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ON METRIZATION AND NEIGHBORHOOD PROPERTIES

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1. Introduction.

In [N] J. Nagata proves the following neighborhood characterization of metrizable spaces, which contains a large number of metrization theorems as direct consequences:

(1.1) Theorem. *A T_1-topological space X is metrizable iff one can assign a countable neighborhood base (c.n.b.)* $\{U_{xn}|n\in N\}$ *for every point x of X such that for every* $n\in N$ *and each point* $x\in X$ *there exist neighborhoods* S_{xn}^1, S_{xn}^2 *of x satisfying*

a) $y\in S_{xn}^1\Rightarrow S_{yn}^2\subset U_{xn}$,

b) $y\notin U_{xn}\Rightarrow S_{yn}^2\cap S_{xn}^1=\emptyset$.

Moreover, J. Nagata gives in that paper an example showing that neither one of the conditions of (1.1) is by itsely sufficient to insure metrizability.

In [W] S. Willard proves the following elegant slight alteration of (1.1) with simpler conditions:

(1.2) Theorem. *A T_0-space X is metrizable iff each* $x\in X$ *posseses a c.n.b.* $\{U_{xn}/n\in N\}$ *with the following properties:*

a) $y\in U_{xn}\Rightarrow U_{yn}\subset U_{xn-1}$,

b) $y\notin U_{xn-1}\Rightarrow U_{yn}\cap U_{xn}=\emptyset$.

Results relating metrization with other topological properties were obtained by Bing, Nagata and Smirnow. In this paper we follow with the study of relations between metrization and properties of neighborhood bases, locking for still more simple conditions. Our main result gives three characterizations of metrizability:

(1.3) Theorem. *For a T_0-space X the following conditions are equivalent:*

i) *X is metrizable.*

ii) *Each* $x\in X$ *posseses a c.n.b.* $\{U_{xn}|n\in N\}$ *such that*

$$y\notin U_{xn-1}\Longrightarrow U_{yn}\cap U_{xn}=\emptyset\,.$$

iii) *Each* $n \in X$ *posseses a c. n. b.* $\{V_{xn} | n \in N\}$ *such that*

$$y \in V_{xn} \Longrightarrow V_{xn} \subset V_{yn-1}.$$

iv) *There exists a function* $f: N \to N$ *and each* $x \in X$ *posseses a c. n. b.* $\{W_{xn}/n \in N\}$ *such that*

$$y \in W_{xk}, \quad k \geqq f(n) \Longrightarrow W_{xk} \subset W_{yn}.$$

Therefore, condition a) of (1.2) can be deleted, and so, Exercise 23H in [W] is not right.

In Section 3 we show how to associate to many metrizable topologies other ones which are finer and not metrizable, obtaining in Section 4 a characterization of the first countable T_1-topologies τ satisfying that any first countable topology finer than τ is metrizable.

2. Proof of (1.3).

i) $\Rightarrow$ ii) is obvious by (1.2).

ii) $\Rightarrow$ iv): From ii) we obtain

(2.1) $$y \in U_{xn} \Longrightarrow x \in U_{yn-1},$$

(2.2) $$U_{xn} \subset U_{xn-1}.$$

Let us take $f(n)=n+2$ and define $W_{xn}=U_{xn}$ for all $x \in X$ and all $n \in N$. For $y, z \in W_{xk}$ with $k \geqq n+2$ we have $x \in W_{yk-1} \cap W_{zk-1}$ by (2.1), then $x \in W_{yn+1} \cap W_{zn+1}$ by (2.2) obtaining that $z \in W_{yn}$ by hypothesis. So iv) is verified.

iv) $\Rightarrow$ iii): From iv) we have

(2.3) $$k \geqq f(n) \Longrightarrow W_{xk} \subset W_{xn}.$$

We can assume that f is a strictly increasing function and take another strictly increasing $g: N \to N$ such that $g(n+1) \geqq fg(n)$. We define $V_{xn}=W_{xg(n)}$ for all $x \in X$ and all $n \in N$, obtaining by (2.3) that $\{V_{xn} | n \in N\}$ is a c. n. b. at x. If we take $y \in V_{xn}$, since $g(n) \geqq fg(n-1)$ we obtain by hypothesis that $V_{xn} \subset V_{yn-1}$.

iii) $\Rightarrow$ i): From iii) we have

(2.4) $$y \in V_{xn} \Longrightarrow x \in V_{yn-1},$$

(2.5) $$V_{xn} \subset V_{xn-1}.$$

Let us define $U_{xn}=\{y \in X | y \in V_{xn} \text{ and } x \in V_{yn}\}$ for all $x \in X$ and all $n \in N$. Using (2.4) and (2.5) we obtain $V_{xn+1} \subset U_{xn} \subset V_{xn}$, so $\{U_{xn} | n \in N\}$ is a c. n. b. at x. It can be proved that

(2.6) $$y \in U_{xn} \Longrightarrow x \in U_{yn},$$

(2.7) $$y \in U_{xn} \Longrightarrow U_{xn} \subset U_{yn-1}.$$

From (2.6) and (2.7) we obtain that these last countable neighborhood bases verify (1.2)-a). Now let us suppose $U_{yn} \cap U_{xn} \neq \emptyset$ and take $z \in U_{yn} \cap U_{xn}$. Then $y \in U_{zn} \subset U_{xn-1}$ by (2.6) and (2.7) obtaining that (1.2)-b) is also verified and X is metrizable. □

3. A way of obtaining no metrizable spaces.

Let (X, τ) be a topological space and let $E \subset X$. The coarsest topology containing $\tau \cup \{E\}$ is given by

(3.1) $$\tau_E = \{U \cup (V \cap E) \mid U, V \in \tau\}.$$

(3.2) Proposition. *If (X, τ) is metrizable and E is dense but not in τ then (X, τ_E) is not metrizable.*

Proof. Let us assume (X, τ_E) metrizable and let d and δ be metrics in X generating τ and τ_E respectively.

For each $n \in N$ and each $x \in X$ we define

$$G_{xn} = \begin{cases} B_d(x, 1/n) & \text{if } x \notin E \\ B_d(x, 1/n) \cap E & \text{if } x \in E, \end{cases}$$

being $\{G_{xn} \mid n \in N\}$ a τ_E-c. n. b. at x.

Since $E \notin \tau$ we can take a point $x \in E$ which is a cluster point of $X-E$ (considering τ). Since $E \in \tau_E$ there exists $n \in N$ such that $B_\delta(x, 1/n) \subset E$. We obtain that there exists $k \geqq n$ such that $G_{xk} \subset B_\delta(x, 1/2n)$, there exists $y \in X-E$ such that $d(y, x) < 1/2k$, there exists $l \geqq 2k$ such that $G_{yl} \subset B_\delta(y, 1/2k)$, and there exists $z \in E$ such that $d(y, z) < 1/l$. Then we have

$$z \in B_d(y, 1/l) = G_{yl} \subset B_\delta(y, 1/2k),$$

and

$$d(x, z) \leqq d(x, y) + d(y, z) < 1/2k + 1/l \leqq 1/k,$$

so

$$z \in B_d(x, 1/k) \cap E = G_{xk} \subset B_\delta(x, 1/2n).$$

Therefore $\delta(y, z)$, $\delta(x, z) < 1/2n$, obtaining $\delta(x, y) < 1/n$, thus $y \in B_\delta(x, 1/n) \subset E$, which is a contradiction. □

4. Almost-discrete topological spaces.

Let us define a topological space (X, τ) to be prediscrete when every point $x \in X$ has a neighborhood U such that $U = \{x\}$ or $U - \{x\}$ is discrete with the induced topology. If (X, τ) is a prediscrete space let $A = \{x \in X \mid \{x\} \in \tau\}$ and

$B=X-A$. Then we say that (X, τ) is almost-discrete when each $x\in B$ has a neighborhood U_x such that $U_x\cap U_y=\emptyset$ whenever x and y are different points in B. Using (1.3) it is proved in the Appendix that a prediscrete topological space is metrizable if and only if it is almost-discrete, first countable and T_1.

(4.1) Proposition. *For a first countable T_1-topological space (X, τ) the following conditions are equivalent:*

i) *(X, τ) is almost-discrete.*

ii) *Every first countable topology in X finer than τ is metrizable.*

Proof. It is obvious that every topology finer than an almost-discrete one is almost-discrete, from which we obtain that i) implies ii).

Let us assume that ii) is verified but (X, τ) is not almost-discrete, thus it is prediscrete because τ is metrizable. Then there exists a point $x\in X$ such that for any neighborhood U of x we have that $U\neq\{x\}$ and $U-\{x\}$ is not discrete. So, we can take a sequence $(x_n)_{n\in N}$ converging to x with its points contained in $X-\{x\}$, and for each $n\in N$ we can take a sequence $(x_{n,m})_{m\in N}$ converging to x_n with its points contained in $X-\{x_n\}$. Therefore, using (3.2) with $E=X-\{x_n\mid n\in N\}$ we obtain that the topology τ_E on X is finer than τ, first countable and not metrizable, which contradicts ii). □

Appendix

(A.1) Lemma. *A prediscrete topological space (X, τ) is metrizable if and only if it is almost-discrete, first countable and T_1.*

Proof. Assume that (X, τ) is metrizable, then it is first countable and T_1. Since (X, τ) is prediscrete each $x\in B$ has a neighborhood V_x such that $V_x-\{x\}$ is discrete with the induced topology, thus $V_x\cap B=\{x\}$. Let d be a metric in X which induces τ and for each $x\in B$ choose $N(x)\in N$ so that

$$B_d(x, 1/2^{N(x)})\subset V_x.$$

Let us take

$$U_x=B_d(x, 1/2^{N(x)+1}).$$

Suppose that x and y are different points in B such that $U_x\cap U_y\neq\emptyset$, and take $z\in U_x\cap U_y$. We may assume that $N(x)\leqq N(y)$ obtaining

$$d(x, y)\leqq d(x, z)+d(y, z)<\frac{1}{2^{N(x)+1}}+\frac{1}{2^{N(y)+1}}\leqq\frac{1}{2^{N(x)}},$$

so $y\in V_x$, which is a contradiction. Therefore (X, τ) is almost-discrete.

Now assume that (X, τ) is almost-discrete, first countable and T_1. Then each $x\in B$ has a neighborhood U_x such that $U_x\cap U_y=\emptyset$ if x and y are different points in B. Since (X, τ) is first countable, for each $x \in B$ we can choose a c.n.b. $\{U_{xn}\}_{n \in N}$ at x so that $U_x \supset U_{xn} \supset U_{xn+1}$ for all $n \in N$. Then, for each $x \in A$ and $n \in N$ we define U_{xn} in the following way:

a) if $x \notin U_{yn}$ for all $y \in B$ then take $U_{xn} = \{x\}$;

b) if there exists some $y \in B$ (which is unique) so that $x \in U_{yn}$ then take $U_{xn} = U_y$;

obtaining that $\{U_{xn}\}_{n \in N}$ is a c.n.b. at x because (X, τ) is T_1. Now it is easy to check that ii) of (1.3) is verified, so (X, τ) is metrizable. □

It is very easy to find examples of prediscrete spaces which are first countable and T_2 but not almost-discrete, so they are not metrizable. For instance, one of them is the space (X, τ) where $X = \{(x, y) \in R^2 \mid y \geqq 0\}$ and τ is defined by assuming that each set $\{(x, y)\}$ is in τ for $y > 0$, and each point $(x, 0)$ posseses a c.n.b. $\{B_{xn}\}_{n \in N}$ given by

$$B_{xn} = \{(z, y) \in R^2 \mid y > 0,\ z^2 + y^2 < 1/n\} \cup \{(x, 0)\}.$$

References

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