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MORITA DUALITY AND INTERMEDIATE TRIANGULAR EXTENSIONS

By

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Abstract. Let $R\leq T\leq S$ be ring extensions such that $R\leq S$ is a finite triangular extension. Let ${}_RE$ be a left R-module. If R has a Morita duality induced by ${}_RE$ then T has a Morita duality induced by ${}_T\mathrm{Hom}_R(T, E)$.

1. Introduction.

Throughout this note all rings have identity, modules are unitary, and ring extensions share the same identity. A ring extension $R\leqq S$ is called a *finite triangular extension* in case there are a finite number of elements $s_1, \cdots, s_n\in S$ such that $S=\sum_{i=1}^{n} Rs_i$ and $\sum_{i=1}^{j} Rs_i=\sum_{i=1}^{j} s_iR$, for $j=1, \cdots, n$. A module M is called *linearly compact* (l. c.) if any finitely solvable system of congruences, $m\equiv m_i \bmod M_i$ ($M_i\leqq M$, $i\in I$), is solvable (see [5], [6]). It is easy to see that (1) submodules and quotient modules of a l. c. module are still l. c., (2) a direct sum of finitely many l. c. modules is l. c., and (3) a semisimple module is l. c. if and only if it is a sum of finitely many simple modules. And it is also well-known that a ring R is semiperfect if the left R-module ${}_RR$ is l. c. (see [5], for example).

In this note we study ring extensions and (Morita) duality that was established by Morita [4] and Azumaya [2]. A presentation of duality can be found in Anderson and Fuller [1, § 23, § 24]. Vámos [6, Prop. (1.4)] showed that a ring R has a duality (induced by ${}_RE$) if and only if ${}_RR$ is l. c. and ${}_RE$ is a l. c. and finitely cogenerated injective cogenerator. E. g., every semisimple ring R has a duality induced by ${}_RR$.

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2. Main Result.

Recently the author [7, Theorem 1(1)] proved that a finite triangular extension over a ring with duality has itself a duality. We shall generalize this result as follows:

Theorem. *Let* $R \leqq T \leqq S$ *be ring extensions such that* $R \leqq S$ *is a finite triangular extension. Let* ${}_RE$ *be a left R-module. If R has a Morita duality induced by* ${}_RE$ *then T has a Morita duality induced by* ${}_T\mathrm{Hom}_R(T, E)$.

3. Proof of the Theorem.

Using the proof of [6, Lemma 2.9] we note

Sublemma. *Suppose* $R \leqq T$ *is a ring extension such that* ${}_RT$ *is l.c.. If* ${}_RE$ *is a finitely cogenerated injective cogenerator, then* ${}_TF = {}_T\mathrm{Hom}_R(T, E)$ *is an essential extension of* $\mathrm{Soc}({}_TF)$.

Now let $F = \mathrm{Hom}_R(T, E)$ and $G = \mathrm{Hom}_R(S, E)$. Since ${}_RR$ is l.c. and ${}_RS$ is finitely generated, ${}_RS$ is l.c., and hence ${}_RT$ is l.c.. Using the injectivity of ${}_RE$, the exact sequence of R-bimodules

$$0 \longrightarrow T \longrightarrow S \longrightarrow S/T \longrightarrow 0$$

induces an exact sequence of left R-modules

$$0 \longrightarrow {}_R\mathrm{Hom}_R(S/T, E) \longrightarrow {}_RG \longrightarrow {}_RF \longrightarrow 0 .$$

Now ${}_RG$ is l.c. from the proof of [7, Theorem 1], so we conclude that ${}_RF$ is l.c.. We have proved that ${}_RT$ and ${}_RF$ are l.c., and hence ${}_TT$ and ${}_TF$ are l.c. since any T-submodule is automatically an R-submodule. Thus $\mathrm{Soc}({}_TF)$, being a submodule of the l.c. module ${}_TF$, is l.c.. Therefore $\mathrm{Soc}({}_TF)$ is finitely generated since it is l.c. and semisimple. It follows from the Sublemma that ${}_TF$ is finitely cogenerated. Since ${}_TF$ is an injective cogenerator by [6, p. 279], so ${}_TF$ induces a Morita duality by Vámos' result [6, Prop. (1.4)].

4. Two Examples of Finite Extensions.

As a generalization of finite triangular extensions, one calls a ring extension $R \leqq S$ a *finite extension* if both ${}_RS$ and S_R are finitely generated modules. For a finite extension $R \leqq S$ there is no connection between the duality of R and the duality of S, as can be shown by the following two examples.

Example 4.1. Let $R=\begin{bmatrix} \boldsymbol{Q} & \boldsymbol{Q} & \boldsymbol{Q} \\ 0 & \boldsymbol{Z} & \boldsymbol{Q} \\ 0 & 0 & \boldsymbol{Q} \end{bmatrix}$ and $S=M_3(\boldsymbol{Q})$. Then $R \leqq S$ is a finite extension. In fact, we have $S=\sum_{i=1}^{4} Rs_i=\sum_{i=1}^{4} s_i R$, where $s_1=1_S$, $s_2=\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $s_3=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, and $s_4=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Now S is semisimple but R is not even semilocal. (i.e. $R/J(R)$ is not semisimple.) It follows that R does not have a duality since ${}_RR$ is not l.c..

Example 4.2. Cohn [3] has shown that there is a division ring extension $C \leqq D$ such that $\dim({}_CD)=\infty$ and $\dim(D_C)<\infty$. It follows that the artinian ring $S=\begin{bmatrix} D & D \\ 0 & C \end{bmatrix}$ does not have a duality (see [1, p. 286]). But S is a finite extension over the semisimple ring $R=\begin{bmatrix} D & 0 \\ 0 & C \end{bmatrix}$.

5. An Open Question.

We conclude this note by asking the following question:

If $R \leqq S$ is a finite triangular extension and S has a duality, does R have a duality? The answer is "yes" if, in addition, both ${}_RS$ and S_R are progenerators [7, Theorem 1(2)].

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