$ is surjective (see, e.g. [2] for the proof), we get the surjectivity of

$$x_1^\nu Q: \mathcal{B}_{x^0} \longrightarrow \mathcal{B}_{x^0}.$$

Hence for any $f \in \mathcal{B}_{x^0}$ there exists $u \in \mathcal{B}_{x^0}$ such that

$$Px_1^\nu u = x_1^\nu Qu = f.$$

This means the local solvability of P.

(iii) Finally we consider general case. First note that $Q=x_1^{m-k}P$ is a Fuchsian elliptic or hyperbolic operator of weight 0. Put $R=Px_1^{m-k}$. Then since

$$Qx_1^{m-k}=x_1^{m-k}R,$$

we know that R is also a Fuchsian elliptic or hyperbolic operator of weight 0 by the same argument as in (ii). Hence by (ii)

$$R: \mathscr{B}_{x0} \longrightarrow \mathscr{B}_{x0}$$

is surjective. This means the surjectivity of

$$P: \mathscr{B}_{x0} \longrightarrow \mathscr{B}_{x0}$$

since $R=Px_1^{m-k}$. This completes the proof.

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