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NON-UNIFORM ESTIMATES IN THE CENTRAL LIMIT THEOREM FOR RANDOM SUMS OF INDEPENDENT RANDOM VARIABLES

By

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Abstract. Let $\{X_n, n\in N\}$ be a sequence of independent random variables with $E[X_n]=0$ and $E[X_n^2]<\infty$, and let $\{N_n, n\in N\}$ be a sequence of random indices defined on the same space as X_n's and independent of the latter. We write $S_{N_n}=\sum_{i=1}^{N_n}X_i$ and $L_n^2=\mathrm{Var}(S_{N_n})<\infty$. We show that for the validity $\lim_{n\to\infty}\|(1+|x|)^{2-1/p}|P(S_{N_n}\leq L_nx)-\Phi(x)|\|_p=0$ it is required that $\lim_{n\to\infty}L_n^{-2-\delta}E[\sum_{i=1}^{N_n}E[|X_i|^{2+\delta}]=0$ for $1\leq p\leq\infty$ and $\delta\in(0, 1)$, and $\lim_{n\to\infty}L_n^{2-2}\cdot\mathrm{Var}(s_{N_n}^2)^{1/2}=0$. Furthermore, we provide conditions under which the series $[\sum_{n=1}^{\infty}(g(L_n)\|(1+|x|)^{2-1/p}|P(S_{N_n}\leq L_nx)-\Phi(x)|\|_p)^s(l_n^2/L_n^2)]^{1/s}$ converges for any $1\leq s<\infty$.

1. Introduction.

This work is concerned with extending the rate of convergence in the central limit theorem of random sums of random variables. This problem, which has attracted the attention of many probabilists, is of interest because it yields extensive applications in actuarial mathematics, stochastic inventory control theory, growth processes and many other fields; see Fotopoulos and Wang (1988) and references cited therein. The first to set up such an extension was Rychlik and Szynal (1975). In a later study, Serova (1978) considered a different extension. In both articles, the authors have studied uniform convergence. Many results about this problem are summarized in a book by Hall (1982) including some general conclusions obtained by this author. Our study will focus on non-uniform estimates of the rate of convergence in the central limit theorem, which have not been emphasized in previous research.

Throughout this paper we shall let $\{X_n, n\in N\}$ (N is the set of natural numbers) be a sequence of real-valued independent random variables ($i.r.v.$'s), not necessarily identical, defined on a probability space $(\Omega, \boldsymbol{F}, P)$, and we assume that $E[X_j]=0$, $E[X_j^2]=\sigma_j^2<\infty$ with distribution functions $F_j(x)$, for $j\in N$.

Key Words: Rate of convergence, central limit theorem, independent random variables, $\mathcal{L}_p$ metrics, Berry-Essen Inequality.

Write $S_n=\sum_{i=1}^{n}X_i$ and $s_n^2=\sum_{i=1}^{n}\sigma_i^2$. Further, let $\{N_n, n\in\boldsymbol{N}\}$ denote a sequence of integer-valued random variables defined on the same probability space $(\Omega, \boldsymbol{F}, P)$ and independent of the sequences $\{X_n, n\in\boldsymbol{N}\}$. The distribution law of the r. v. N_n is defined in the following way: $p_i(n)=P(N_n=i)$, $i\in\boldsymbol{N}$, where $p_i(n)\geqq 0$ and $\sum_{i=1}^{\infty}p_i(n)=1$, for any $n\in\boldsymbol{N}$. Let $\Phi(x)$ denote the standard normal distribution. For any function $a(x)$, write $\|a(x)\|_p=\left(\int|a(x)|^p dx\right)^{1/p}$, $1\leqq p<\infty$, and $d_s(a(x))=\left(\int_1^{\infty}|a(x)|^s(dx/x)\right)^{1/s}$, $1\leqq s<\infty$. Here, $I(E)$ denotes the indicator function of an event E.

Set

$$S_{N_n}=\sum_{i=1}^{N_n}X_i,\quad s_{N_n}^2=\sum_{i=1}^{N_n}\sigma_i^2.$$

By the assumption of independence

$$L_n^2=\mathrm{Var}(S_{N_n})=\sum_{i=1}^{\infty}s_i^2 p_i(n)=E[s_{N_n}^2],\qquad \text{for any}\quad n\in\boldsymbol{N}.$$

We assume that $L_n\to\infty$ as $n\to\infty$.

We define the partial sums of truncated random variables as follows:

$$S_k(D)=\sum_{i=1}^{k}X_iI(|X_i|\leqq(1+|x|)D),$$

where $D>0$ is a specified real number; D may depend on n. In most cases we use $D=L_n$.

We also write

$$m_i(D)=E[X_iI(|X_i|\leqq(1+|x|)D)],\ M_k(D)=\sum_{i=1}^{k}m_i(D)$$

$$s_k^2(D)=\sum_{i=1}^{k}\{E[X_i^2I(|X_i|\leqq(1+|x|)D)]-m_i^2(D)\}$$

and

$$B_k(D)=\sum_{i=1}^{k}E[|X_iI(|X_i|\leqq(1+|x|)D)]-m_i(D)|^3].$$

We shall consider a series of questions with the magnitude of $|P(S_{N_n}\leqq L_nx)-\Phi(x)|$. A considerable amount of recent literature has been devoted to problems of this nature. The most complete results are known for the indendent and identically distributed (i. i. d.) case and for N_n non-random (see e. g., Hall, 1982; a rich bibliography can be found there). But, relatively little work appears to have been done so far on the non-identically distributed case. For N_n non-random, the quantity $|P(S_{N_n}\leqq L_nx)-\Phi(x)|$ has been investigated by Bikyalis (1966), Feller (1968), Hall (1982), Hertz (1969), Galstyan (1971), Maejima (1978a and b) Osipov and Petrov (1967), to name a few. A natural generalization of the above case is when N_n is a random variable. This case was examined by Rychlik and Szynal (1975) and Serova (1978). Rychlik and Szynal (1975) derive upper bounds of the uniform distance of the distribution of the process S_{N_n}/L_n from the normal law. They provide sufficient conditions under which the bounds are the best possible. Serova (1978) describes a method of obtaining bounds when L_n^2 increases more slowly than n. These bounds in turn yield characterizations exactly as those revealed by the previous authors. In this paper we undertake the study of the non-uniform case. Such non-uniform estimates have wider applicability than uniform, e.g., for obtaining inequalities for $\| \ \|_p$ norms and for the theory of moderate deviations.

We begin with a strengthening of the classical Berry-Essen Theorem. The theorems proposed herein are designed to cover random sums of random variables in the absence of third moments, and the estimates given are also sharper than the classical one's, even when the third moments do in fact exist. The main idea of the approach actually goes back to Heyde (1975). We will show that the method developed by Heyde (1975) for problems in the i.i.d. case can be adapted to study the rate of convergence in the central limit theorem for random sums of independent (not necessarily identical) random variables. Overall, it involves simple computations, as in the previous method, and consequently seems potentially capable of characterizing the rate of convergence, like those obtained by Ibragimov (1966) and Heyde (1967). The results are also useful for obtaining probabilities of moderate deviations of random sums of independent random variables, normed in an arbitrary fashion, from the normal law. In addition, our aim is to establish sufficient conditions for the convergence of the series of the type $(\sum_{n=1}^{\infty}(a_n p_n)^s)^{1/s}$, where a_n's are some specified constants and p_n is any one of the quantities $q_n=\sup_{x\in R}(1+|x|)^2|P(S_N\leqq xL_n)-\Phi(x)|$ and $r_n=\|(1+|x|)^{2-1/p}|P(S_{N_n}\leqq xL_n)-\Phi(x)\|_p$ for $1\leqq p<\infty$. It is shown that the generalities of the theorems presented by Rychlik and Szynal (1975) and Serova (1978) are deceptive inasmuch as they can be derived as simple corollaries of Theorem 4 below.

The layout of the remainder of this article is as follows. Section 2 contains the main results of the investigation. Section 3 presents several useful technical lemmas on obtaining generalization of the Berry-Essen theorem. These lemmas are then exploited to obtain some of the desired results.

2. Main Results.

The following two Theorems extend results contained in Bikyalis (1966), Feller (1968), Hall (1982), Heyde (1975) and Osipov and Petrov (1968), which deal with non-uniform bounds on the convergence to normality for independent random variables. We extend their results for the random sums of i. r. v., resulting in more accurate approximations for the distribution of S_{N_n}/L_n.

Theorem 1. *Let* $\{X_i, i \in N\}$ *be a sequence of i. r. v.'s and let* N_n *be an integer valued random variable. Suppose* $E[|X_j|^{2+\alpha}]<\infty$ *for any* $\alpha \in [0, 1]$ *and* $j \in N$. *Then, for all* $x \in (-\infty, \infty)$ *there exists a positive universal constant* c_1 *and* c_2 *such that*

$$|P(S_{N_n} \leqq x L_n) - \Phi(x)| \leqq c_1(1+|x|)^{-2-\alpha}\Big\{L_n^{-3} E\Big[\sum_{j=1}^{N_n} E[|X_j|^3 I(|X_j| \leqq L_n)]\Big]$$
$$+ L_n^{-2-\alpha} E\Big[\sum_{j=1}^{N_n} E[|X_j|^{2+\alpha} I(|X_j|>L_n)]\Big]\Big\}$$
$$+ c_2(\mathrm{Var}(s_{N_n}^2)^{1/2}/L_n^2).$$

Theorem 2. *Let* $\{X_i, i \in N\}$ *be a sequence of i. r. v.'s and let* N_n *be an integer-valued random variable. Suppose* $E[|X_j|^{2+\delta}]<\infty$ *for and* $\delta \in (0, 1)$ *and* $j \in N$. *There exists positive constants* c_1 *and* c_2 *such that whenever* $x \in (-\infty, \infty)$

$$|P(S_{N_n} \leqq L_n x) - \Phi(x)|$$
$$\leqq c_1(L_n(1+|x|))^{-2-\delta} E\Big[\sum_{j=1}^{N_n} E[|X_j|^{2+\delta}]\Big] + c_2(\mathrm{Var}(s_{N_n}^2)^{1/2}/L_n^2).$$

The next result expands the above theorems to the $\mathcal{L}_p$ versions for the central limit theorem.

Theorem 3. *Let* $\{X_i, i \in N\}$ *be a sequence of i. r. v.'s and let* N_n *be an integer-valued random index. Suppose* $E[|X_j|^{2+\delta}]<\infty$ *for any* $\delta \in (0, 1)$ *and* $j \in N$. *There exists positive constants* c_1 *and* c_2 *such that*

$$\|(1+|x|)^{2-1/p}|P(S_{N_n} \leqq L_n x) - \Phi(x)|\|_p$$
$$\leqq c_1 L_n^{-2-\delta} E\Big[\sum_{j=1}^{N_n} E[|X_j|]^{2+\delta}\Big] + c_2(\mathrm{Var}(s_{N_n}^2)^{1/2}/L_n^2).$$

To obtain the following Theorem, some additional notations and remarks are needed.

Let us denote a class of functions $g(x)$ as follows.

$\mathfrak{G} = \{g(x) | g(x)$ is even on the real line with $g(x) \geqq 0$, $g(x) \uparrow \infty$ as $x \uparrow \infty$ and $\exists q \in [0, 1)$ and $x_{q,g} \in [1, \infty)$ such that $g(x)/x^q$ does not increase $\forall x \geqq x_{q,g}\}$.

For our purposes, we shall assume that functions $g \in \mathfrak{G}$ defined for sufficiently small values of the argument are such that we do not have to be concerned about the convergence of integrals which appear. Without loss of generality, we may assume that $x_{q,g}=1$.

Some of the best known and most elegant results on rates of convergence are formulated specifically for the case of independent and identically distributed summands. Some of them can be generalized to sums of "almost identically distributed" (Hall, 1982) variables which we study next.

We impose the following condition (Galstyan, 1971).

Condition A. A sequence $\{X_i, i\in N\}$ of i.r.v.'s is said to satisfy condition A if there exists random variable X with $\mathrm{Var}(X)<\infty$ such that

(2.1) $$n^{-1}\sum_{i=1}^{n} P(|X_i|>x)\leqq c_1P(|X|>x), \quad n\geqq 1 \text{ and } x\geqq 0, \text{ and}$$

(2.2) $$s_n^2\geqq c_2 n, \quad n\geqq 1,$$

where c_1 and c_2 are suitable positive constants.

Note that (2.1) is effectively the same condition with $x>0$ "replaced by $x>x_0$" (x_0 is a fixed constant), since (2.1) can be obtained from a weaker version by altering the distribution of X in a neighborhood of the origin (see, e.g., Hall, p. 78, 1982).

We also observe that under (2.1)

(2.3) $$n^{-1}s_n^2=2\int_0^\infty x\Big(n^{-1}\sum_{i=1}^{n} P(|X_i|>x)\Big)dx\leqq c_1E[X^2].$$

Therefore, under (2.2), $n^{-1}s_n^2$ is bounded away from zero and infinity. Osipov (1968) uses the notation $s_n^2\asymp n$, for this case. Now, in conjunction with (2.1) and (2.2), it can easily be seen that

(2.4) $$L_n^{-1}E\Big[\sum_{i=1}^{N_n} P(|X_i|>x)\Big]<c_0P(|X|>x) \qquad \text{for any} \quad x\geqq 0.$$

Finally, we write

$$l_n^2=L_n^2-L_{n-1}^2,$$

$$I(s,\alpha)=\Big(\sum_{n=1}^{\infty}(g(L_n)(1+|x|)^{2+\alpha}|P(S_{N_n}\leqq xL_n)-\Phi(x)|)^s(l_n^2/L_n^2)\Big)^{1/s},$$

for $1\leqq s<\infty$

and

$$J(s,p)=\Big(\sum_{n=1}^{\infty}(g(L_n))\|(1+|x|)^{2-1/p}|P(S_{N_n}\leqq xL_n)-\Phi(x)|\|_p)^s(l_n^2/L_n^2)\Big)^{1/s},$$

for $1\leqq s<\infty$ and $1\leqq p<\infty$,

where $g\in\mathfrak{C}$.

Then we have the following Theorem.

Theorem 4. *Let $\{X_i, i\in N\}$ be a sequence of i.r.v.'s and let N_n be an integer-valued random index. Suppose that condition A is satisfied and if*

$$\limsup_{n\to\infty}(L_{n+1}/L_n)<\infty \quad \text{and} \quad \left(\sum_{n=1}^{\infty}(g(L_n)(\operatorname{Var}(s_{N_n}^2)^{1/2}/L_n^2)^s(l_n^2/L_n^2)\right)^{1/s}<\infty,$$

where $g\in\mathfrak{C}$, then the sufficient condition for any $1<s<\infty$ that

i. $I(s, 0)<\infty$

and

ii. $J(s, p)<\infty$ *for* $1\leqq p<\infty$

is $d_s(x^2g(x)P(|X|>x))<\infty$.

In particular, when $s=1$, we obtain a similar version of Hall's Theorem 2.11, p. 78 (1982).

Corollary. *Let $\{X_i, i\in N\}$ be a sequence of i.r.v.'s and let N_n be an integer-valued random index. Suppose that condition A is satisfied and if*

$$\limsup_{n\to\infty}(L_{n+1}/L_n) \quad \text{and} \quad \sum_{n=1}^{\infty} g(L_n)(\operatorname{Var}(s_{N_n}^2)^{1/2}/L_n^2)<\infty,$$

where $g\in\mathfrak{C}$, then the sufficient condition that

i. $I(1, 0)<\infty$

and

ii. $J(1, p)<\infty$ *for* $1\leqq p<\infty$

is $E[X^2G(|X|)]<\infty$, *where* $G(x)=\int_1^x \frac{g(x)}{x}dx \; I(|x|\geqq 1)$.

Theorem 1, 2 and 3 give a direct answer to the need for an estimate of the worst error which can be expected in the central limit theorem for random sums of independent random variables. Theorem 4 is of considerable theoretical interest, and provides a method of describing why the upper bound obtained from Theorems 1, 2 and 3 are the best possible and why improvement can be made. However, characterizations of rates of convergence are usually restricted to the case of almost identically distributed summands.

3. Proofs.

The inclusion of Theorem 5 below (which is a restatement of Theorem 1 from Osipov and Petrov, 1967) is given here without proof to assist us in deducing Theorems 1 and 2 above.

Theorem 5. (Osipov-Petrov) *The following inequality holds:*

$$\sup_{x\in R}\left|P\Big(\frac{1}{b}\sum_{i=1}^{n}X_i-h<x\Big)-\Phi(x)\right|$$

$$\leqq A_n+\frac{c_0B_n(D)}{s_n^3(D)}+\frac{|hb-M_n(D)|}{s_n(D)\sqrt{2\pi}}+\frac{1}{2\sqrt{2\pi e}}\left|1-\frac{s_n^2(D)}{b^2}\right|\max\Big(1,\ \frac{b^2}{s_n^2(D)}\Big),$$

where $A_n=\sum_{j=1}^{n}P(|X_j|>D(1+|x|))$, $B_n(D)=\sum_{j=1}^{n}E[|X_jI(|X_j|\leqq D(1+|x|))-m_j(D)|^3]$ *and* $b>0$ *and* h *are arbitrary real numbers.*

Remark. b and h may depend on n.

To demonstrate Theorems 1 and 2, we will begin by establishing 4 lemmas. In our proofs, the symbol c denotes a generic, positive constant, not necessarily the same at each appearance, while c_0, c_1, c_2 and c_3 denote particular versions of c.

Lemma 1. *If* $E[|X_i|^{2+\alpha}]<\infty$ *for* $\alpha\in[0,1]$ *for any* $i\in N$, *then for any* $|x|\geqq 0$ *and* $D>0$,

$$\sum_{j=1}^{n}E[I(|X_j|>(1+|x|)D)]\leqq(D(1+|x|))^{-(2+\alpha)}\sum_{j=1}^{n}E[|X_j|^{2+\alpha}I(|X_j|>(1+|x|)D)]$$

$$\leqq(D(1+|x|))^{-(2+\alpha)}\sum_{j=1}^{n}E[|X_j|^{2+\alpha}I(|X_j|>D)].$$

Proof. This proof is closely related to the result given by Heyde (1975). We have that for any $j\in N$,

$$P(|X_j|>(1+|x|)D)\leqq(D(1+|x|))^{-(2+\alpha)}\sup_{u>D(1+|x|)}u^{2+\alpha}P(|X_j|>u)$$

$$\leqq(D(1+|x|))^{-(2+\alpha)}\sup_{u>D(1+|x|)}E[|X_j|^{2+\alpha}I(|X_j|>u)]$$

$$\leqq(D(1+|x|))^{-(2+\alpha)}E[|X_j|^{2+\alpha}I(|X_j|>(1+|x|)D)]$$

$$\leqq(D(1+|x|))^{-(2+\alpha)}E[|X_j|^{2+\alpha}I(|X_j|>D)].$$

Hence, summing over j, the result follows.

Lemma 2.

$$\frac{|M_n(D)|}{D(1+|x|)}\leqq(D(1+|x|))^{-(2+\alpha)}\sum_{j=1}^{n}E[|X_j|^{2+\alpha}I(|X_j|>(1+|x|)D)]$$

$$\leqq(D(1+|x|))^{-(2+\alpha)}\sum_{j=1}^{n}E[|X_j|^{2+\alpha}I(|X_j|>D)].$$

Proof. Since $E[X_j]=0$ for any $j\in N$, we obtain that

$$|E[X_jI(|X_j|\leqq(1+|x|)D)]|=|E[X_jI(|X_j|>(1+|x|)D]|$$
$$\leqq(D(1+|x|))^{-(1+\alpha)}E[|X_j|^{2+\alpha}I(|X_j|>(1+|x|)D)]$$
$$\leqq(D(1+|x|))^{-(1+\alpha)}E[|X_j|^{2+\alpha}I(|X_j|>D)].$$

The result follows immediately.

Lemma 3.

$$\frac{B_n(D)}{(D(1+|x|))^3}\leqq\frac{4}{(D(1+|x|))^3}\sum_{j=1}^{n}E[|X_j|^3I(|X_j|\leqq(1+|x|)D)]$$
$$\leqq\left\{\frac{1}{(D(1+|x|))^3}\sum_{j=1}^{n}E[|X_j|^3I(|X_j|\leqq D)]\right.$$
$$\left.+\frac{1}{(D(1+|x|))^{2+\alpha}}\sum_{j=1}^{n}E[|X_j|^{2+\alpha}I(|X_j|>D)]\right\}.$$

Proof. The first inequality comes from the fact that $E[|X-E[X]|^3\leqq 4E[|X|^3]$. With respect to the second inquality, we have only to observe that

$$E[|X_j|^3I(|X_j|\leqq(1+|x|)D)]\leqq E[|X_j|^3I(|X_j|\leqq D)]$$
$$+(D(1+|x|))^{1-\alpha}E[|X_j|^{2+\alpha}I(|X_j|>D)],$$

and then the result can easily be pursued.

To show the following Lemma, we set

$$y_n(D)=\frac{Dx-M_n(D)}{s_n(D)}.$$

Lemma 4. *For* $\frac{M_n(D)}{D}<\frac{|x|}{2}$ *and* $|x|\geqq 2$

$$|\Phi(y_n(D))-\Phi(x)|\leqq c\left\{(1+|x|)^{-2}\left|1-\frac{s_n^2(D)}{D}\right|\right.$$
$$\left.+((1+|x|)D)^{-(2+\alpha)}\sum_{j=1}^{n}E[|X_j|^{2+\alpha}I(|X_j|>(1+|x|D)]\right\}.$$

Proof. For $|x|\geqq 0$, $e^{1/2x^2}>\frac{1}{8}x^4\geqq\frac{1}{4}|x|^3$. We also note that the variables, $y_n(D)=\frac{D}{s_n(D)}\left(x-\frac{M_n(D)}{D}\right)$ and x, have the same sign, for $|x|\geqq 2$. Therefore,

$$|\Phi(y_n(D))-\Phi(x)|<\frac{4}{\sqrt{2\pi}}\left|\int_x^{y_n(D)}\frac{dt}{t^3}\right|=\frac{2}{\sqrt{2\pi}}\left|\frac{1}{x^2}-\frac{1}{y_n^2(D)}\right|$$
$$=\frac{2}{\sqrt{2\pi}}\left(x-\frac{M_n(D)}{D}\right)^{-2}\left\{1-\frac{2M_n(D)}{Dx}+\frac{M_n(D)^2}{D^2x^2}-\frac{s_n^2(D)}{D^2}\right\}$$

$$=\frac{2}{\sqrt{2\pi}}\Big(x-\frac{M_n(D)}{D}\Big)^{-2}\Big(1-\frac{s_n^2(D)}{D^2}\Big)$$

$$+\frac{2}{\sqrt{2\pi}}\Big(x-\frac{M_n(D)}{D^2x^2}\Big)^{-2}\Big(-2\frac{M_n(D)}{Dx}+\frac{M_n^2(D)}{D^2x^2}\Big)$$

$=I_1+I_2$ say.

Since $\frac{M_n(D)}{D}<\frac{|x|}{2}$ and $|x|\geqq 2$, it can be noticed that

$$\Big|x-\frac{M_n(D)}{D}\Big|^{-2}<\Big(|x|-\frac{|x|}{2}\Big)^{-2}<\frac{4}{|x|^2}<\frac{16}{(1+|x|)^2}.$$

Hence

$$I_1\leqq\frac{32}{\sqrt{2\pi}}(1+|x|)^{-2}<\Big(1-\frac{s_n^2(D)}{D^2}\Big).$$

Using the assumption $\frac{M_n(D)}{D}<\frac{|x|}{2}$ and $|x|\leqq 2$, it can also be seen that

$$\Big(x-\frac{M_n(D)}{D}\Big)^{-2}<\frac{4}{|x|^2}\leqq 1 \quad\text{and}\quad \frac{M_n^2(D)}{D^2}<\frac{|M_n(D)|}{2D}|x|.$$

Therefore, from Lemma 2, it follows that

$$I_2<\frac{10}{\sqrt{2\pi}}\frac{|M_n(D)|}{D|x|}\leqq\frac{10}{\sqrt{2\pi}}((1+|x|)D)^{-(2+\alpha)}\sum_{j=1}^{n}E[|X_j|^{2+\alpha}I(|X_j|>(1+|x|)D)].$$

Combining I_1 and I_2, the proof of Lemma 4 is now completed.

Proof of Theorem 1 and 2. By dividing the domain of x, we can confirm Theorems 1 and 2 by adopting some of the steps stated by Osipov and Petrov (1967). First, it is noted that

$$|P(S_{N_n}\leqq L_nx)-\Phi(x)|=\sum_{i=1}^{\infty}p_i(n)|P(S_i\leqq L_nx)-\Phi(x)|. \tag{3.1}$$

To this end, we shall investigate the behavior of $|P(S_i\leqq L_nx)-\Phi(x)|$ by utilizing the classical Berry-Essen Theorem.

We shall start with $|x|\leqq 2$ and $L_n^2<2s_k^2(L_n)$. We note that

$$\Big|1-\frac{s_k^2(L_n)}{L_n^2}\Big|\leqq\frac{|L_n^2-s_k^2|}{L_n^2}+\frac{\sum_{i=1}^{k}E[|X_i^2I(|X_i|>(1+|x|)L_n)]}{L_n^2} \tag{3.2}$$

$$\leqq\frac{|L_n^2-s_k^2|}{L_n^2}+\frac{\sum_{i=1}^{k}E[|X_i|^{2+\alpha}I(|X_i|>(1+|x|)L_n)]}{L_n^{2+\alpha}(1+|x|)^\alpha}$$

$$\leqq \frac{|L_n^2 - s_k^2|}{L_n^2} + 3^2 \frac{\sum_{i=1}^k E[|X_i|^{2+\alpha} I(|X_i| > (1+|x|)L_n)]}{(L_n(1+|x|))^{2+\alpha}}$$

$$\leqq \frac{|L_n^2 - s_k^2|}{L_n^2} + 3^2 \frac{\sum_{i=1}^k E[|X_i|^{2+\alpha} I(|X_i| > L_n)]}{(L_n(1+|x|))^{2+\alpha}}.$$

Hence, in view of Lemmas 1-3, Theorem 5 at $h=0$ and $b \equiv L_n \equiv D$, inequality (3.2), and the conditions $|x| \leqq 2$ and $L_n^2 < 2s_k^2(L_n)$, we obtain that

(3.3) $|P(S_k \leqq L_n x) - \Phi(x)|$

$$\leqq A_k + \frac{|M_k(L_n)|}{s_k(L_n)\sqrt{2\pi}} + \frac{1}{2\sqrt{2\pi e}} \left| 1 - \frac{s_k^2(L_n^2)}{L_n^2} \right| \max\left(1, \frac{L_n^2}{s_k^2(L_n)}\right) + \frac{c_0 B_k(L_n)}{s_k^3(L_n)}$$

$$\leqq A_k + \frac{\sqrt{2}\,|M_k(L_n)|}{L_n\sqrt{2\pi}} + \frac{1}{\sqrt{2\pi e}} \left| 1 - \frac{s_k^2(L_n)}{L_n^2} \right| + \frac{2\sqrt{2} c_0 B_k(L_n)}{L_n^3}$$

$$\leqq \left(1 + \frac{3}{\sqrt{\pi}} + \frac{3^2}{\sqrt{2\pi e}}\right) \frac{\sum_{i=1}^k E[|X_i|^{2+\alpha} I(|X_i| > (1+|x|)L_n)]}{(L_n(1+|x|))^{2+\alpha}}$$

$$+ \frac{(4\sqrt{2})3^3 c_0 \sum_{i-1}^k E[|X_i|^3 I(|X_i| \leqq (1+|x|)L_n)]}{(L_n(1+|x|))^3} + \frac{|L_n^2 - s_k^2|}{\sqrt{2\pi e} L_n^2}$$

$$\leqq \left(1 + \frac{3}{\sqrt{\pi}} + \frac{3^2}{\sqrt{2\pi e}}\right) \frac{\sum_{i=1}^k E[|X_i|^{2+\alpha} I(|X_i| > L_n)]}{(L_n(1+|x|))^{2+\alpha}}$$

$$+ \frac{4(2\sqrt{2})3^3 c_0 \sum_{i=1}^k E[|X_i|^3 I(|X_i| \leqq L_n)]}{(L_n(1+|x|))^3} + \frac{|L_n^2 - s_k^2|}{\sqrt{2\pi e} L_n^2}.$$

By inserting (3.3) into (3.1), Theorem 1 follows, for $|x| \leqq 2$ and $L_n^2 < 2s_k^2(L_n)$.

Moreover, if $E[|X_i|^{2+\delta}] < \infty$, for some $\delta \in (0, 1)$, we have that for $\alpha = 0$,

(3.4)
$$\frac{\sum_{i=1}^k E[|X_i^2| I(|X_i| > (1+|x|)L_n)]}{(L_n(1+|x|))^2} + \frac{\sum_{i=1}^k E[|X_i^3| I(|X_i| \leqq (1+|x|)L_n)]}{(L_n(1+|x|))^3}$$

$$\leqq \frac{\sum_{i=1}^k E[|X_i|^{2+\delta}]}{(L_n(1+|x|))^{2+\delta}},$$

and hence Theorem 2 follows immediately, for this case.

If $|x| \leqq 2$ and $L_n^2 \geqq 2s_k^2(L_n)$, then $1 - \frac{s_k^2(L_n)}{L_n^2} \geqq 1/2$ and consequently using the fact that $E[X_i^2 I(|X_i| \leqq (1+|x|)L_n)] = \sigma_i^2 - E[X_i^2 I(|X_j| > (1+|x|)L_n)]$ and some of the steps of the proof of Lemma 2, we have the following

(3.5) $$|P(S_k \leqq L_n x)-\Phi(x)| \leqq 1 \leqq 2\left(1-\frac{s_k^2(L_n)}{L_n^2}\right)$$

$$\leqq 2\left|1-\frac{s_k^2}{L_n^2}\right|+\frac{\sum_{i=1}^{k}\left\{E[X_i^2 I(|X_i|>(1+|x|)L_n)]+m_i(L_n)^2\right\}}{L_n^2}$$

$$\leqq 2\left|1-\frac{s_k^2}{L_n^2}\right|+\frac{(2)3^{2+\alpha}\sum_{i=1}^{k}E[|X_i|^{2+\alpha}I(|X_i|>(1+|x|)L_n)]}{(L_n(1+|x|))^{2+\alpha}}.$$

From (3.5), Theorems 1 and 2 follow in a straightforward fashion for $|x|\leqq 2$ and $L_n^2 \geqq 2s_k^2(L_n)$.

For $|x|>2$ and $\frac{|M_k(L_n)|}{L_n}\geqq\frac{|x|}{2}$ and by incorporating Lemma 2, it is easily seen that

(3.6) $$|P(S_k \leqq L_n x)-\Phi(x)|$$

$$\leqq 1 \leqq \frac{2}{|x|}\frac{|M_k(D)|}{L_n} \leqq 3^{2+\alpha}\frac{\sum_{i=1}^{k}E[|X_i|^{2+\alpha}I(|X_i|>(1+|x|)L_n)]}{(L_n(1+|x|))^{2+\alpha}}$$

which is the same as in (3.3) or (3.5).

Finally, taking $|x|>2$ and $\frac{|M_k(L_n)|}{L_n}<\frac{|x|}{2}$, we need to obtain a similar expression to those derived above.

We define

$$S_{N_n}^*(L_n)=\frac{S_{N_n}(L_n)-M_{N_n}(L_n)}{s_{N_n}(L_n)} \quad \text{and} \quad y_{N_n}(L_n)=\frac{L_n x-M_{N_n}(L_n)}{s_{N_n}(L_n)}$$

As in Heyde (1975), it is not hard to verify the inequality.

(3.7) $$|P(S_{N_n}\leqq L_n x)-\Phi(x)| \leqq |P(S_{N_n}^*(L_n)\leqq y_{N_n}(L_n))-\Phi(y_{N_n}(L_n))|$$

$$+|\Phi(y_{N_n}(L_n))-\Phi(x)|+E\left[\sum_{i=1}^{N_n}P(|X_i|>L_n(1+|x|))\right].$$

Clearly,

(3.8) $$|P(S_{N_n}^*(L_n)\leqq y_{N_n}(L_n))-\Phi(y_{N_n}(L_n))|=\sum_{k=1}^{\infty}p_k(n)\Delta_k(L_n),$$

where

$$\Delta_n(L_n)=|P(S_k^*(L_n)\leqq y_k(L_n))-\Phi(y_k(L_n))|.$$

Now, calling upon Bikyalis' Theorem (1966), we have that

(3.9) $$\Delta_k(L_n)\leqq\frac{c_0}{s_k^3(L_n)\left(1+\left|\frac{L_n}{s_k(L_n)}-\frac{M_k(L_n)}{s_k(L_n)}\right|^3\right)}B_k(L_n)$$

$$=\frac{c_0}{L_n^3\left(\frac{s_k^3(L_n)}{L_n^3}+\left|x-\frac{M_k(L_n)}{L_n}\right|^3\right)}B_k(L_n).$$

Since $\frac{M_k(L_n)}{L_n}<\frac{|x|}{2}$, it can be seen that

$$\left(\frac{s_k^3(L_n)}{L_n^3}+\left|x-\frac{M_k(L_n)}{L_n}\right|^3\right)\geqq\left(|x|-\frac{|x|}{2}\right)^3$$

$$\geqq\frac{1}{(10)2^3}(1+|x|)^3.$$

Thus, (3.9) yields

$$\Delta_k(L_n)\leqq\frac{(10)2^3c_0}{(L_n(1+|x|))^3}B_k(L_n).$$

In conjunction with Lemma 3, the last inequality is bounded by

$$\Delta_k(L_n)\leqq\frac{4(10)2^3c_0\sum_{i=1}^{k}E[|X_i|^3I(|X_i|\leqq(1+|x|)L_n)]}{(L_n(1+|x|))^3}. \tag{3.10}$$

To prove the rest of Theorems 1 and 2, we use Lemmas 1 and 4 and then equation (3.7). Completion of Theorems 1 and 2 are now accomplished.

Proof of Theorem 3. Keeping in mind some of the steps and the techniques used to show Theorems 1 and 2, it turns out that for $\alpha=0$

$$|P(S_{N_n}\leqq L_nx)-\Phi(x)|\leqq c\left\{\frac{E\left[\sum_{i=1}^{N_n}E[X_i^2I(|X_i|>(1+|x|)L_n)]\right]}{(L_n(1+|x|))^2}\right. \tag{3.11}$$

$$\left.+\frac{E\left[\sum_{i=1}^{N_n}E[|X_i^3|I(|X_i|\leqq(1+|x|)L_n)]\right]}{(L_n(1+|x|))^3}+\frac{\mathrm{Var}(s_{N_n}^2)^{1/2}}{L_n^2}I(|x|\leqq2)\right\}.$$

Now, for any $1\leqq p<\infty$, (3.11) can be written as

(3.12)

$$(1+|x|)^{2-1/p}|P(S_{N_n}\leqq L_nx)-\Phi(x)|\leqq c\left\{\frac{E\left[\sum_{i=1}^{N_n}E[X_i^2I(|X_i|>(1+|x|)L_n)]\right]}{L_n^3(1+|x|)^{1/p}}\right.$$

$$\left.+\frac{E\left[\sum_{i=1}^{N_n}E[|X_i|^3I(|X_i|\leqq(1+|x|)L_n)]\right]}{L_n^3(1+|x|)^{1+1/p}}+(1+|x|)^{-1-1/p}\frac{\mathrm{Var}(s_{N_n})}{L_n^2}\right\}$$

$$=c(A_j+A_2+A_3) \text{ say.}$$

Next, it is known that for any $x_1(u), \cdots, x_d(u)$ positive functions of u, the Minkowski inequality can be written as

$$(3.13)\qquad \left(\int\left(\sum_{i=1}^{d} x_i(u)\right)^p du\right)^{1/p} \leqq \sum_{i=1}^{d}\left(\int x_i^p(u)du\right)^{1/p} \quad \text{for any} \quad 1\leqq p\leqq\infty .$$

Hence, invoking (3.12), we obtain that

$$(3.14)\qquad \|(1+|x|)^{2-1/p}|P(S_{N_n}\leqq L_n x)-\Phi(x)|\|_p$$
$$\leqq c\left(\int_{-\infty}^{\infty}(A_1+A_2+A_3)^p dx\right)^{1/p} \leqq c\left(\left(\int_{-\infty}^{\infty}A_1^p dx\right)^{1/p}\right.$$
$$\left.+\left(\int_{-\infty}^{\infty}A_2^p dx\right)^{1/p}+\left(\int_{-\infty}^{\infty}A_3^p dx\right)^{1/p}\right)$$

To verify Theorem 3, we note that for any $\delta\in(0, 1)$

(3.15)
$$\left(\int_{-\infty}^{\infty}A_1^p dx\right)^{1/p}=\left(\int_{-\infty}^{\infty}(1+|x|)^{-1}\frac{1}{L_n^{2p}}E\left[\sum_{i=1}^{N_n}E[X_i^2 I(|X_i|>(1+|x|)L_n)]\right]^p dx\right)^{1/p}$$
$$\leqq\left(\int_{-\infty}^{\infty}(1+|x|)^{-1-\delta p}\frac{1}{L_n^{(2+\delta)p}}E\left[\sum_{i=1}^{N_n}E[|X_i|^{2+\delta}I(|X_i|>L_n)]\right]^p dx\right)^{1/p}$$
$$\leqq c_1\frac{1}{E_n^{2+\delta}}E\left[\sum_{i=1}^{N_n}E[|X_i|^{2+\delta}I(|X_i|>L_n)]\right].$$

Following a similar argument as in Lemma 3, and utilizing Minokowski's inequality, we can extract that

$$(3.16)\ \left(\int_{-\infty}^{\infty}A_2^p dx\right)^{1/p}$$
$$=\left(\int_{-\infty}^{\infty}(1+|x|)^{-(p+1)}L_n^{-3p}E\left[\sum_{i=1}^{N_n}E[|X_i|^3 I(|X_i|\leqq(1+|x|)L_n)]\right]^p dx\right)^{1/p}$$
$$\leqq\left(\int_{-\infty}^{\infty}(1+|x|)^{-(p+1)}L_n^{-3p}\left\{E\left[\sum_{i=1}^{N_n}E[|X_i|^{2+\delta}I(|X_i|\leqq L_n)\right]L_n^{1-\delta}\right.\right.$$
$$\left.\left.+((1+|x|)L_n)^{1-\delta}E\left[\sum_{i=1}^{N_n}E[|X_i|^{2+\delta}I(L_n<|X_i|<(1+|x|)L_n)\right]\right\}^p dx\right)^{1/p}$$
$$\leqq\left(\int_{-\infty}^{\infty}(1+|x|)^{-(p+1)}L_n^{-(2+\delta)p}E\left[\sum_{i=1}^{N_n}E[|X_i|^{2+\delta}I(|X_i|\leqq L_n)]\right]dx\right)^{1/p}$$
$$+\left(\int_{-\infty}^{\infty}A_1^p dx\right)^{1/p}$$
$$\leqq c_2 L_n^{-(2+\delta)}E\left[\sum_{i=1}^{N_n}E[|X_i|^{2+\delta}]\right].$$

Finally,

$$(3.17)\qquad \left(\int_{-\infty}^{\infty} A_3^p dx\right)^{1/p} \leqq c_3 \frac{\mathrm{Var}(s_{N_n}^2)^{1/2}}{L_n^2}.$$

Combining (3.15), (3.16) and (3.17), the result can clearly be assured.

Proof of Theorem 4. For this part of our work, some of Rozovski's (1978) ideas are taken into consideration. Under condition A, in precisely the same way as Hall (p. 79, 1980) implies,

$$(3.18)\qquad \frac{1}{L_n^2} E\left[\sum_{i=1}^{N_n} E[X_i^2 I(|X_i|>L_n)]\right] \leqq c_0 E[X^2 I(|X|>L_n)]$$

and

$$(3.19)\qquad \frac{1}{L_n^3} E\left[\sum_{i=1}^{N_n} E[|X_i^3| I(|X_i|\leqq L_n)]\right] \leqq c_0 L_n^{-1} \int_0^{L_n} y^2 P(|X|>y) dy.$$

Recalling (3.18), (3.19), Theorem 1 and utilizing Minkowski's inequality, it can be seen that, for any $g \in \mathfrak{C}$,

$$(3.20)\qquad \begin{aligned} I(s,0) &\leqq c\Big(\sum_{i=1}^{\infty}\Big(g(L_n)\Big\{E[X^2 I(|X|>L_n)] \\ &\quad + L_n^{-1}\int_0^{L_n} y^2 P(|X|>y)dy + \frac{\mathrm{Var}(s_{N_n}^2)^{1/2}}{L_n^2}\Big)^s \frac{l_n^2}{L_n^2}\Big)^{1/s} \\ &\leqq c(\Sigma_1+\Sigma_2+\Sigma_3) \text{ say.} \end{aligned}$$

We have that

$$(3.21)\qquad \begin{aligned} \Sigma_1 &= \Big(\sum_{n=1}^{\infty} (g(L_n) E[X^2 I(|X|>L_n)])^s \frac{l_n^2}{L_n^2}\Big)^{1/s} \\ &\leqq c_1\Big(\int_1^{\infty} (g(u) E[X^2 I(|X|>u)])^s \frac{du}{u}\Big)^{1/s} \\ &= c_1\Big(\int_1^{\infty} g^s(u)\Big(u^2 P(|X|>u) + 2\int_u^{\infty} y P(|X|>y)dy\Big)^s \frac{du}{u}\Big)^{1/s} \\ &\leqq c_1\Big(\int_1^{\infty} g^s(u) u^{2s-1} P(|X|>u)^s du\Big)^{1/s} \\ &\quad + 2\Big(\int_1^{\infty} g^s(u)\Big(\int_u^{\infty} y P(|X|>y)dy\Big)^s \frac{du}{u}\Big)^{1/s} \\ &\leqq c_1\Big(\int_1 + \int_2\Big) \text{ say,} \end{aligned}$$

because of Minkowski's inequality.

We first note that for any $\beta<1$, and using Holder's inequality,

(3.22) $$\left(\int_u^\infty yP(|X|>y)dy\right)^s=\left(\int_u^\infty y^{-\beta}(y^{1+\beta}P(|X|>y)dy\right)^s$$
$$\leqq\left(\int_u^\infty y^{-(\beta s/s-1)}dy\right)^{s-1}\int_u^\infty(y^{1+\beta}P(|X|>y))^s dy$$
$$=\left(1-\frac{\beta s}{s-1}\right)^{-(s-1)}u^{s-1-\beta s}\int_u^\infty(y^{1+\beta}P(|X|>y))^s dy.$$

We choose β such that $g(u)u^{1-\beta-(2/s)}$ is non-decreasing for any $\beta>1$. Therefore, by virtue of (3.22), let us proceed as follows:

(3.23) $$\int_2\leqq c\left(\int_1^\infty g^s(u)u^{-2+s-\beta s}\int_u^\infty(y^{1+\beta}P(|X|>y))^s dy\,du\right)^{1/s}$$
$$\leqq c\left(\int_1^\infty y^{(1+\beta)s}P(|X|>y)^s dy\int_1^y g(u)u^{-2+s-s\beta}du\right)^{1/s}$$
$$\leqq c\left(\int_1^\infty y^{2s-1}g^s(y)P(|X|>y)^s dy\right)^{1/s}$$

From (3.21) and (3.23), it is readily seen that

(3.24) $$\Sigma_1\leqq c\left(\int_1^\infty y^{2s-1}g^s(y)P(|X|>y)^s dy\right)^{1/s}.$$

Treating Σ_2 in the same way as above, we obtain successively

(3.25) $$\Sigma_2=\left(\sum_{n=1}^\infty\left(g(L_n)L_n^{-1}\int_0^{L_n}y^2P(|X|>y)dy\right)^s\frac{l_n^2}{L_n^2}\right)^{1/s}$$
$$\leqq c\left(\int_1^\infty\left(g(u)u^{-1}\int_0^u y^2P(|X|>y)dy\right)^s\frac{du}{u}\right)^{1/s}$$

Now, in connection with (3.22), we analogously show that

(3.26) $$\left(\int_1^u y^2P(|X|>y)dy\right)^s\leqq\left(1-\frac{sq}{s-1}\right)^{-(s-1)}u^{s-1-qs}\int_1^u(y^{2+q}P(|X|>y)^s dy.$$

Then, (3.25) is deduced as follows.

(3.27) $$\Sigma_2\leqq c\left(\int_1^\infty g^s(u)u^{-2-qs}du\int_1^u y^{(2+q)s}P(|X|>y)^s dy\right)^{1/s}$$
$$\leqq c\left(\int_1^\infty y^{(2+q)s}P(|X|>y)^s dy\int_y^\infty g^s(u)u^{-2-qs}du\right)^{1/s}$$
$$\leqq c\left(\int_1^\infty y^{2s-1}g^s(y)P(|X|>y)^s dy\right)^{1/s}.$$

Here, the choice of q is such that $\frac{g(u)}{u^q}$ is non-increasing. This completes the proof of Theorem 4.

References

[1] Bikyalis, A.: *Estimates of the remainder term in the central limit theorem.* Litovsk. Mat. Sb. **6**, (1966), 323-346. (In Russian).

[2] Feller, W.: *On the Berry-Essen Theorem.* Z. Wahrscheinlichkeittstheorie venr. Geb. **10**, (1968), 261-268.

[3] Fotopoulos, S.B. and Wang, M.C.: *Limiting behavior of weighted random sums of independent random variables.* Working paper at the Department of Management and Systems, Washington State University, Pullman, WA. (.988).

[4] Galstyan, F.N.: *On the rate of convergence in the central limit theorem.* Theor. Prob. and Math. Statist. **5**, (1975), 13-24.

[5] Hall, P.: *Rates of convergence in the central limit theorem.* Pitman Advanced Publishing Program. Boston. (1982).

[6] Hertz, E.S.: *On convergence rates in the central limit theorem.* Ann. Math. Statist. **40**, (1969), 475-479.

[7] Heyde, C.C.: *On the influence of moments on the rate of convergence to the normal distribution.* Z. Wahrscheinlichkeitstheorie Vern. Geb. **8**, (1967), 12-18.

[8] Heyde, C.C.: *A non-uniform bound on convergence to normality.* Ann. Probab. **3**, (1975), 903-907.

[9] Ibragimov, I.A.: *On the accuracy of Gaussian approximation to the distribution functions of sums of independent variables.* Theor. Prob. Appl. **11**, (1966), 556-576.

[10] Maejima, M.: *Non-uniform estimates in the central limit theorem.* Yokohama Math. Jour. **26**, (1978), 137-149.

[11] Maejima, M.: *Some L_p versions for the central limit theorem.* Ann. Probab. **6**, (1978), 341-344.

[12] Osipov, L.V.: *Accuracy of the approximation of the distribution of a sum of independent random variables to the normal distribution.* Dokl. Akad. Nauk. SSSR **178**, (1968), 233-236.

[13] Osipov, L.V. and Petrov, V.V.: *On an estimate of the remainder term in the central limit theorem.* Theor. Probab. Appl. **12**, (1967), 281-286.

[14] Rozovski, L.V.: *On the precision of an estimate of the remainder term in the central limit ȷheorem.* Theor. probab. Appl. **23**, (1978), 712-730.

[15] Rychlik, Z. and Szynal, D.: *Convergence rates in the central limit theorem for sums of a random number of independent random variables.* Theor. Probab. Appl. **20**, (1975), 351-362.

[16] Serova, G.V.: *On the rate of convergence in the central limit theorem.* Theor. Prob. and Math. Statist. **17**, (1978), 121-130.

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