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THE BERNSTEIN PROBLEM FOR TIMELIKE SURFACES

By

MARTIN A. MAGID*

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Bernstein's Theorem, which states that the only entire minimal surfaces in $\boldsymbol{R}^3$ are planes, is one of the most striking theorems in global geometry.

It has been known for some time (e.g. [G]) that such a result does not hold for entire timelike surfaces in $\boldsymbol{R}_1^3$, three dimensional Lorentz space. T. Milnor discusses the indefinite Bernstein problem in [Mi] and proves a conformal Bernstein's Theorem.

This paper gives one version of a solution to the indefinite Bernstein problem. We look at entire timelike surfaces in $\boldsymbol{R}_1^3$ which are critical points of the area functional, i.e., which have zero mean curvature. This is equivalent to finding global solutions $f(x, y)$ or $h(x, y)$ to the partial differential equations $(1+f_y^2)f_{xx}-2f_xf_yf_{xy}+(f_x^2-1)f_{yy}=0$ or $(1-h_y^2)h_{xx}+2h_xh_yh_{xy}+(1-h_x^2)h_{yy}=0$. We show that such a graph over a timelike or spacelike plane is a global translation surface (Theorems 1 and 1′). We also obtain a standard form for all such graphs (Theorems 2 and 2). These last two theorems give an answer to the question "What do all the entire timelike surfaces with zero mean curvature in $\boldsymbol{R}_1^3$ look like?" As an application we calculate the sectional curvature of these surfaces and find that the sectional curvature can be negative. Along the way we give all solutions to a hyperbolic Monge-Ampère equation: $\varphi_{xx}\varphi_{yy}-(\varphi_{xy})^2=-1$.

1. Introduction

We assume that the metric in $\boldsymbol{R}_1^3$, has the standard form $g((x, y, z), (x, y, z)) = -x^2+y^2+z^2$. Thus a graph over a timelike plane has the form

$$F(x, y)=(x, y, f(x, y)), \tag{1}$$

where $f: \boldsymbol{R}^2 \rightarrow \boldsymbol{R}$, while a graph over a spacelike plane can be written

$$H(y, z)=(h(y, z), y, z) \tag{1'}$$

for some $h: \boldsymbol{R}^2 \rightarrow \boldsymbol{R}$.

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(We will not consider the case of a graph over a light-like plane, which can be written as

$$(1'') \qquad P(u, v)=\left(\frac{u+p(u, v)}{\sqrt{2}}, \frac{u-p(u, v)}{\sqrt{2}}, v\right).$$

From equations (1) and (1′) we have

$$\frac{\partial F}{\partial x}=(1, 0, f_x) \qquad \frac{\partial F}{\partial y}=(0, 1, f_y)$$

$$\frac{\partial H}{\partial y}=(h_y, 1, 0) \qquad \frac{\partial H}{\partial z}=(h_z, 0, 1),$$

where, for example, f_x denotes $\frac{\partial f}{\partial x}$.

Therefore the metrics induced on the graphs have determinants

$$(2) \qquad g\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial x}\right)g\left(\frac{\partial F}{\partial y}, \frac{\partial F}{\partial y}\right)-g\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right)^2=-1-f_y^2+f_x^2$$

and

$$(2') \qquad g\left(\frac{\partial H}{\partial y}, \frac{\partial H}{\partial y}\right)g\left(\frac{\partial H}{\partial z}, \frac{\partial H}{\partial z}\right)-g\left(\frac{\partial H}{\partial y}, \frac{\partial H}{\partial z}\right)^2=1-h_y^2+h_z^2.$$

We want the induced metric to be timelike, so these determinants must be negative. Therefore we require

$$(3) \qquad 1+f_y^2-f_x^2>0 \quad \text{and}$$

$$(3') \qquad h_y^2+h_z^2-1>0.$$

The unit normal vector in each case is

$$(4) \qquad \xi(x, y)=\frac{(f_x, -f_y, 1)}{\sqrt{(1+f_y^2-f_x^2)}}$$

$$(4') \qquad \xi(y, z)=\frac{(1, h_y, h_z)}{\sqrt{(h_y^2+h_z^2-1)}}.$$

A tedious, but straight-forward, calculation of the shape operator shows that the trace of the shape operator is zero iff

$$(5) \qquad (1+f_y^2)f_{xx}-2f_xf_yf_{xy}+(f_x^2-1)f_{yy}=0 \quad \text{or}$$

$$(5') \qquad (1-h_z^2)h_{yy}+2h_yh_zh_{yz}+(1-h_y^2)h_{zz}=0.$$

We will find all global solutions to (5) and (5′) satisfying (3) and (3′).

Some simple solutions are found in [Mi].

(i) $f(x, y)=k(x\pm y)$, for any C^2 function k,

(ii) $f(x, y)=x\tanh(y)$,

(iii) $f(x, y)=x+k(y)$, for $k\neq 0$ and
(iv) $h(y, z)=y+k(z)$, for $k'\neq 0$.

Note that (i), (iii), and (iv) have zero sectional curvature κ, while (ii) has positive sectional curvature. We will see that it is possible to have $\kappa<0$.

2. Reducing the problem to the classification of isometric immersions from E^2 to H_1^3.

We begin with a function $F(x, y)$ as in (1), satisfying (3) and (5), and employ the standard abbreviations: $p=f_x$, $q=f_y$ and $W=(1+q^2-p^2)^{1/2}$, the positive root. Since

$$\frac{\partial\left(\frac{1+q^2}{W}\right)}{\partial x}=\frac{\partial\left(\frac{pq}{W}\right)}{\partial y} \quad \text{and} \quad \frac{\partial\left(\frac{pq}{W}\right)}{\partial x}=\frac{\partial\left(\frac{p^2-1}{W}\right)}{\partial y},$$

we can find a $\varphi: \boldsymbol{R}^2\to\boldsymbol{R}$ satisfying

$$\varphi_{xx}=\frac{p^2-1}{W},\ \varphi_{xy}=\frac{pq}{W} \quad \text{and} \quad \varphi_{yy}=\frac{q^2+1}{W}. \tag{6}$$

Such a φ satisfies the hyperbolic Monge-Ampère equation

$$\varphi_{xx}\varphi_{yy}-(\varphi_{xy})^2=-1. \tag{7}$$

For $H(y, z)$ satisfying (3′) and (5′) we set $p=h_y$, $q=h_z$, $W=(p^2+q^2-1)^{1/2}>0$

$$\frac{\partial\left(\frac{1-q^2}{W}\right)}{\partial y}=\frac{\partial\left(-\frac{pq}{W}\right)}{\partial z} \quad \text{and} \quad \frac{\partial\left(-\frac{pq}{W}\right)}{\partial y}=\frac{\partial\left(\frac{1-p^2}{W}\right)}{\partial z}.$$

This yields $\phi: \boldsymbol{R}^2\to\boldsymbol{R}$ with

$$\phi_{yy}=\frac{1-p^2}{W},\ \phi_{yz}=-\frac{pq}{W} \quad \text{and} \quad \phi_{zz}=\frac{1-q^2}{W}, \tag{6′}$$

as well as

$$\phi_{yy}\phi_{zz}-(\phi_{yz})^2=-1. \tag{7′}$$

Given such a φ or ϕ we can construct a 2×2 matrix A using the second partials:

$$A=\begin{bmatrix}\varphi_{xx} & \varphi_{xy}\\ \varphi_{xy} & \varphi_{yy}\end{bmatrix} \quad \text{or} \quad \begin{bmatrix}\phi_{yy} & \phi_{yz}\\ \phi_{yz} & \phi_{zz}\end{bmatrix}.$$

The matrices both have determinant -1.

From the Fundamental Theorem of Surfaces it follows that A is the shape operator of an isometric immersion of E^2 into H_1^3, the 3-dimensional Lorentzian space form with constant curvature -1. We use the standard rectangular coordinates $\{x, y\}$ or $\{y, z\}$ in E^2. The Gauss equation holds because $\det A = -1$, while the equations $(\varphi_{xx})_y = (\varphi_{xy})_x$ and $(\varphi_{xy})_y = (\varphi_{yy})_x$ etc. gives Codazzi's equation. (See [D-N], pp. 74-75.) [D-N] also shows that the same shape operator gives an isometric immersion from E^2 into S^3, while $B = \begin{bmatrix} \varphi_{xx} & \varphi_{xy} \\ -\varphi_{xy} & \varphi_{yy} \end{bmatrix}$ or $\begin{bmatrix} \psi_{yy} & \psi_{yz} \\ -\psi_{yz} & \psi_{zz} \end{bmatrix}$ yields an isometric immersion from the Lorentzian plane into H_1^3.

Thus each global solution to (3) and (5) or (3′) and (5′) gives an isometric immersion from E^2 into H_1^3.

3. The classification of isometric immersions of E^2 into H_1^3.

In [B] S. Buyske proves a generalization of a theorem due to J. D. Moore: Let M be a complete, simply connected n-dimensional Riemannian manifold of constant curvature κ isometrically immersed in a $2n-1$ dimensional semi-Riemannian manifold $\tilde{M}$ of constant curvature $\tilde{\kappa}$ such that the metric restricted to the normal space is negative definite. If $\kappa - \tilde{\kappa} > 0$, then there exist n linearly independent unit-length asymptotic vector fields $Z_1, \cdots, Z_n$ on M which determine a global coordinate system whose coordinate vectors are the Z_i's.

Using this theorem isometric immersions from E^2 into H_1^3 are classified. It is shown that the shape operator of such an isometric immersion must take the form

$$\frac{1}{\sin(U+V)}\begin{bmatrix} -\cos(U+V)+\cos(U-V) & \sin(U-V) \\ \sin(U-V) & -\cos(U+V)-\cos(U-V) \end{bmatrix} \tag{8}$$

with respect to a Euclidean coordinate system $\{x, y\}$. Here $0 < U+V < \pi$. U and V are initially given as pure functions of global asymptotic coordinates $\{u, v\}$ on E^2, that is, $U = U(u)$ and $V = V(v)$. This gives all solutions to (7).

The relationship between $\{x, y\}$ and $\{u, v\}$ is given by

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{-\sin V}{\sin(U+V)}\frac{\partial}{\partial u} - \frac{\sin U}{\sin(U+V)}\frac{\partial}{\partial v} \\ \frac{\partial}{\partial y} &= \frac{-\cos V}{\sin(U+V)}\frac{\partial}{\partial u} + \frac{\cos U}{\sin(U+V)}\frac{\partial}{\partial v}. \end{aligned} \tag{9}$$

In [B] the author begins with functions $U(u)$ and $V(v)$ satisfying $0 < \varepsilon < U+V < \pi - \varepsilon < \pi$ and constructs a metric $g = du^2 + 2\cos(U+V)dudv + dv^2$ on $\boldsymbol{R}^2$ which he shows to be complete. In fact we do not need the ε. With the above change of variable the metric g is transformed to $dx^2 + dy^2$, which is clearly complete.

We note in passing that U and V, considered as functions of x and y, must satisfy the Codazzi equations. These are equivalent to the system:

$$-(\cos V)U_x+(\sin V)U_y=0$$
$$(\cos U)V_x+(\sin U)V_y=0\,.$$

Using (9) we can generate the entries of the Jacobian matrices of the coordinate transformations:

$$\text{(10)}\qquad \begin{aligned} u_x&=\frac{-\sin V}{\sin(U+V)} & u_y&=\frac{-\cos V}{\sin(U+V)}\\ v_x&=\frac{-\sin U}{\sin(U+V)} & v_y&=\frac{\cos U}{\sin(U+V)}\end{aligned}$$

$$\text{(11)}\qquad \begin{aligned} x_u&=-\cos(U) & x_v&=-\cos(V)\\ y_u&=-\sin(U) & y_v&=\sin(V)\,.\end{aligned}$$

To summarize, if we begin with $F(x, y)$ satisfying (3) and (5) then we can find, using equation (8), functions φ, U, and V satisfying (6) and (7), as well as

$$\text{(12)}\qquad \begin{aligned} \varphi_{xx}&=\frac{-\cos(U+V)+\cos(U-V)}{\sin(U+V)}=\frac{2\sin(U)\sin(V)}{\sin(U+V)}\\ \varphi_{xy}&=\frac{\sin(U-V)}{\sin(U+V)}\\ \varphi_{yy}&=\frac{-\cos(U+V)-\cos(U-V)}{\sin(U+V)}=\frac{-2\cos(U)\cos(V)}{\sin(U+V)}\end{aligned}$$

We now look at the metric F induces on $\boldsymbol{R}^2$ using the variables $\{u, v\}$. That is, start with $F: \boldsymbol{R}^2 \to \boldsymbol{R}^3_1$ with zero mean curvature. This determines functions U and V by (12). Change to the coordinate system $\{u, v\}$ using equations (9). Then we have

Proposition 1. *$\{u, v\}$ is a global null cordinate system on $\boldsymbol{R}^2$ with the metric induced by F.*

Proof. We look at $F(u, v)=(x(u, v), y(u, v), f(x(u, v), y(u, v)))$. Thus

$$F_u=(x_u,\ y_u,\ f_x x_u+f_y y_u) \quad\text{and}\quad F_v=(x_v,\ y_v,\ f_x x_v+f_y y_v)\,.$$

Using (11) these become

$$\text{(13)}\qquad \begin{aligned} F_u&=(-\cos U,\ -\sin U,\ -\cos(U)f_x-\sin(U)f_y) \quad\text{and}\\ F_v&=(-\cos V,\ \sin V,\ -\cos(V)f_x+\sin(V)f_y)\,.\end{aligned}$$

Using (6) and (12) we calculate:

$$\begin{aligned} &g(F_u, F_u)\\ &\quad=(f_x^2-1)\cos^2 U+(f_y^2+1)\sin^2 U+2f_x f_y\sin U\cos U\end{aligned}$$

$$=W(\varphi_{xx}\cos^2 U+\varphi_{yy}\sin^2 U+2\varphi_{xy}\sin U\cos U)$$
$$=\frac{W}{\sin(U+V)}(2\cos^2 U\sin U\sin V-2\sin^2 U\cos U\cos V+2\cos U\sin U\sin(U-V))$$
$$=0.$$

In the same way we can see that $g(F_v, F_v)=0$ and $g(F_u, F_v)=W\sin(U+V)\neq 0$.
Q. E. D.

The crucial point is that, according to [Mc], $F(u, v)$ has zero mean curvature with respect to a (local) null coordinate system iff each coordinate function of $F(u, v)$ is a sum of the form $\alpha(u)+\beta(v)$. Roughly speaking this is so because the Laplacian has the form $\partial^2/\partial u\partial v=0$ in this coordinate system. Thus we can write

$$F(u, v)=(\alpha_1(u)+\beta_1(v), \alpha_2(u)+\beta_2(v), \alpha_3(u)+\beta_3(v)). \tag{14}$$

We record this as

Theorem 1. *Every timelike graph over a timelike plane in $\boldsymbol{R}_1^3$ with zero mean curvature is a global translation surface.*

There is more information to be gleaned about the α_j's and β_j's. It is also shown in [Mc] that $(\alpha_1(u), \alpha_2(u), \alpha_3(u))$ and $(\beta_1(v), \beta_2(v), \beta_3(v))$ are linearly independent null curves. Therefore $(\alpha_1')^2=(\alpha_2')^2+(\alpha_3')^2$ and $(\dot\beta_1)^2=(\dot\beta_2)^2+(\dot\beta_3)^2$, where $'$ and $\dot{}$ denote differentiation with respect to u and v respectively. From equation (13) we then have

Theorem 2. *If $F: \boldsymbol{R}^2\to\boldsymbol{R}_1^3$ define a timelike graph over a timelike plane with zero mean curvature then there are coordinates $\{u, v\}$ in $\boldsymbol{R}^2$ and functions $U(u)$ and $V(v)$ so that*

$$F_u=(-\cos U, -\sin U, \pm(\cos(2U))^{1/2})$$
$$F_v=(-\cos V, \sin V, \pm(\cos(2V))^{1/2}).$$

The same change of variables gives a more striking result in the case of a graph over a spacelike plane with zero mean curvature $H(y, z)=(h(y, z), y, z)$. Given such an H we have found ψ, U and V so that ψ satisfies (6′), (7′) and

$$\begin{aligned}\psi_{yy}&=\frac{-\cos(U+V)+\cos(U-V)}{\sin(U+V)}=\frac{2\sin(U)\sin(V)}{\sin(U+V)}\\ \psi_{yz}&=\frac{\sin(U-V)}{\sin(U+V)}\\ \psi_{zz}&=\frac{-\cos(U+V)-\cos(U-V)}{\sin(U+V)}=\frac{-2\cos(U)\cos(V)}{\sin(U+V)}.\end{aligned}\tag{12′}$$

We then change variables using

$$(9') \quad \begin{aligned} \frac{\partial}{\partial y} &= \frac{-\sin V}{\sin(U+V)} \frac{\partial}{\partial u} - \frac{\sin U}{\sin(U+V)} \frac{\partial}{\partial v} \\ \frac{\partial}{\partial z} &= \frac{-\cos V}{\sin(U+V)} \frac{\partial}{\partial u} + \frac{\cos U}{\sin(U+V)} \frac{\partial}{\partial v}. \end{aligned}$$

This gives

$$\begin{aligned} y_u &= -\cos U \qquad y_v = -\cos V \\ z_u &= -\sin U \qquad z_v = \sin V. \end{aligned}$$

Writing $H(u, v)=(h(y(u, v), z(u, v)), y(u, v), z(u, v))$, we get

$$H_u=(-h_y \cos U - h_z \sin U,\ -\cos U,\ -\sin U) \quad \text{and}$$

$$H_v=(-h_y \cos V + h_z \sin V,\ -\cos V,\ \sin V).$$

As before we have

Proposition 1′. *$\{u, v\}$ is a global null cordinate system on $\boldsymbol{R}^2$ with the metric induced by H.*

Theorem 1′. *Every timelike graph over a spacelike plane in $\boldsymbol{R}^3_1$ with zero mean curvature is a global translation surface.*

If we write $H(u, v)=(\gamma_1(u)+\delta_1(v), \gamma_2(u)+\delta_2(v), \gamma_3(u)+\delta_3(v))$, then $(\gamma_1')^2 = (\gamma_2')^2+(\gamma_3')^2$ implies that $(\gamma_1')^2=\cos^2 U+\sin^2 U=1$. Similarly $(\delta_1')^2=1$. Thus $h(u, v)= \pm u \pm v+\text{constant}$. In this case then the first function is a linear function. This is perhaps not so surprising if we realize that we have chosen two null vectors H_u and H_v which happen to lie in the null cone at the circle of height ± 1. We again have

Theorem 2′. *If $H: \boldsymbol{R}^2 \rightarrow \boldsymbol{R}^3_1$ defines a timelike graph over a spacelike plane with zero mean curvature then there are coordinates $\{u, v\}$ in $\boldsymbol{R}^2$ and functions $U(u)$ and $V(v)$ so that*

$$\begin{aligned} H_u &= (\pm 1,\ -\cos U,\ -\sin U) \\ H_v &= (\pm 1,\ -\cos V,\ \sin V). \end{aligned}$$

4. Necessary and sufficient conditions to generate entire surfaces with zero mean curvature.

In order to obtain explicit, or at least computable, examples it is necessary to know which function $U(u)$ and $V(v)$ generate surfaces with zero mean curvature. We first give the result for the case of a graph over a timelike plane.

Proposition 2. *Given non-constant functions $U(u)$ and $V(v): \boldsymbol{R}^2 \to \boldsymbol{R}$ with $0<U+V<\pi$ there exist functions p and $q: \boldsymbol{R}^2 \to \boldsymbol{R}$ satisfying*

(i) $1+q^2-p^2>0$

(ii) $\dfrac{p^2-1}{\sqrt{1+q^2-p^2}}=\dfrac{2\sin U\sin V}{\sin(U+V)}$

(iii) $\dfrac{q^2+1}{\sqrt{1+q^2-p^2}}=\dfrac{-2\cos U\cos V}{\sin(U+V)}$

iff (1) $0>\cos U\cos V$

(2) $0\geqq(\sin U+\cos U)(\sin V+\cos V)$

(3) $\cos(2U)\geqq 0$ and $\cos(2V)\geqq 0$.

We need a lemma before beginning the proof of this proposition.

Lemma 1. *If* $0\geqq(\sin U+\cos U)(\sin V+\cos V)$ *and* $0<U+V<\pi$ *then* $\cos(2U)\cos(2V)\geqq 0$.

Proof of Lemma 1. Condition (2) is equivalent to $-\sin U\sin V-\cos U\cos V \geqq \sin U\cos V+\cos U\sin V$. Because $\sin(U+V)>0$ condition (2) then implies

$$(\sin U\sin V+\cos U\cos V)^2\geqq(\sin U\cos V+\cos U\sin V)^2$$

or

$$\cos^2U\cos^2V+\sin^2U\sin^2V-\sin^2U\cos^2V-\cos^2U\sin^2V\geqq 0$$

or

$$\cos(2U)\cos(2V)\geqq 0. \qquad \text{Q.E.D.}$$

Proof of Proposition 2. We will use the abbreviations

$$\varphi_{xx} \text{ for } \frac{2\sin U\sin V}{\sin(U+V)} \quad \text{and} \quad \varphi_{yy} \text{ for } \frac{-2\cos U\cos V}{\sin(U+V)}.$$

Assume that we have p and q satisfying (i), (ii) and (iii). Let $W=(1+q^2-p^2)^{1/2}$ be the positive square root. Then $\cos U\cos V<0$ because $\dfrac{-(q^2+1)}{W}\dfrac{\sin(U+V)}{2}<0$. W is a positive real number satisfying $\varphi_{yy}-\varphi_{xx}=\dfrac{q^2+1-p^2+1}{W}=\dfrac{W^2+1}{W}$ or

$$W^2+W(\varphi_{xx}-\varphi_{yy})+1=0. \tag{15}$$

From the quadratic formula we see that the solution W is positive iff $\varphi_{xx}-\varphi_{yy}\leqq -2$. Thus we have

$$\frac{2\sin U\sin V+2\cos U\cos V}{\sin(U+V)}\leqq -2$$

or $2\sin U\sin V+2\cos U\cos V+2\sin U\cos V+2\cos U\sin V\leqq 0$, which is (2).

To see that (3) holds we note that $q^2\geqq 0$ implies $\varphi_{yy}W\geqq 1$. Solving for W

in (15) and substituting in $\varphi_{yy}W\geqq 1$ gives

$$\frac{-2\cos U\cos V}{\sin(U+V)}\left[\frac{-(\cos U\cos V+\sin U\sin V)+\sigma\sqrt{\cos(2U)\cos(2V)}}{\sin(U+V)}\right]\geqq 1,$$

for $\sigma=\pm 1$. Simplifying, this becomes

$$2\cos^2 U\cos^2 V+2\cos U\sin U\cos V\sin V-2\sigma\cos U\cos V\sqrt{\cos(2U)\cos(2V)}$$
$$\geqq(\sin U\cos V+\cos U\sin V)^2$$

or

$$2\cos^2 U\cos^2 V-\sin^2 U\cos^2 V-\cos^2 U\sin^2 V\geqq 2\sigma\cos U\cos V\sqrt{\cos(2U)\cos(2V)}.$$

This reduces to

$$(*)\qquad \cos^2 U\cos(2V)+\cos^2 V\cos(2U)\geqq 2\sigma\cos U\cos V\sqrt{\cos(2U)\cos(2V)}.$$

By Lemma 1 $\cos(2U)$ and $\cos(2V)$ have the same sign. If both were negative then $2\sigma\cos U\cos V\leqq 0$. By squaring both sides of $(*)$ we get

$$\cos^4 U\cos^2(2V)+\cos^4 V\cos^2(2U)+2\cos^2 U\cos^2 V\cos(2U)\cos(2V)$$
$$\leqq 4\cos^2 U\cos^2 V\cos(2U)\cos(2V),$$

which implies that

$$(\cos^2 U\cos(2V)-\cos^2 V\cos(2U))^2=0.$$

This yields $\sin^2 U=\sin^2 V$. Since U and V are pure functions of u and v this cannot occur unless both are constant. Suppose (1), (2) and (3) hold. We can set

$$W=\left[\frac{-(\cos(U-V))+\sigma\sqrt{\cos(2U)\cos(2V)}}{\sin(U+V)}\right],\ \text{for } \sigma=\pm 1.$$

W is a positive solution to (15). In order to define p and q we must have $\varphi_{xx}W+1\geqq 0$ and $\varphi_{yy}W-1\geqq 0$. If this were so then we could set $p=\pm(\varphi_{xx}W+1)^{1/2}$ and $q=\pm(\varphi_{yy}W-1)^{1/2}$.

$\varphi_{xx}W+1\geqq 0$ can be rewritten

$$\frac{2\sin U\sin V}{\sin(U+V)}\left[\frac{-(\cos U\cos V+\sin U\sin V)+\sigma\sqrt{\cos(2U)\cos(2V)}}{\sin(U+V)}\right]\geqq -1$$

or

$$-2\sin^2 U\sin^2 V+2\sigma\sin U\sin V\sqrt{\cos(2U)\cos(2V)}\geqq -\sin^2 U\cos^2 V-\cos^2 U\sin^2 V.$$

Equivalently

$$\sin^2 U\cos(2V)+\sin^2 V\cos(2U)\geqq -2\sigma\sin U\sin V\sqrt{\cos(2U)\cos(2V)}$$

This holds because $(\sin U\sqrt{\cos(2V)}+\sigma\sin V\sqrt{\cos(2U)})^2\geqq 0$.

In the same way $\varphi_{yy}\geqq 1$ is equivalent to the valid inequality

$$\cos^2 U\cos(2V)+\cos^2 V\cos(2U)-2\sigma\cos U\cos V\sqrt{\cos(2U)\cos(2V)}\geqq 0.$$

Finally if $p^2=\varphi_{xx}W+1$ and $q^2=\varphi_{yy}W-1$ then $1+q^2-p^2=(\varphi_{yy}-\varphi_{xx})W-1=W^2>0$. Q.E.D.

From $p^2=\varphi_{xx}W+1$ and $q^2=\varphi_{yy}W-1$ we obtain:

$$p=\pm\left(\frac{\sin U\sqrt{\cos(2V)}+\sigma\sin V\sqrt{\cos(2U)}}{\sin(U+V)}\right) \text{ and}$$

$$q=-\pm\left(\frac{\sigma\cos(V)\sqrt{\cos(2U)}-\cos U\sqrt{\cos(2V)}}{\sin(U+V)}\right).$$

In order to check that $p=f_x$ and $q=f_y$ for some $f: \boldsymbol{R}^2\to\boldsymbol{R}$, we must of course check that $p_y=q_x$, which is true.

In the case of a graph over a spacelike plane there are no conditions on U and V beyond $\pi>U+V>0$.

Proposition 2′. *Given functions U and V with $\pi>U+V>0$ there exist functions p and q satisfying*

(i′) $p^2+q^2-1>0$

(ii′) $\dfrac{1-p^2}{\sqrt{p^2+q^2+1}}=\dfrac{2\sin U\sin V}{\sin(U+V)}$

(iii′) $\dfrac{1-q^2}{\sqrt{p^2+q^2+1}}=\dfrac{-2\cos U\cos V}{\sin(U+V)}$.

Proof. Again we use the abbreviations

$$\phi_{yy} \text{ for } \frac{2\sin U\sin V}{\sin(U+V)} \quad\text{and}\quad \phi_{zz} \text{ for } \frac{-2\cos U\cos V}{\sin(U+V)}.$$

Set $W=\dfrac{-(\phi_{yy}+\phi_{zz})+\sqrt{(\phi_{yy}+\phi_{zz})^2+4}}{2}$. This is the only positive solution of:

$$W^2+W(\phi_{yy}+\phi_{zz})-1=0. \tag{15′}$$

We need to show that $1-W\phi_{yy}\geqq 0$ and $1-W\phi_{zz}\geqq 0$, so that p and q can be defined, and that (i′) holds. $1\geqq W\phi_{yy}$ iff

$$1\geqq\frac{2\sin U\sin V}{\sin(U+V)}\left[\frac{-(\sin U\sin V-\cos U\cos V)+1}{\sin(U+V)}\right].$$

Equivalently

$$\sin^2 U\cos^2 V+\cos^2 U\sin^2 V\geqq -2\sin^2 U\sin^2 V+2\sin V\sin U$$

or $\sin^2 U+\sin^2 V-2\sin U\sin V\geqq 0$, which always holds.

In the same way $1\geqq\phi_{zz}W$ iff

$$1\geqq\frac{-2\cos U\cos V}{\sin^2(U+V)}[-\sin U\sin V+\cos U\cos V+1]$$

iff

$$\sin^2 U\cos^2 V+\cos^2 U\sin^2 V\geqq -2\cos^2 U\cos^2 V-2\cos U\cos V$$

iff

$$\cos^2 V+\cos^2 U+2\cos U\cos V\geqq 0.$$

Once p and q are defined $p^2+q^2-1=1-W\phi_{yy}+1-W\phi_{zz}-1=1-W(\phi_{yy}+\phi_{zz})$ $=W^2>0$. Q.E.D.

Here we must verify that $p_z=q_y$, using the formulas:

$$p=\pm\left(\frac{\sin U-\sin V}{\sin(U+V)}\right)\quad\text{and}\quad q=-\pm\left(\frac{\cos U+\cos V}{\sin(U+V)}\right).$$

5. Sectional curvature of global Lorentzian graphs with zero mean curvature.

Using the representation of Theorem 2 and 2′ we can calculate the sectional curvature of global Lorentzian graphs in terms of U and V.

We first consider the graph over a timelike plane. We let $\sigma=\pm1$ and $\tau=\pm1$. A unit normal vector is in the direction of the Lorentzian cross product $N=(-\cos U,\ -\sin U,\ \sigma\sqrt{\cos(2U)})\times(-\cos V,\ \sin V,\ \tau\sqrt{\cos(2V)})=(\tau\sin U\sqrt{\cos(2V)}$ $+\sigma\sin V\sqrt{\cos(2U)},\ \tau\cos U\sqrt{\cos(2V)}-\sigma\cos V\sqrt{\cos(2U)},\ -\sin(U+V))$. $g(N, N)=$ $(\tau\cos(U-V)-\sigma\sqrt{\cos(2U)\cos(2V)})^2$, so the unit normal vector is

$$\xi=\frac{N}{\tau\cos(U-V)-\sigma\sqrt{\cos(2U)\cos(2V)}}.$$

With respect to the null basis $\{\partial/\partial u, \partial/\partial v\}$ the shape operator $A_\xi=\begin{bmatrix}0 & b\\ c & 0\end{bmatrix}$, so that the sectional curvature

$$\begin{aligned}\kappa=-bc&=-g\left(D_{\partial/\partial u}\frac{\partial}{\partial u},\ \xi\right)g\left(D_{\partial/\partial v}\frac{\partial}{\partial v},\ \xi\right)\\ &=\frac{-1}{g(N, N)}g\left(D_{\partial/\partial u}\frac{\partial}{\partial u},\ N\right)g\left(D_{\partial/\partial v}\frac{\partial}{\partial v},\ N\right).\end{aligned}$$

Now

$$D_{\partial/\partial u}\frac{\partial}{\partial u}=U'\left(\sin U,\ -\cos U,\ \frac{-\sigma\sin(2U)}{\sqrt{\cos(2U)}}\right)$$

and

$$D_{\partial/\partial v}\frac{\partial}{\partial v}=V^{\cdot}\left(\sin V,\ \cos V,\ \frac{-\tau\sin(2V)}{\sqrt{\cos(2V)}}\right).$$

In the end we get

$$\kappa=\frac{\sigma\tau U'V^{\cdot}}{\sqrt{\cos(2U)\cos(2V)}}.$$

The sign of κ depends on $U'V^{\cdot}$ and can be chosen to be whatever one wishes.

If we consider the graph $H(y, z)$ and assume that $H_u=(\sigma, -\cos U, -\sin U)$ and $H_v=(\tau, -\cos V, \sin V)$, then the normal vector

$$\xi=\frac{1}{\tau-\sigma\cos(U+V)}(\sin(U+V),\ -\tau\sin U-\sigma\sin V,\ -\sigma\cos V+\tau\cos U).$$

The shape operator is

$$\begin{pmatrix} 0 & \dfrac{-\tau V^{\cdot}}{\tau-\sigma\cos(U+V)} \\ \dfrac{-\sigma U'}{\tau-\sigma\cos(U+V)} & 0 \end{pmatrix}.$$

Thus the sectional curvature is

$$\frac{-\sigma\tau U'V^{\cdot}}{(\tau-\sigma\cos(U+V))^2}.$$

6. A new example.

The only obstacle to calculating new examples is to find functions U and V so that F_u, F_v or H_u, H_v can be explicitly integrated. We find an $H(u, v)$.

Example. Let $U=\arccos\left(\frac{e^u}{1+e^u}\right)$ and $V=\arccos\left(\frac{e^v}{1+e^v}\right)$. Since $1>\left(\frac{e^x}{1+e^x}\right)>0$ we have $\frac{\pi}{2}>U, V>0$.

$$H_u=\left(\sigma,\ \frac{-e^u}{1+e^u},\ -\frac{\sqrt{1+2e^u}}{1+e^u}\right)\quad\text{and}\quad H_v=\left(\tau,\ \frac{-e^v}{1+e^v},\ \frac{\sqrt{1+2e^v}}{1+e^v}\right).$$

$$H(u, v)=(\sigma u+\tau v,\ -\ln((1+e^u)(1+e^v)),$$
$$\ln\left(\frac{(\sqrt{1+2e^v}-1)(\sqrt{1+2e^u}+1)}{(\sqrt{1+2e^v}+1)(\sqrt{1+2e^u}-1)}\right)+2\arctan(\sqrt{1+2e^v})-2\arctan(\sqrt{1+2e^u}).$$

Note. After this work was completed I learned that Professor T. K. Milnor obtained many of these results, as well as others, using quite different methods.

These are contained in a preprint entitled: "Entire timelike minimal surfaces in E_1^3".

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Department of Mathematics
Wellesley College
Wellesley, MA 02181 USA