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LIMITING BEHAVIOR OF GENERALIZED QUADRATIC FORMS GENERATED BY ABSOLUTELY REGULAR SEQUENCES II

By

KEN-ICHI YOSHIHARA

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Summary. As a continuation of the previous result, sufficient conditions for the asymptotic normality of various "triangular" scheme of degenerate U-statistics generated by absolutely regular sequence are considered. The results contain extensions of some known results. As a special case, the *-mixing sequence cases are also considered by proving a new inequality. Some application are also considered.

1. Introduction. This paper is a continuation of [9]. Let $\{\xi_n, -\infty<n<\infty\}$ be a nonstationary sequence of random variables defined on a probability space $(\Omega, \mathcal{A}, P)$. Let F_n be the distribution function (df) of ξ_n. Let $w_{i,j,n}(\cdot, \cdot)$ be Borel functions such that

(1.1) $$\sup_{i,j,n} E|w_{i,j,n}(\xi_i, \xi_j)|^{2+\delta}<\infty$$

for some δ ($\geqq 0$). Put

(1.2) $$W_{i,j,n}(x, y)=w_{i,j,n}(x, y)+w_{j,i,n}(y, x)$$

and

(1.3) $$W_{i,j,n}=W_{i,j,n}(\xi_i, \xi_j).$$

Define

(1.4) $$W(n)=\sum_{1\leq i<j\leq n} W_{i,j,n}.$$

Assumption (A). For all i, j ($1\leqq i<j\leqq n$; $n\geqq 1$)

(1.5) $$\int W_{i,j,n}(x, y)dF_i(x)=0, \quad \text{and}$$

(1.6) $$\sigma_{i,j}^2=\iint W_{i,j,n}^2(x, y)dF_i(x)dF_j(y)<\infty.$$

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Let $\{\eta_n\}$ and $\{\zeta_n\}$ be two absolutely regular sequence of random variables such that $\{\eta_n\}$ and $\{\zeta_n\}$ are independent and have the same mixing coefficient as that of $\{\xi_n\}$ and for each n $\eta_n \overset{d}{=} \xi_n$ and $\zeta_n \overset{d}{=} \xi_n$. Denote

$$\hat{H}(A, B, C) = \sum_{j=B}^{C} \sum_{i=1}^{A} W_{i,j,n}(\eta_i, \zeta_j). \tag{1.7}$$

The following theorem was proved in [9].

Theorem A. *Let $\{\xi_i\}$ be an absolutely regular nonstationary sequence of random variables. Suppose that* (1.1), (1.6) *and Assumption* (*A*) *hold. Suppose that there exists a δ $(0<\delta<1)$ for which*

$$\|W(n)\|_2^{-2} \max\{\max_{1\le i<j\le n} \|W_{i,j,n}\|_2^2, \max_{1\le i<j\le n} \|W_{i,j,n}\|_{2+\delta}^2\} = O(n^{-2}), \tag{1.8}$$

$$\|W(n)\|_2^{-2} \sum_{1\le i<j\le n} \|W_{i,j,n}\|_{1+\delta}^2 = o(n^{-2}) \tag{1.9}$$

and

$$\beta(n) = O(n^{-8(2+\delta)/\delta}). \tag{1.10}$$

Suppose further that for arbitrary nondecreasing seqnences $\{A_n\}$, $\{B_n\}$ and $\{C_n\}$ of integers such that $1\le A_n \le B_n \le C_n \le n$ the sequence $\{\hat{H}^2(A_n, B_n, C_n)/E\hat{H}^2(A_n, B_n, C_n): n\ge 1\}$ is uniformly integrable.

Then, the central limit theorem holds, i.e., as $n\to\infty$

$$W(n)/\|W(n)\|_2 \xrightarrow{D} N(0, 1). \tag{1.11}$$

Here, we write $\|X\|_r = \{E|X|^r\}^{1/r}$ $(r\ge 1)$ if the r-th absolute moment of X is finite.

Remark. In the description of Theorem in [9] and its proof there are some minor errors. (i) The conclusion of Lemma 3.3 in [9] is not correct. The inequality (3.8) should be as follows:

$$E\left|\sum_{j=a+q+1}^{a+q+m} \sum_{i=j-q}^{j-1} W_{i,j,n}\right|^2 \le cmq \max\{M_n(0), M_n(\delta)\} + cm^2\rho_n \tag{1.12}$$

where

$$\rho_n = \max_{1\le i<j\le n} \|W_{i,j,n}\|_{1+\delta}^2. \tag{1.13}$$

(1.12) is easily obtained from Lemma 3.1 in [9] and the fact that by virtue of (1.10)

$$\sum_{1\le j<j'\le m} \sum_{i=j-q+1}^{j-1} |EW_{i,j,n}| \cdot \sum_{i'=j'-q+1}^{j'-1} |EW_{i',j',n}| \le cm^2\rho_n^2.$$

(ii) Hence, (4.3) in [9] should be as follows:

(1.14) $$|EV_k^{(1)}V_{k'}^{(1)}|$$

$$\leqq \sum_{j=(k-1)(p+q)+q+1}^{k(p+q)} \sum_{i=j-q+1}^{j-1} \sum_{j'=(k'-1)(p+q)+q+1}^{k'(p+q)} \sum_{i'=j'-q+1}^{j'-1} EW_{ij}W_{i'j'}|$$

$$+\sum_{j'=(k'-1)(p+q)+1}^{k'(p+q)} \Big\{ \sum_{i'=(k'-1)(p+q)+1}^{j'-1} \sum_{j=(k-1)(p+q)+q+1}^{k(p+q)} \sum_{i=j-q+1}^{j-1} |EW_{i,j}W_{i',j'}|$$

$$+\sum_{i'=(k'-1)(p+q)+1}^{j'-1} \sum_{j=(k-1)(p+q)+q+1}^{k(p+q)} \sum_{i=(k-1)(p+q)+1}^{j-q} |EW_{i,j}W_{i',j'}| \Big\}$$

$$\leqq cq(p+q)\{M_n(0)+M_n(\delta)\}+c(p+q)^2\rho_n$$

$$+c(p+q)^4\beta^{\delta/(2+\delta)}(q)M_n(\delta).$$

In this paper, we consider limit theorems on some degenerate U-statistics. The results obtained correspond to the results in [4] and extend the results in [3].

2. Limit theorems on degenerate U-statistics. Let $\{\xi_i\}$ be a strictly stationary absolutely regular sequence of random variables with mixing coefficient $\beta(n)$. Let $F(x)$ be the df of ξ_1. We will study the "triangular" scheme of U-statistics defined by

(2.1) $$U(n)=\sum_{1\leq i<j\leq n} f_n(\xi_i, \xi_j)$$

where $f_n(\cdot, \cdot)$ are symmetric Borel functions. Define

$$\mu_n=\int f_n(x, y)dF(x)dF(y),$$

(2.2) $$g_n(x)=Ef_n(x, \xi_1)-\mu_n,$$

$$W_n(x, y)=f_n(x, y)-g_n(x)-g_n(y)-\mu_n.$$

Then, it is obvious that $W_n(x, y)$ is symmetric and

(2.3) $$Eg_n(\xi_1)=0=EW_n(x, \xi_1)=EW_n(\xi_1, y).$$

Put

(2.4) $$V(n)=\sum_{i=1}^{n} g_n(\xi_i), \quad \text{and}$$
$$W(n)=\sum_{1\leq i<j\leq n} W_n(\xi_i, \xi_j).$$

Then, we can rewrite $U(n)$ as

$$U(n)=\sum_{1\leq i<j\leq n}\{W_n(\xi_i,\ \xi_j)+g_n(\xi_i)+g_n(\xi_j)+\mu_n\}$$

(2.5)

$$=W(n)+(n-1)V(n)+\binom{n}{2}\mu_n.$$

We put

$$M_{1n}(\delta)=\max\{\|g_n(\xi_1)\|_2^2,\ \|g_n(\xi_1)\|_{2+\delta}^2\},\quad \text{and}$$

(2.6)

$$M_{2n}(\delta)=\max_{2\leq j\leq n}[\max\{\|W_n(\xi_1,\ \xi_j)\|_2^2,\ \|W_n(\xi_1,\ \xi_j)\|_{2+\delta}^2\}]$$

where δ is some nonnegative number. Let

$$\sigma_1^2(n)=\left\|U(n)-\binom{n}{2}\mu_n\right\|_2^2. \tag{2.7}$$

Theorem 1. *Let* $\{\xi_i\}$ *be an absolutely regular strictly stationary sequence of random variables with mixing coefficient* $\beta(n)$. *Suppose there exists a* δ $(0<\delta<1)$ *such that*

(i) $\sigma_1^{-2}(n)\max\{nM_{1n}(\delta),\ M_{2n}(\delta)\}=O(n^{-2})$ *and*

(ii) $\sigma_1^{-2}(n)\max_{2\leq j\leq n}\|W_n(\xi_1,\ \xi_j)\|_{1+\rho}^2=o(n^{-2})$ *for some* ρ $(\delta\leq\rho<1)$ *and*

(iii) $\beta(n)=O(n^{-8(2+\delta)/\delta})$.

Suppose further that as $n\to\infty$

(iv) $n^2EV^2(n)/\sigma_1^2(n)\to\gamma_1^2$ $(0\leq\gamma_1\leq 1)$,

(v) $EW^2(n)/\sigma_1^2(n)\to\gamma_2^2$ $(0\leq\gamma_2\leq 1)$,

(vi) *the sequence* $\left\{\left(\sum_{i=1}^n g_n(\xi_i)\right)^2\Big/E\left(\sum_{i=1}^n g_n(\xi_i)\right)^2:\ n\geq 1\right\}$ *is uniformly integrable, and*

(vii) *for arbitrary nondecreasing sequences* $\{A_n\}$, $\{B_n\}$, $\{C_n\}$ *of integers such that* $1\leq A_n\leq B_n\leq C_n\leq n$ *the sequence* $\{\hat{H}^2(A_n,\ B_n,\ C_n)/E\hat{H}^2(A_n,\ B_n,\ C_n):\ n\geq 1\}$ *is uniformly integrable, where* $\hat{H}(\cdot,\ \cdot,\ \cdot)$ *is the one defined by* (1.7). *Then, as* $n\to\infty$

$$\sigma_1^{-1}(n)\begin{pmatrix} nV(n) \\ W(n)\end{pmatrix}\xrightarrow{D}N\left(\begin{pmatrix}0\\0\end{pmatrix},\ \begin{pmatrix}\gamma_1^2 & 0\\ 0 & \gamma_2^2\end{pmatrix}\right). \tag{2.8}$$

Remark. It is obvious that $\gamma_1^2+\gamma_2^2=1$.

The following corollary is easily obtained from Theorem 1.

Corollary. *Suppose conditions of Theorem* 1 *hold. Then*

$$\left\{U(n)-\binom{n}{2}\mu_n\right\}\Big/\sigma_1(n)\xrightarrow{D}N(0,\ 1). \tag{2.9}$$

To prove Theorem 1, we consider the following lemma.

Lemma 1. *Suppose conditions of Theorem* 1 *hold. Then, as* $n\to\infty$

$$EV^2(n)=n\|g_n(\xi_1)\|_2^2+2\sum_{1\le i<j\le n}Eg_n(\xi_i)g_n(\xi_j)=O(nM_{1n}(\delta))$$

(2.10) $$EW^2(n)=\sum_{1\le i<j\le n}\|W_n(\xi_i,\ \xi_j)\|_2^2$$

$$+2\sum_{\substack{1\le i<j\le n\\|i-i'|+|j-j'|\ge 1}}\sum_{1\le i'<j'\le n}EW_n(\xi_i,\ \xi_j)W_n(\xi_{i'},\ \xi_{j'})=O(n^2M_{2n}(\delta))$$

$$|EV(n)W(n)|\leqq\sum_{i=1}^{n}\sum_{1\le j<k\le n}|Eg_n(\xi_i)W_n(\xi_j,\ \xi_k)|=O(nM_{1n}^{1/2}(\delta)M_{2n}^{1/2}(\delta)).$$

Proof. The first two relations in (2.10) are obtained by the method in [7] and so are omitted. To prove the last relation let $J(i;\ j,\ k)=Eg_n(\xi_i)\ W_n(\xi_j,\ \xi_k)$. If $|j-i|\geqq\max\{|k-i|,\ k-j\}$, then by (2.3)

$$|J(i;\ j,\ k)|\leqq c\|g_n(\xi_1)\|_2\|W_n(\xi_j,\ \xi_k)\|_{2+\delta}\beta^{\delta/2(2+\delta)}(|j-i|).$$

If $|k-i|\geqq\max\{|j-i|,\ k-j\}$, then by (2.3) again

$$|J(i;\ j,\ k)|\leqq c\|g_n(\xi_1)\|_2\|W_n(\xi_j,\ \xi_k)\|_{2+\delta}\beta^{\delta/(2+\delta)}(|k-i|).$$

If $k-j\geqq\max\{|k-i|,\ |j-i|\}$, then by (2.3), Lemma 2 in [7] and the Schwarz inequality

$$\begin{aligned}|J(i;\ j,\ k)|&\leqq c\|g_n(\xi_i)W_n(\xi_j,\ \xi_k)\|_{1+\delta/2}\beta^{\delta/(2+\delta)}(k-j)\\&\leqq c\|g_n(\xi_1)\|_{2+\delta}\|W_n(\xi_j,\ \xi_k)\|_{2+\delta}\beta^{\delta/(2+\delta)}(k-j)\\&\leqq cM_{1n}^{1/2}(\delta)M_{2n}^{1/2}(\delta)\beta^{2/(2+\delta)}(k-j).\end{aligned}$$

Hence, using the fact that $\|g_n(\xi_1)\|_2\leqq\|g_n(\xi_1)\|_{2+\delta}$ for $\delta>0$, we obtain

$$\begin{aligned}&|EV(n)W(n)|\\&\leqq\Big\{\sum_{\substack{i=1\\|j-i|\ge\max(k-j,\ |k-i|)}}^{n}\sum_{1\le j<k\le n}+\sum_{\substack{i=1\\|k-i|\ge\max(|j-i|,\ k-j)}}^{n}\sum_{1\le j<k\le n}\ \sum_{\substack{i=1\\k-j\ge\max(|k-i|,\ |j-i|)}}^{n}\sum_{1\le j<k\le n}\Big\}J(i;\ j,\ k)\\&\leqq c\Big\{n\sum_{l=1}^{n}l\beta^{\delta/2(2+\delta)}(l)\Big\}M_{1n}^{1/2}(\delta)M_{2n}^{1/2}(\delta)=O(nM_{1n}^{1/2}(\delta)M_{2n}^{1/2}(\delta)),\end{aligned}$$

which completes the proof.

Proof of Theorem 1. We note first that by (i), (ii), (iii), (v), Lemma 1 and Theorem 1 in [7]

$$nV(n)/\sigma_1(n)\xrightarrow{D}N(0,\ \gamma_1^2)$$

as $n\to\infty$. Further, by (i), (ii), (iv), (vi), Lemma 1 and Theorem A, we see that as $n\to\infty$

$$W(n)/\sigma_1(n)\xrightarrow{D}N(0,\ \gamma_2^2).$$

Hence, it remains to show that $nV(n)/\sigma_1(n)$ and $W(n)/\sigma_1(n)$ are asymptotically uncorrelated. But the fact is easily shown by (i) and the last relation in (2.10).

3. The *-mixing case. We say that $\{\xi_n\}$ is $*$-mixing if

$$\psi(n)=\sup_k \sup_{A\in\mathcal{M}_0^k,\ B\in\mathcal{M}_{k+n}^\infty}\left|\frac{P(AB)-P(A)P(B)}{P(A)P(B)}\right|\longrightarrow 0, \tag{3.1}$$

where, as before, $\mathcal{M}_a^b$ denotes the σ-algebra generated by $\xi_a, \cdots, \xi_b$. In this case we can prove the analogous result to Lemma 2 in [7].

We consider the following lemma, which is new.

Lemma 2. *Let $\{\xi_n\}$ be $*$-mixing. Let η be an $\mathcal{M}_0^k$-measurable random variable taking values in a measurable space $(Y, \mathfrak{F}_Y)$ and ζ an $\mathcal{M}_{k+n}^\infty$-measurable random variables in a measurable space $(Z, \mathfrak{F}_Z)$. Let $g(y, z)$ $(y\in Y, z\in Z)$ be a Borel measurable function such that*

$$M=\iint|g(y, z)|\,dF_\eta(y)dF_\zeta(z)<\infty \tag{3.2}$$

where F_η and F_ζ are distribution functions of η and ζ, respectively. Then

$$\Delta=E|E\{g(\eta, \zeta)|\eta\}-G(\eta)|\leq M\psi(n) \tag{3.3}$$

where

$$G(y)=Eg(y, \zeta). \tag{3.4}$$

Proof. Since we can approximate η and ζ by simple random variables, so it is enough to prove that (3.3) holds for random variables of the form

$$\eta=\sum_{i=1}^m a_i I_{A_i} \qquad (a_i\in Y,\ i=1, \cdots, m) \tag{3.5}$$

and

$$\zeta=\sum_{i=1}^n b_j I_{B_j} \qquad (b_j\in Z,\ j=1, \cdots, n) \tag{3.6}$$

where $\{A_i\}$ is an $\mathcal{M}_0^k$-measurable finite partition of Ω, $\{B_j\}$ is an $\mathcal{M}_{k+n}^\infty$-measurable finite partition of Ω and $P(A_iB_j)>0$ $(i=1, \cdots, m;\ j=1, \cdots, n)$.

We note that

$$\begin{aligned}\Delta&\leq\sum_{i=1}^m\sum_{j=1}^n|g(a_i, b_j)|\,|P(A_iB_j)-P(A_i)P(B_j)|\\&\leq\sum_{i=1}^m\sum_{j=1}^n|g(a_i, b_j)|P(A_i)P(B_j)\left|\frac{P(A_iB_j)-P(A_i)P(B_j)}{P(A_i)P(B_j)}\right|.\end{aligned}$$

Since $\{\xi_n\}$ is $*$-mixing, so

$$\left|\frac{P(A_iB_j)-P(A_i)P(B_j)}{P(A_i)P(B_j)}\right| \leqq \psi(n).$$

Hence, we have

$$\Delta \leqq \psi(l) \sum_{i=1}^{m} \sum_{j=1}^{n} |g(a_i, b_j)| P(A_i)P(B_j) = M\psi(l). \qquad \square$$

The next corollary is a special case of Lemma 2.

Corollary. *Let ξ be an $\mathcal{M}_a^k$-measurable random variable and η an $\mathcal{M}_{k+n}^b$-measurable random variable. Suppose that $E|\eta|<\infty$ and $E|\zeta|<\infty$. Then*

$$|\operatorname{cov}(\eta, \zeta)| = |E\eta\zeta - E\eta E\zeta| \leqq \psi(n) E|\eta| E|\zeta| . \tag{3.7}$$

By Lemma 2 and the methods in the preceeding sections we have the following results.

(I) Firstly, we consider the limit distribution of $W(n)/\|W(n)\|_2$ where $W(n)$ is the one defined by (1.4). Put

$$L_n = \max_{1 \leq i < j \leq n} \left[\max \left\{ \iint |W_{i,j,n}(x, y)|^2 dF_i(x) dF_j(y),\ \|W_{i,j,n}\|_2^2 \right\} \right]. \tag{3.8}$$

and for some ρ $(1/2 \leqq \rho < 1)$

$$L_n(\rho) = \max_{1 \leq i < j \leq n} \left\{ \iint |W_{i,j,n}(x, y)|^{1+\rho} dF_i(x) dF_j(y) \right\}^{2/(1+\rho)} \tag{3.9}$$

Theorem 2. *Let $\{\xi_i\}$ be a nonstationary $*$-mixing sequence with mixing coefficient $\psi(n)$. Suppose that*

(i) $\|W(n)\|_2^{-2} L_n = O(n^{-2})$

(ii) $\|W(n)\|_2^{-2} L_n(\rho) = o(n^{-1})$ *for some* ρ $(1/2 \leqq \rho < 1)$, and

(iii) $\sum_{n=1}^{\infty} n^3 \psi(n) < \infty$.

Suppose further that the rest of conditions in Theorem A are satisfied. Then, the conclusion of Theorem A remains true.

(II) Next, as in Section 2, we consider the triangular scheme of U-statistics generated by a strictly stationary $*$-mixing sequence. We use the notations in Section 2.

Theorem 3. *Let $\{\xi_i\}$ be a strictly stationary $*$-mixing sequence with mixing coefficient $\psi(n)$. Suppose that as $n \to \infty$*

(i) $\sigma_1^{-2}(n) \max\{nM_{1n}(0), M_{2n}(0)\} = O(n^{-2})$,

(ii) $\sigma_1^{-2}(n)\left\{\iint |W_n(x, y)|^{1+\rho} dF(x)dF(y)\right\}^{2/(1+\rho)} = o(n^{-2})$ *for some* ρ $(1/3 \leqq \rho < 1)$, *and*

(iii) $\sum_{n=1}^{\infty} n^3 \phi(n) < \infty$.

Suppose further that (iv)-(vii) *in Theorem* 1 *hold. Then, the conclusion of Theorem* 1 *remains true.*

4. Some applications.

(I) *U*-statistics generated by some strictly stationary sequences. Let $\{\xi_i\}$ be a strictly stationary $*$-mixing sequence of random variables which are defined on a probability space $(\Omega, \mathcal{F}, P)$ and take values in a measurable space $(X, \mathcal{A})$. Let $W_n(x, y)$ be a symmetric function, defined on $X \times X$, such that

$$EW_n(x, \xi_1) = 0 \quad \text{for all} \quad x \in X. \tag{4.1}$$

Define

$$W(n) = \sum_{1 \leq i < j \leq n} W_n(\xi_i, \xi_j). \tag{4.2}$$

The following theorem corresponds to Theorem 1 in [3].

Theorem 4. *Let* $\{\xi_j\}$, $W_n(\cdot, \cdot)$ *and* $W(n)$ *be the ones defined above. Suppose the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} n^3 \phi(n) < \infty$.

(ii) $\|W_n(X_1, X_2)\|_{1+\rho}^2 = o(\|W_n(X_1, X_2)\|_2^2)$ *for some* ρ $(1/2 \leqq \rho < 1)$.

(iii) $\liminf_{n\to\infty} \dfrac{\|W(n)\|_2}{n\|W_n(X_1, X_2)\|_2} > 0$.

(iv) $\limsup_{n\to\infty} \dfrac{\|W_n(X_1, X_2)\|_4}{\|W_n(X_1, X_2)\|_2} < \infty$.

Then $W(n)/\|W(n)\|_2 \xrightarrow{D} N(0, 1)$.

Here, X_1 and X_2 are independent random variables such that $X_1 \stackrel{d}{=} \xi_1$ and $X_2 \stackrel{d}{=} \xi_2$.

Remark. *The analogous conclusion to Theorem* 4 *can be proved by the same method when* $W(n)$ *is constructed by a strictly stationary absolutely regular sequence.*

Proof. We note first that from (4.1), Lemma 2 and the proof of Lemma 2 in [7] we can show that

(4.3) $$\|W(n)\|_2 \leqq cn\|W_n(X_1, X_2)\|_2$$

holds. Therefore, from (4.3) and condition (iii) we have

(4.4) $$\|W(n)\|_2 \sim c_0 n\|W_n(X_1, X_2)\|,$$

which, together with (ii), implies that (ii) in Theorem 2 holds. It is obvious that (i) in Theorem 2 holds. It remains to show the uniform integrability of $\{\hat{H}^2(A_n, B_n, C_n)/E\hat{H}^2(A_n, B_n, C_n): n \geqq 1\}$. Let $\{\eta_i\}$ and $\{\zeta_j\}$ be two strictly stationary $*$-mixing sequences such that $\{\eta_i\}$ and $\{\zeta_j\}$ are independent and have the same mixing coefficient as that of $\{\xi_n\}$ and for each i $\eta_i \overset{d}{=} \xi_1$ and $\zeta_i \overset{d}{=} \xi_1$. Then, using (4.1), Lemma 2 and the proof of Lemma 3 in [7] we have that for arbitrary positive integers p and q

(4.5) $$E\left|\sum_{i=1}^{p}\sum_{j=1}^{q} W_n(\eta_i, \zeta_j)\right|^4 \leqq cp^2q^2\|W_n(X_1, X_2)\|_4^4$$

Hence, the desired conclusion follows from (4.3) and (4.4), and the proof of Theorem 4 is completed. □

(II) A global measure of density estimators. Let $\{\xi_j\}$ be a $*$-mixing strictly stationary sequence of random variables with density f. We assume that f and its derivative are bounded and uniformly continuous on $(-\infty, \infty)$. Let K be a function which satisfies the following conditions:

(i) $K(x)$ is symmetric, bounded, continuous and nonnegative.

(ii) $K(x)=0$ for $|x| \geqq 1$.

(iii) $\int_{-1}^{1} K(x)dx=1$.

As in [3], we consider a nonparametric estimator of f, written in the form

(4.6) $$f_n(x)=\sum_{i=1}^{n} K_n(x, \xi_i)=\sum_{i=1}^{n} \frac{1}{nh} K\left(\frac{x-\xi_i}{h}\right)$$

where $h=h(n)$ is a function of n such that $h(n) \downarrow 0$ and $nh(n) \to \infty$ as $n \to \infty$.

We will study the asymptotic normality of random variables

(4.7) $$\begin{aligned} T_n &= \int \{f_n(x)-Ef_n(x)\}^2 w(x)dx \\ &= 2\sum_{1 \leq i < j \leq n} \int \{K_n(x, \xi_i)-EK_n(x, \xi_i)\}\{K_n(x, \xi_j)-EK_n(x, \xi_j)\} a(x)dx \\ &\quad + \sum_{i=1}^{n} \int \{K_n(x, \xi_i)-EK_n(x, \xi_i)\}^2 a(x)dx, \end{aligned}$$

where $a(x)$ is a bounded function.

Theorem 5. *Let* $\{\xi_i\}$, K_n *and* T_n *be the ones defined above. Suppose*

$$\sum n^3\beta(n)<\infty . \tag{4.8}$$

Then, as $n\to\infty$

$$(T_n-ET_n)/V^{1/2}(T_n)\xrightarrow{D} N(0, 1) \tag{4.9}$$

provided

$$\liminf_{n\to\infty} n^4h^{-4}\left\|\int\sum_{i=1}^{2}\{K_n(x, X_i)-EK_n(x, X_i)\}\,a(x)dx\right\|_2^2>0 . \tag{4.10}$$

Proof. Let

$$H_n(u, v)=\int\{K_n(x, \xi_i)-EK_n(x, \xi_i)\}\{K_n(x, \xi_j)-EK_n(x, \xi_j)\}\,a(x)dx .$$

Then, for each n $H_n(u, v)$ is a bounded symmetric function and satisfies the condition

$$EH_n(u, \xi_1)=0 \qquad \text{for all } u . \tag{4.11}$$

Further, let

$$H_{n,i}=\int\{K_n(x, \xi_i)-EK_n(x, \xi_i)\}^2a(x)dx \quad (1\leqq i\leqq n).$$

Then, $\{H_{n,i}\}$ satisfies the $*$-mixing condition with the same mixing coefficient as that of $\{\xi_i\}$ and each $H_{n,i}$ is bounded and $V(H_{n,i})\leqq cn^{-2}$.

Now, we note that T_n-ET_n may be rewritten as

$$T_n-ET_n=2\sum_{1\leq i<j\leq n}H(\xi_i, \xi_j)+2\sum_{1\leq i<j\leq n}EH(\xi_i, \xi_j)+\sum_{i=1}^{n}(H_{n,i}-EH_{n,i})$$

$$=L_{1,n}+L_{2,n}+L_{3,n}, \text{ (say)}.$$

Therefore, to prove (4.9) it is enough to show that as $n\to\infty$

(i) $L_{1,n}/V^{1/2}(T_n)\xrightarrow{D}N(0, 1)$,

(ii) $L_{2,n}/V^{1/2}(T_n)\to 0$, and

(iii) $L_{3,n}/V^{1/2}(T_n)\xrightarrow{P}0$

hold.

Firstly, we remark that by the previous methods we obtain

$$E|L_{1,n}|^2\sim n^2h^{-4}\|H_n(X_1, X_2)\|_2^2, \tag{4.12}$$

$$|L_{2,n}|\leqq cn^{-2}h^{-2}\|H_n(X_1, X_2)\|_1^2=O(n^{-1}h^2) \tag{4.13}$$

and

$$E\left|\sum_{i=1}^{n}(H_{n,i}-EH_{n,i})\right|^2\sim n^{-3}h^{-2}\|H_{n,1}-EH_{n,1}\|_2^2=O(n^{-3}h). \tag{4.14}$$

From (4.10) and (4.12)-(4.14) we obtain that

$$V(T_n)\sim n^{-2}, \tag{4.15}$$

which, in turn, implies that (ii) and (iii) hold.

Now, we put

$$W_n(u, v)=\frac{n(n-1)}{2}H_n(u, v). \tag{4.16}$$

Then, it is clear that $\{W_n(\xi_i, \xi_j)\}$ satisfies conditions (i)-(iv) in Theorem 4. Hence, (i) is obtained from Theorem 4 and the proof is completed.

(III) Asymptotic normality of some cross-validatory estimator. Let $\{\xi_i\}$ be a strictly stationary $*$-mixing sequence of random variables with probability density $f(x)$ which is unknown. Let $K(x)$ and $f_n(x)$ the ones defined in (II). Put

$$\theta_h=\frac{1}{h}EK\left(\frac{X_1-X_2}{h}\right) \tag{4.17}$$

where X_1 and X_2 are independent random variables each having density f. Hence

$$E\int f_n(x)f(x)dx=E\frac{1}{n(n-1)h}\sum_{i\neq j}\sum K\left(\frac{\xi_i-\xi_j}{h}\right)=\theta_h \tag{4.18}$$

if $\{\xi_i\}$ is a sequence of i. i. d. random variables with probability density function $f(x)$, and in this case

$$S_n=\frac{2}{n(n-1)h}\sum_{i\neq j}\sum K\left(\frac{\xi_i-\xi_j}{h}\right) \tag{4.19}$$

is the unbiased estimate of θ_h.

We consider the asymptotic normality of S_n defined by (4.19) when $\{\xi_i\}$ is $*$-mixing. Put

$$p_h(x)=\frac{1}{h}EK\left(\frac{x-\xi_1}{h}\right) \tag{4.20}$$

and

$$W_n(x, y)=\frac{1}{h}K\left(\frac{x-y}{h}\right)-p_h(x)-p_h(y)+\theta_h. \tag{4.21}$$

Then, $W_n(x, y)$ is a bounded symmetric function and satisfies the equation

$$\int W_n(x, y)f(y)dy=0 \qquad \text{for all } x. \tag{4.22}$$

Further, $\binom{n}{2}$ may be rewritten as

$$\binom{n}{2}S_n=\binom{n}{2}\theta_h+(n-1)V(n)+W(n) \tag{4.23}$$

where

$$V(n)=\sum_{i=1}^{n}(p_h(\xi_i)-\theta_h) \tag{4.24}$$

and

$$W(n)=\sum_{1\le i<j\le n}W_n(\xi_i, \xi_j). \tag{4.25}$$

Since for all r ($\geqq 1$)

$$\max\{E|p_h(X_1)|^r, E|W_n(X_1, X_2)|^r\}\leqq ch, \tag{4.26}$$

so, using the method of the proof of Lemma 1, from Lemma 2 and (4.22) we obtain the following relations:

$$EV^2(n)\sim n\|p_h(X_1)-\theta_h\|_2^2. \tag{4.27}$$

$$EW^2(n)\sim n^2\|W_n(X_1, X_2)\|_2^2. \tag{4.28}$$

$$|EV(n)W(n)|=O(n\|p_h(X_1)-\theta_h\|_2\|W_n(X_1, X_2)\|_2). \tag{4.29}$$

Consequently, we have

$$\sigma_2^2(n)=E|S_n-\theta_h|^2\sim\max\{n\|p_h(X_1)-\theta_h\|_2^2, \|W_n(X_1, X_2)\|_2^2\}. \tag{4.30}$$

Theorem 6. *Let $\{\xi_n\}$ be a *-mixing strictly stationary sequence of random variables. Suppose $nh\to\infty$ as $n\to\infty$. If*

$$\liminf_{n\to\infty}\inf\{n\|p_h(X_1)-\theta_h\|_2^2, \|W_n(X_1, X_2)\|_2^2\}/h>0 \tag{4.31}$$

and

$$\sum n^3\psi(n)<\infty, \tag{4.32}$$

then

$$\frac{1}{\sigma_2(n)}(T_n-\theta_h)\xrightarrow{D}N(0, 1). \tag{4.33}$$

To prove Theorem 6, we need the following two lemmas.

Lemma 3. *Suppose conditions of Theorem* 6 *are satisfied. Then, as $n\to\infty$*

$$EV^4(n)=O(\|V(n)\|_2^4). \tag{4.34}$$

Proof. For brevity, let

$$Y_j = p_h(\xi_j) - \theta_h \quad (j=1, \cdots, n).$$

Then

$$EV^4(n) = \sum_i EY_i^4 + \sum_{i\neq j} EY_i^2Y_j^2 + \sum_{i\neq j} EY_i^3Y_j$$

$$+\sum_{i\neq j\neq k} EY_i^2Y_jY_k + \sum_{i\neq j\neq k\neq l} EY_iY_jY_kY_l.$$

Since $\{Y_j\}$ is a $*$-mixing strictly stationary sequence of bounded zero-mean random variables with the same mixing coefficient as that of $\{\xi_i\}$, so by (4.26), (4.32) and (3.7) we have the following inequalities:

$$\sum_i EY_i^4 = nEY_1^4 \leqq cnh.$$

$$\sum_{i\neq j} EY_i^2Y_j^2 = 2\sum_{1\leq i<j\leq n} EY_i^2Y_j^2$$

$$\leqq 2\sum_{1\leq i<j\leq n}(1+\psi(j-i))EY_i^2EY_j^2 \leqq cn^2h^2.$$

$$\sum_{i\neq j} EY_i^3Y_j \leqq c\sum_{1\leq i<j\leq n}\psi(j-i)E|Y_i|^3E|Y_j| \leqq cnh^2$$

$$\sum_{\substack{1\leq i<j<k\leq n\\ k-j>j-i}} |EY_i^2Y_jK_k| \leqq c\sum_{\substack{1\leq i<j<k\leq n\\ k-j>j-i}}\psi(k-j)E|Y_i^2Y_j|\cdot E|Y_k|$$

$$\leqq cnh^2\sum_{l=1}^{n} l\psi(l) \leqq cnh^2$$

$$\sum_{\substack{1\leq i<j<k\leq n\\ j-i>k-j}} |EY_i^2Y_jY_k| \leqq \sum_{\substack{1\leq i<j<k\leq n\\ k-j>j-i}}\{EY_i^2EY_iY_k + \psi(j-i)EY_i^2\cdot E|Y_jE_k|\}$$

$$\leqq \sum_{\substack{1\leq i<j<k\leq n\\ j-i>k-j}}\psi(k-j)EY_i^2E|Y_j|E|Y_k| + cnh^2$$

$$\leqq cn^2h^2$$

$$\sum_{1\leq i<j<k\leq n} |EY_iY_jY_k^2| \leqq cn^2h^2$$

$$\sum_{1\leq i<j<k\leq n} |EY_iY_j^2Y_k \leqq c\sum_{1\leq i<j<k\leq n}\psi(\max(j-i,\ k-j))h^2 \leqq cnh^2.$$

Finally, by using the method of the proof of Lemma 3 in [7] we have

$$\sum_{i\neq j\neq k\neq l} |EY_iY_jY_kY_l| \leqq cn^2\max_{1\leq i\leq n}\{E|Y_i|^4\} \leqq cn^2h^4.$$

Hence, noting that by (4.31) $EV^2(n)\sim nh$ as $n\to\infty$, we have the desired conclusion. □

Lemma 4. *Let $\{\eta_i\}$ and $\{\zeta_i\}$ be two copies of $\{\xi_i\}$ such that $\{\eta_i\}$ and $\{\zeta_i\}$ are independent. Suppose conditions of Theorem* 6 *hold. Then, for all integers p and q sufficiently large*

$$E\left|\sum_{i=1}^{p}\sum_{j=1}^{q}W_n(\eta_i,\zeta_j)\right|^4=O\left(\left\|\sum_{i=1}^{p}\sum_{j=1}^{q}W_n(\eta_i,\zeta_j)\right\|_2^4\right). \tag{4.35}$$

Proof. We note that for arbitrary fixed $y_1,\cdots,y_p$ the collection $\left\{\sum_{i=1}^{p}W_n(y_i,\zeta_1),\cdots,\sum_{i=1}^{p}W_n(y_i,\zeta_q)\right\}$ constitutes a $*$-mixing sequence of identically distributed bounded zero-mean random variables with the same mixing coefficient as that of $\{\xi_i\}$. Hence, by the above method we have

$$\begin{aligned}E\left|\sum_{i=1}^{p}\sum_{j=1}^{q}W_n(\eta_i,\zeta_j)\right|^4&=E\left[E\left\{\left|\sum_{j=1}^{q}\sum_{i=1}^{p}W_n(\eta_i,\zeta_j)\right|^4\middle|\eta_1,\cdots,\eta_p\right\}\right]\\&\leqq cq^2E\left[E\left\{\left|\sum_{i=1}^{p}W_n(\eta_i,\zeta_1)\right|^4\middle|\eta_1,\cdots,\eta_p\right\}\right]\\&\leqq cq^2E\left|\sum_{i=1}^{p}W_n(\eta_i,\zeta_1)\right|^4.\end{aligned} \tag{4.36}$$

(cf. [9]).

Further, using the method of the proof of Lemma 3 and the fact that $\{\eta_i\}$ and ζ_1 are independent we obtain from (4.26) that for all p sufficiently large

$$\begin{aligned}E\left|\sum_{i=1}^{p}W_n(\eta_i,\zeta_1)\right|^4&=E\left[E\left\{\left|\sum_{i=1}^{p}W_n(\eta_i,\zeta_1)\right|^4\middle|\zeta_1\right\}\right]\\&=O(p^2h^2).\end{aligned} \tag{4.37}$$

Now, (4.35) follows from (4.28), (4.31), (4.36) and (4.37). □

Proof of Theorem 6. It follows from (4.26)-(4.30) that conditions (i)-(iii) in Theorem 3 and conditions (iv) and (v) in Theorem 1 hold. Further, conditions (vi) and (vii) in Theorem 1 follow from Lemmas 3 and 4. Hence, by Theorem 3 we have the desired conclusion. □

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Department of Mathematics
Faculty of Engineering
Yokohama National University
156, Tokiwadai, Hodogaya,
Yokohama, JAPAN