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REAL HYPERSURFACES WITH CYCLIC η-PARALLEL RICCI TENSOR OF A COMPLEX SPACE FORM

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Introduction.

The study of real hypersurfaces of a complex projective space P_nC was initiated by Takagi [15], who proved that all homogeneous hypersurfaces of P_nC could be divided into six types which are said to be *of type* A_1, A_2, B, C, D and E. He showed also in [15, 16] that if a real hypersurface M of P_nC has two or three distinct constant principal curvatures, then M is locally congruent to one of the homogeneous ones of type A_1, A_2 and B. This result is generalized by Kimura [5], who proves that a real hypersurface M of P_nC has constant principal curvatures and the structure vector ξ is principal if and only if M is locally congruent to one of the homogeneous real hypersurfaces. In particular, real hypersurfaces of type A_1, A_2 and B of P_nC have been studied by several authors (cf. Cecil and Ryan [2], Kimura [5], Maeda [6] and Okumura [10]).

On the other hand, real hypersurfaces of a complex hyperbolic space H_nC have also been investigated from different points of view and there are some studies by the authors [9], Montiel [11], Montiel and Romero [12]. In particular, real hypersurfaces of H_nC, which are said *of type* A_0, A_1 and A_2 were treated by Montiel and Romero [12].

Now, let M be a real hypersurface of a complex space form $M_n(c)$, $c\neq 0$. The Ricci tensor S of M is said to be *cyclic-parallel* if it satisfies

$$(*) \qquad \mathfrak{S}\nabla S'(X, Y, Z)=\mathfrak{S}g(\nabla_X S(Y), Z)=0$$

for any vector fields X, Y and Z, where $\mathfrak{S}$ and ∇ denote the cyclic sum and the Riemannian connection, respectively. It is seen that the Ricci tensor of a real hypersurface of type A_1 or A_2 (resp. A_0, A_1 or A_2) of P_nC (resp. H_nC) is cyclic-parallel. In a previous paper [9], this converse is investigated. On the other hand, the notion of a η-parallel shape operator is recently introduced by Kimura and Maeda [8]. A shape operator A is said to be η-parallel, if it satisfies $g(\nabla_X A(Y), Z)=0$ for any vector fields orthogonal to ξ. They prove

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that in a real hypersurface of P_nC the shape operator is η-parallel and ξ is principal if and only if it is of type A_1, A_2 or B.

The Ricci tensor S is said to be *cyclic-η-parallel*, if it satisfies (*) for any vector fields X, Y and Z orthogonal to ξ. The purpose of this note is to prove the following

Theorem. *Let M be a real hypersurface of $M_n(c)$, $c\neq 0$, $n\geqq 3$, on which the structure vector ξ is principal. Then the Ricci tensor is cyclic-η-parallel if and only if M is locally congruent to one of real hypersurfaces of type $A_1\sim B$ of P_nC or of type $A_0\sim B$ of H_nC.*

1. Preliminaries.

Let M be a real hypersurface of an $n(\geqq 2)$-dimensional complex space form $M_n(c)$ of constant holomorphic curvature c $(\neq 0)$ and let C be a unit normal field on a neighborhood of a point x in M. We denote by J an almost complex structure of $M_n(c)$. For a local vector field X on a neighborhood of x in M, the transformations of X and C under J can be represented as

$$JX=\phi X+\eta(X)\xi\,, \qquad JC=-\xi\,,$$

where ϕ defines a skew-symmetric transformation on the tangent bundle TM of M, while η and ξ denote a 1-form and a vector field on a neighborhood of x in M, respectively. Moreover, it is seen that $g(\xi, X)=\eta(X)$, where g denotes the induced Riemannian metric on M. By properties of the almost complex structure J, a set (ϕ, ξ, η, g) of tensors satisfies then

$$\phi^2=-I+\eta\otimes\xi, \quad \phi\xi=0, \quad \eta(\phi X)=0, \quad \eta(\xi)=1\,, \tag{1.1}$$

where I denotes the identity transformation. Accordingly, the set defines an almost contact metric structure. Furthermore the covariant derivatives of the structure tensors are given by

$$\nabla_X\phi(Y)=\eta(Y)AX-g(AX, Y)\xi\,, \qquad \nabla_X\xi=\phi AX\,, \tag{1.2}$$

where ∇ is the Riemannian connection of g and A denotes the shape operator with respect to C on M.

Since the ambient space is of constant holomorphic curvature c, the equations of Gauss and Codazzi are respectively given as follows:

$$\begin{aligned}R(X, Y)Z=c\{&g(Y, Z)X-g(X, Z)Y\\&+g(\phi Y, Z)\phi X-g(\phi X, Z)\phi Y-2g(\phi X, Y)\phi Z\}/4\\&+g(AY, Z)AX-g(AX, Z)AY\,,\end{aligned} \tag{1.3}$$

$$\nabla_X A(Y)-\nabla_Y A(X)=c\{\eta(X)\phi Y-\eta(Y)\phi X-2g(\phi X, Y)\xi\}/4, \tag{1.4}$$

where R denotes the Riemannian curvature tensor of M and $\nabla_X A$ denotes the covariant derivative of the shape operator A with respect to X.

The Ricci tensor S' of M is the tensor of type (0, 2) given by $S'(X, Y)=\operatorname{tr}\{Z\to R(Z, X)Y\}$. But it may be also regarded as the tensor of type (1, 1) and denoted by $S: TM\to TM$; it satisfies $S'(X, Y)=g(SX, Y)$. By the Gauss equation, (1.1) and (1.2) the Ricci tensor S is given by

$$S=c\{(2n+1)I-3\eta\otimes\xi\}/4+hA-A^2, \tag{1.5}$$

where h is the trace of the shape operator A. The covariant derivative of S is also given by

$$\begin{aligned}\nabla_X S(Y)=&-3c\{g(\phi AX, Y)\xi+\eta(Y)\phi AX\}/4\\ &-dh(X)AY+(hI-A)\nabla_X A(Y)+\nabla_X A(AY).\end{aligned} \tag{1.6}$$

Now, some fundamental properties about the structure vector ξ are stated here for later use. First of all, we have the following fact, which is proved by Maeda [10] and Ki and Suh [4], according as $c>0$ and $c<0$.

Proposition A. *Let M be a real hypersurface of $M_n(c)$, $c\neq 0$. If the structure vector ξ is principal, then the corresponding principal curvature α is locally constant.*

In the sequel, that the structure vector ξ is principal, that is, $A\xi=\alpha\xi$ is assumed. It follows from (1.4) that we have

$$2A\phi A=c\phi/2+\alpha)A\phi+\phi A) \tag{1.7}$$

and therefore, if $AX=\lambda X$ for any vector field X, then we have

$$(2\lambda-\alpha)A\phi X=(\alpha\lambda+c/2)\phi X. \tag{1.8}$$

Accordingly, it turns out that in the case where $\alpha^2+c\neq 0$, ϕX is also a principal vector with principal curvature $\lambda'=(\alpha\lambda+c/2)/(2\lambda-\alpha)$, namely, we have

$$\begin{cases} 2\lambda-\alpha\neq 0, \\ A\phi X=\lambda'\phi X, \qquad \lambda'=(\alpha\lambda+c/2)/(2\lambda-\alpha). \end{cases} \tag{1.9}$$

On the other hand, for any principal curvature λ we find

$$d\lambda(\xi)=0 \tag{1.10}$$

by the Codazzi equation (1.4) and Proposition A. In fact, the Codazzi equation gives $\nabla_X A(\xi)-\nabla_\xi A(X)=-c\phi X/4$ for any X orthogonal to ξ. Accordingly, for any principal vector X in $\xi^\perp$ with principal curvature λ, we have $g(\nabla_X A(\xi)-$

$\nabla_\xi A(X), X)=(\alpha-\lambda)g(\nabla_X\xi, X)+d\lambda(\xi)g(X, X)$, which implies that $d\lambda(\xi)=0$, because of (1.2). This is due to Kimura and Maeda [8].

Let $A(\lambda)$ be an eigenspace of A with the eigenvalue λ. A subspace $\xi_x{}^\perp$ of the tangent space T_xM at x consisting of vectors orthogonal to ξ_x can be then decomposed as

$$\xi_x{}^\perp=A(\lambda_1)\oplus A(\lambda_2)\oplus\cdots\oplus A(\lambda_p)\,. \tag{1.11}$$

By P the operator defined by A^2-hA is denoted. Then, for any vector fields X, Y and Z in $\xi^\perp$ we have

$$g(\nabla_X S(Y), Z)=-g(\nabla_X P(Y), Z)\,, \tag{1.12}$$

where

$$\begin{aligned} g(\nabla_X P(Y), Z)=&g(\nabla_X A(AY), Z)+g(\nabla_X A(Y), AZ)\\ &-dh(X)g(AY, Z)-hg(\nabla_X A(Y), Z)\,. \end{aligned} \tag{1.13}$$

In particular, for any $X\in A(\lambda)$, $Y\in A(\mu)$ and $Z\in A(\sigma)$ we get

$$g(\nabla_X P(Y), Z)=(\mu+\sigma-h)g(\nabla_X A(Y), Z)-dh(X)g(AY, Z)\,. \tag{1.14}$$

When we define $\nabla P(X, Y, Z)=g(\nabla_x P(Y), Z)$, it follows from (1.14) that we have

$$\begin{aligned} \mathfrak{S}\nabla P(X, Y, Z)=&\{2(\lambda+\mu+\sigma)-3h\}g(\nabla_X A(Y), Z)\\ &-\{\mu dh(X)g(Y, Z)+\sigma dh(Y)g(Z, X)+\lambda dh(Z)g(X, Y)\}. \end{aligned} \tag{1.15}$$

On the other hand, it is easily seen that we get

$$g(\nabla_X A(Y), Z)=d\mu(X)g(Y, Z)+(\mu-\sigma)g(\nabla_X Y, Z)\,. \tag{1.16}$$

2. Proof of Theorem.

Let M be a real hypersurface of a complex space form $M_n(c)$, $c\neq 0$, on which the structure vector ξ is principal. Assume that the Ricci tensor S is cyclic-η-parallel, that is, it satisfies

$$\mathfrak{S}\nabla S'(X, Y, Z)=0$$

for any vector fields in $\xi^\perp$. This is equivalent to $\mathfrak{S}\nabla P(X, Y, Z)=0$. Putting $X=Y=Z$ in (1.15), we have

$$(2\lambda-h)g(\nabla_X A(X), X)-\lambda dh(X)=0$$

where X is a unit vector in $A(\lambda)$. By (1.16) it is reformed to

$$(2\lambda-h)d\lambda(X)-\lambda dh(X)=0\,. \tag{2.1}$$

On the other hand, putting again $Y=Z$ in (1.15) and supposing that X and Y

are orthonormal, and making use of (1.16), we get

(2.2) $$(2\lambda+4\mu-3h)d\mu(X)-\mu dh(X)=0\,.$$

First of all, the following property is verified.

Lemma 2.1. *Let M be a real hypersurface of $M_n(c)$, $c\neq 0$, $n\geqq 3$, on which ξ is principal. If the Ricci tensor is cyclic-η-parallel, then the mean curvature of M is constant.*

Proof. Suppose that $\alpha^2+c=0$. Then, without less of generality, we may suppose that there are at least one principal curvatures, say λ, different from $\alpha/2$. For any X in $A(\lambda)$, it is seen that ϕX belongs to $A(\lambda')$, where $\lambda'=\alpha/2$ by means of (1.8). Applying (2.2) to the pair $(X, \phi X)$, one gets $\alpha dh(X)=0$. Furthermore, for any $Y\in A(\alpha/2)$, it follows from (2.1) that $\alpha dh(Y)=0$. On the other hand, since h is constant along the ξ-direction, the mean curvature is constant.

The case where $\alpha^2+c\neq 0$ is next considered. For any vector X in $A(\lambda)$, ϕX is also principal and the corresponding principal curvature λ' is given by $(\alpha\lambda+c/2)/(2\lambda-\alpha)$. Because of

$$d\lambda'(X)=-(\alpha^2+c)d\lambda(X)/(2\lambda-\alpha)^2\,,$$

combining it together with (2.2), we have

$$(\alpha^2+c)(2\lambda+4\lambda'-3h)d\lambda(X)+\lambda'(2\lambda-\alpha)^2dh(X)=0$$

for any X in $A(\lambda)$. Since $d\lambda(X)$ can be substituted from (2.1) and the above equation, the following equation holds:

$$\{(\alpha^2+c)\lambda(2\lambda+4\lambda'-3h)+\lambda'(2\lambda-h)(2\lambda-\alpha)^2\}dh(X)=0$$

for any $X\in A(\lambda)$. For any fixed X in $A(\lambda)$, a connected component of a subset of M consisting of points x at which $dh(X)(x)\neq 0$ is denoted by $M(X)$. Suppose that $M(X)$ is not empty. Then we have

(2.3) $$8\alpha\lambda^4-4(\alpha^2+h\alpha-2c)\lambda^3+(4\alpha^3-2h\alpha^2-2c\alpha-8ch)\lambda^2$$
$$+(2h\alpha^3+3c\alpha^2+5ch\alpha+2c^2)\lambda-ch\alpha^2/2=0\,.$$

Suppose that $\alpha=0$. Then the corresponding principal curvatures λ and λ' satisfy $\lambda\lambda'=c/4$ and on $M(X)$ they satisfy also $\lambda(\lambda^2-h\lambda+c/4)=0$ by (2.3). Because of $2\lambda-\alpha\neq 0$, we have $\lambda\neq 0$ and hence $\lambda^2-h\lambda+c/4=\lambda(\lambda-h+\lambda')=0$, which yields that $\lambda+\lambda'=h$. Now, for any principal curvature μ different from α, λ and λ', it follows from the equation (2.2) that we have

$$(2\lambda\mu'+c-3h\mu')d\mu(X)=cdh(X)/4\,,$$

where μ' denotes the principal curvature corresponding to μ and they satisfy $\mu\mu'=c/4$. Similarly, we get

$$(2\lambda\mu+c-3h\mu)d\mu'(X)=cdh(X)/4\,.$$

By adding these two equations and by making use of the fact that $\mu\mu'$ is constant, the following relationship $d(\mu+\mu')(X)=dh(X)/2$ is given, because of $c\neq 0$. Let $\lambda_1, \cdots, \lambda_{2p}$ be mutually different principal curvatures except for α such that $\lambda_1=\lambda$, $\lambda_{p+1}=\lambda'$ and $\lambda_{p+r}=\lambda_r'$. Then h is given by $\sum_{r=1}^{p} n_r(\lambda_r+\lambda_{p+r})$, where n_r denotes the multiplicity of λ_r. Accordingly we have $(n_1-1)h+\sum_{r=2}^{p}(\lambda_r+\lambda_r')=0$, from which it follows that we have $(n_1-1)dh(X)+\sum_{r=2}^{p}d(\lambda_r+\lambda_r')(X)=0$. Combining these two equations, we have

$$(n+n_1-3)dh(X)=0\,.$$

on $M(X)$, which contradicts to the assumption $n\geqq 3$, because of $n_1\geqq 1$. This means that the fact $\alpha\neq 0$ holds. Thus, by differentiating (2.3) in the direction of X and by taking account of (2.1), the simple straightforward calculation gives rise to

$$24\alpha\lambda^4-8(2\alpha^2+h\alpha-c)\lambda^3+2(6\alpha^3-h\alpha^2+3c\alpha-4ch)\lambda^2 \tag{2.4}$$
$$+2(c\alpha^2+c^2)\lambda+ch\alpha^2/2=0\,.$$

Since (2.3) and (2.4) can be regarded as linear equations with the variable h and they are also linearly independent, we can eliminate the function h from these two equations and the argument gives us an equation with the variable λ of degree 7 and with constant coefficients. This means that λ must be constant on $M(X)$ and hence it turns out that $\lambda dh(X)=0$ by (2.1), that is, $\lambda=0$ on the subset $M(X)$. Accordingly, we get $c\alpha^2h=0$ by (2.3) and hence the function h vanishes identically on $M(X)$, a contradiction.

Consequently, the subset $M(X)$ is empty and we have $dh(X)=0$ for any vector field $X\in A(\lambda)$ and any principal curvature λ, which completes the proof.

Proof of Theorem. By Lemma 2.1 the mean curvature may be assumed to be constant, and hence the function h is constant. Then (2.1) and (2.2) are simplified as

$$(2\lambda-h)d\lambda(X)=0\,, \qquad (2\lambda+4\mu-3h)d\mu(X)=0\,. \tag{2.5}$$

For any fixed distinct principal curvatures λ and μ, let M_0 be a connected component of a subset of M consisting points x at which $(2\lambda-h)(x)\neq 0$ holds. Since M_0 is open, $d\lambda(X)$ vanishes identically on M_0. Let M_1 be a connected component of the interior of the complement $M-M_0$ of M_0, if there exists. Then λ is equal to $h/2$ on M_1 and it is constant, so we get $d\lambda(X)=0$ on M_1, which means by the continuity of principal curvatures that λ is constant along the distribution $A(\lambda)$. Next, let M_2 be a subset of M consisting of point x such that $(2\lambda+4\mu-3h)(x)\neq 0$. Then (2.2) implies that $d\mu(X)=0$ on M_2. Since we have $4\mu=3h-2\lambda$ on a connected component of the interior of the complement of M_2, we have $2d\mu(X)=-d\lambda(X)$ on it, which means that $d\mu(X)$ vanishes identically on M. Thus the principal curvature μ different from λ is also constant along the distribution $A(\lambda)$ and hence it yields that any principal curvature λ is constant along the $\xi^{\perp}$-direction. While it is already seen that λ is constant along the ξ-direction, any principal curvature is constant on M. By the classification theorems of real hypersurfaces of $M_n(c)$, $c\neq 0$, due to Takagi [15], Kimura [5] and Berndt [1], M is locally congruent to one of real hypersurfaces of type $A_1\sim E$ of $P_n\boldsymbol{C}$ or of type $A_0\sim B$ of $H_n\boldsymbol{C}$.

Let M be a real hypersurface of type $A_1\sim B$ of $P_n\boldsymbol{C}$ or of type $A_0\sim B$ of $H_n\boldsymbol{C}$. By the characterization theorems of the η-parallel shape operator due to Kimura and Maeda [8] and Suh [14], the shape operator A is η-parallel. Since the subspace $\xi^{\perp}$ is A-invariant, it follows from (1.12) and (1.13) that P is η-parallel and hence so is the Ricci tensor S. Accordingly, the Ricci tensor of M is cyclic-η-parallel.

In order to prove the theorem, it suffices to show that the real hypersurface of type C, D or E can not occur. Let M be a real hypersurface of type C, D or E. Suppose that the Ricci tensor S is cyclic-η-parallel. By the classification theorem due to Takagi [15], M has mutually distinct five constant principal curvatures, say $\alpha=\sqrt{c}\cot 2\theta$, $0<\theta<\pi/4$, $\lambda_1=\sqrt{c}/2\cot\theta$, $\mu_1=-\sqrt{c}/2\tan\theta$, $\lambda_2=\sqrt{c}/2\cot(\theta-\pi/4)$ and $\mu_2=-\sqrt{c}/2\tan(\theta-\pi/4)$. Thus we have for any $X\in A(\lambda)$, $Y\in A(\mu)$ and $Z\in A(\sigma)$

$$\mathfrak{S}\nabla P(X, Y, Z)=\{2(\lambda+\mu+\sigma)-3h\}g(\nabla_x A(Y), Z)=0 \tag{2.6}$$

by (1.15), because h is constant.

Suppose that $\mu\neq\sigma$. There exists a 1-parameter family of real hypersurfaces $M(\theta)$ of type C, D or E of $P_n\boldsymbol{C}$ and $2(\lambda+\mu+\sigma)-3h$ and $g(\nabla_X A(Y), Z)$ are both smooth functions. Moreover, there is a unique value θ_0 for each of type C, D or E and such that $\{2(\lambda+\mu+\sigma)-3h\}(\theta_0)=0$, because of $h=p\alpha-cq/\alpha$, where $0<\alpha<\infty$ and, $p=n-2$, 4 or 8 and $q=2$, 4 or 6, according as M is of type C, D or E. This means that $g(\nabla_X A(Y), Z)=0$ except for θ_0 and hence $g(\nabla_X A(Y), Z)=0$ for any $Y\in A(\mu)$, $Y\in A(\sigma)$, $\mu\neq\sigma$. Since it is trivial by (1.16) that $g(\nabla_X A(Y), Z)=0$ for any $Y, Z\in A(\mu)$, the shape operator A must be η-parallel.

This completes the proof.

Let M be a real hypersurface of $M(c)$, $c\neq 0$, whose Ricci tensor is cyclic-parallel and whose structure vector is principal. Then it is already shown by authors [9] that all principal curvatures are constant. Accordingly, we can apply our theorem to this situation, which means that the real hypersurfaces of type C, D and E can not occur. Theorem 5.3 in [9] together with this fact shows that the following theorem holds.

Theorem 2.2. *Let M be a complete and connected real hypersurface of P_nC. If the Ricci tensor is cyclic-parallel and if the structure vector is principal, then M is congruent to one of $M_0(2n-1, t)$, $M(2n-1, m, t)$ and $M(2n-1, 1/(3n-2))$.*

Remark 2.1. For the definition of the above hypersurfaces, refer to cf. the authors [9]. It is seen in Cecil and Ryan [2] that the hypersurface $M_0(2n-1, t)$ is a tube of radius $r=\cos^{-1}(t/(t+1))^{1/2}$ over a totally geodesic $P_{n-1}C$, which is of type A_1. The hypersurface $M(2n-1, m, t)$ is a tube of radius $r=\cos^{-1}(t/(t+1))^{1/2}$ over a totally geodesic P_mC, $0<m<n-1$, which is of type A_2. The hypersurface of $M(2n-1, t)$ is a tube of radius $r=(1/2)\cos^{-1}t$ over a complex quadric Q_{n-1}, which is of type B,

3. A complex hyperbolic space.

In this section, we are concerned with real hypersurfaces of a complex hyperbolic space H_nC. Some properties about real hypersurfaces of H_nC are already investigated by Montiel [11], Montiel and Romero [12] and so on. In particular, it is proved in [12] that a complete and connected real hypersurface of H_nC is congruent to $M_{p,q}(t)$ or M_n if it satisfies $A\phi-\phi A=0$. These hypersurfaces are explained in the next paragraph. We shall here investigate whether or not Theorem 2.2 holds in the case of H_nC.

In order to investigate real hypersurfaces of H_nC with cyclic-parallel Ricci tensors, some standard examples of real hypersurfaces of H_nC whose Ricci tensors are cyclic-parallel are given. In a complex Euclidean space C^{n+1} with the standard basis, let F be a Hermitian form F defined by

$$F(z, w)=-z_0\bar{w}_0+\sum_{k=1}^{n} z_k\bar{w}_k\,,$$

where $z=(z_0, \cdots, z_n)$ and $w=(w_0, \cdots, w_n)$ are in C^{n+1}. Then (C^{n+1}, F) is a Minkowski space, which is simply denoted by C_1^{n+1}. A scalar product given by $F(z, w)$ is a semi-Riemannian metric of index 2 in C_1^{n+1}. Let H_1^{2n+1} be a real hypersurface of C_1^{n+1} defined by

$$H_1^{2n+1}=\{z\in C_1^{n+1}: F(z, z)=-1\},$$

and let G be a semi-Riemannian metric of H_1^{2n+1} induced from the complex Lorentz metric $\mathrm{Re}\,F$ of C_1^{n+1}. Then (H_1^{2n+1}, G) is the Lorentz manifold of constant curvature -1, which is called an *anti-de Sitter space*. For any point z of H_1^{2n+1} the tangent space $T_zH_1^{2n+1}$ can be identified with $\{w\in C_1^{n+1}: \mathrm{Re}\,F(z, w)$

$=0\}$. Let T'_z be an orthogonal complement of the vector iz in $T_zH_1^{2n+1}$. When the anti-de Sitter space H_1^{2n+1} is considered as a principal fiber bundle over $H_n\boldsymbol{C}$ with the structure group S^1 and the projection π, there is a connection such that T'_z is the horizontal subspace at z which is invariant under the S^1-action. The metric g of constant holomorphic sectional curvature -4 is given by $g_p(X, Y)=\mathrm{Re}\, F_z(X^*, Y^*)$ for any tangent vectors X and Y in $T_p(H_n\boldsymbol{C})$, where z is any point of H_1^{2n+1} with $\pi(z)=p$ and, X^* and Y^* are vectors in T'_z such that $d\pi X^*=X$ and $d\pi Y^*=Y$. On the other hand, a complex structure $J: w\to iw$ in T'_z is compatible with the action of S^1 and induces the almost complex structure J on $H_n\boldsymbol{C}$ such that $d\pi\circ i=J\circ d\pi$. Thus $H_n\boldsymbol{C}$ is a complex hyperbolic space with constant holomorphic curvature -4 and it is seen that H_1^{2n+1} is a principal S^1-bundle over a complex hyperbolic space $H_n\boldsymbol{C}$ with the projection $\pi: H_1^{2n+1}\to H_n\boldsymbol{C}$, which is a semi-Riemannian submersion with the fundamental tensor J and time-like totally geodesic fibres.

Now, for given integers p and q with $p+q=n-1$ and $t\in\boldsymbol{R}$ with $0<t<1$, a Lorentz hypersurface $N_{p,q}(t)$ of H_1^{2n+1} is defined by

$$N_{p,q}(t)=\{(z_0, \cdots, z_n)\in H_1^{2n+1}: r(-|z_0|^2+\textstyle\sum_{j=1}^{p}|z_j|^2)=-\sum_{j=p+1}^{n}|z_j|^2\}$$

and a Lorentz hypersurface N_n of H_1^{2n+1} is given by

$$N_n=\{(z_0, \cdots, z_n)\in H_1^{2n+1}: |z_0-z_1|^2=1\}.$$

Lastly, for any fixed $t\in\boldsymbol{R}$ with $t>1$, let $N(t)$ be a real hypersurface of $H_n\boldsymbol{C}$ defined by

$$N(t)=\{(z_0, \cdots, z_n)\in H_1^{2n+1}\subset\boldsymbol{C}^{n+1}: |-z_0^2+\textstyle\sum_{j=1}^{n}z_j^2|^2=t\}.$$

Then it is seen that $N_{p,q}(t)$, N_n and $N(t)$ are all isoparametric hypersurface of H_1^{2n+1}.

For a real hypersurface M of $H_n\boldsymbol{C}$ it is known that we can construct a real hypersurface N of H_1^{2n+1} which is a principal S^1-bundle over M with totally geodesic fibres and the projection π. Moreover, the projection is compatible with the Hopf fibration $\pi: H_1^{2n+1}\to H_n\boldsymbol{C}$, that is, the diagram

$$\begin{array}{ccc} N & \xrightarrow{\ i'\ } & H_1^{2n+1} \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ M & \xrightarrow[\ i\ ]{} & H_n\boldsymbol{C} \end{array}$$

is commutative (i' and i being the respective immersions). Then $M_{p,q}(t)=\pi(N_{p,q}(t))$ is a real hypersurface of $H_n\boldsymbol{C}$, which is a tube of radius r over a totally geodesic submanifold $H_p\boldsymbol{C}$ imbedded in $H_n\boldsymbol{C}$, where $\sqrt{t}=\tanh r$. It is said to be of type A_1 or A_2, according as $p=0$, $n-1$ or $0<p<n-1$ and it is

seen by the authors [9] that these Ricci tensors are cyclic-parallel. $M_n=\pi(N_n)$ and $M(t)=\pi(N(t))$ ($n\geqq 3$) are examples of real hypersurfaces of H_nC. The former is totally η-umbilical with principal curvatures 1 and 2, which is said to be *of type* A_0. The real hypersurface M_n of type A_0 has also cyclic-parallel Ricci tensor, which is also called a *Montiel tube.* The latter $M(t)$ is a tube of radius r over a totally real n-dimensional submanifold H_nR of H_nC, where $t=\cosh^2 2r$, which is said to be *of type* B.

Theorem 3.1. *Let M be a complete and connected real hypersurface of H_nC. If the Ricci tensor is cyclic-parallel and if the structure vector is principal, then M is congruent to one of M_n, $M_{2n-1,0}(t)$, $M_{0,2n-1}(t)$ or $M_{p,q}(t)$, $0<p,\ q<n-1$.*

Proof. Let M be a real hypersurface of H_nC. Assume that ξ is principal and the Ricci tensor is cyclic-parallel. Then it is in [9, Theorem 3.2] that all principal curvatures are constant. So h is constant and M is locally congruent to one of real hypersurfaces of type A_0, A_1, A_2 or B by Theorem. It is essentially due to Berndt [1]. In order to prove Theorem 3.1, it suffices to show that the ease of type B can not occur, because it is seen in [9] that M_n and $M_{p,q}(t)$ have cyclic-parallel Ricci tensors. Let M be a real hypersurface of type B of H_nC. Suppose that the Ricci tensor is cyclic-parallel. Since α is constant by Ki and Suh [4], the equation (3.3) in [9] means that $\mathfrak{S}\nabla S'(X,U,\xi)=0$ for any X, Y in $\xi^{\perp}$ is equivalent to

$$2\alpha(A^2\phi-\phi A^2)-k(A\phi-\phi A)=0\,,\qquad k=(3h\alpha-2c-2\alpha^2)\,.$$

Because of $\alpha^2+c\neq 0$, it is reduced to

$$(\lambda-\mu)\{\alpha(\lambda+\mu)-k\}=0\,,$$

where $\alpha=\sqrt{-c}\tanh r$, $\lambda=\sqrt{-c}/2\coth r$ and $\mu=\sqrt{-c}/2\tanh r$. Accordingly, $\lambda\neq\mu$ and $\lambda+\mu=-c/\alpha$, and hence it follows from the above equation that $h=2\alpha/3$. On the other hand, since the multiplicities of λ and μ are equal in the real hypersurface of type B, the constant h is given by $h=\alpha+(n-1)(\lambda+\mu)$. Thus we get $\alpha^2-3c(n-1)=0$, a contradiction.

This completes the proof.

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