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THE SYMMETRIC SPACE $SO(2n)/U(n)$

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1. Introduction and statement of main theorem.

Our aim here is to characterise the Riemannian Hermitian Symmetric space $SO(2n)/U(n)$ by means of a particular parallel tensor field T of type (1, 3) and the Weingarten map on geodesic spheres. This continues earlier analogous results on Grassmannians [1], [4], [5] which, in turn, extended the characterisation of spaces of constant curvature and spaces of constant holomorphic sectional curvature due to L. Vanhecke and T. J. Willmore [7].

The tangent space at any point $m \in SO(2n)/U(n)$ can be identified with the vector space $S(n)$ of all complex skew-symmetric matrices of order n considered as a real vector space with inner product

(1.1) $$g(X, Y)=\frac{1}{2}\mathrm{re}\ \mathrm{tr}\ X\bar{Y}^t .$$

This is Hermitian with respect to the complex structure J corresponding to multiplication by $i=\sqrt{-1}$, and an invariant Kaehler metric g is then defined on $SO(2n)/U(n)$. We remark that $U(n)$ acts on $S(n)$ as a group of congruences. The corresponding Riemannian curvature tensor at m is represented by its action on $S(n)$ by

(1.2) $$R(X, Y)Z=X\bar{Y}^tZ+Z\bar{Y}^tX-Y\bar{X}^tZ-Z\bar{X}^tY .$$

For the non-compact dual $SO^*(2n)/U(n)$ the curvature tensor is just the negative of the above expression and it is sufficient to consider only the compact case. Of course the metric g can be replaced by any metric homothetic to it without affecting R.

The tensor T of type (1, 3) defined at m by

(1.3) $$T(X, Y, Z)=X\bar{Y}^tZ+Z\bar{Y}^tX$$

is invariant by $U(n)$ and so extends to a parallel tensor field on $SO(2n)/U(n)$ alsd denoted by T. We define endomorphisms T_{XY} and T^X_Y of the tangent space at m by

$$T_{XY}Z=T(X, Y, Z) \quad \text{and} \quad T^X_YZ=T(X, Z, Y).$$

Also, for any tangent vector X at m we may regard the (1, 1) tensor JT_{XX}, defined by $JT_{XX}Y=J(T_{XX}Y)$, as a derivation on the tensor algebra at m. Then it is easily verified that T has the following properties at m and hence on $SO(2n)/U(n)$:

P_1: $T(X, Y, Z)=T(Z, Y, X)$;

P_2: $JT(X, Y, Z)=T(JX, Y, Z)=-T(X, JY, Z)$;

P_3: (i) $JT_{XX}g=0$, (ii) $JT_{XX}T=0$;

P_4: (i) $\mathrm{tr}\, T_{XX}=4(n-1)g(X, X)$,

(ii) $\mathrm{tr}\,(T^X_X)^2=16\,(g(X, X))^2-4g(T(X, X, X), X)$.

We remark that the coefficients of g in P_4 could be simplified by writing $4g$ in place of g but the present choice is preferred because of later work. Particular use will be made of unit vectors at m satisfying $T(X, X, X)=2X$. These are characterised by the following lemma which is easily proved using elementary matrix methods.

Lemma 1.1. *Let X be any complex $p\times q$ matrix of rank $r>0$ and write $a=(1/r)\,\mathrm{tr}\, XX^t$. Then $X\bar{X}^tX=\alpha X$, for some $\alpha\in \boldsymbol{C}$, if and only if $X\bar{X}^t$ has a as its only non-zero eigenvalue; in this case $\alpha=a$ and is an r-fold eigenvalue of $X\bar{X}^t$. In particular suppose X is skew-symmetric. Then $X\bar{X}^tX=(1/2)\,\mathrm{tr}\,(X\bar{X}^t)X$ if and only if X has rank 2 or, equivalently, if and only if $X=(a_ib_j-a_jb_i)$ where $\sum a_i\bar{b}_i=0$.*

Now choose a geodesic γ through m with unit tangent vector field N such that $T(N, N, N)=2N$ on γ. This relation holds if and only if it is satisfied at m, and clearly such vectors exist at m because of the above lemma. Then from (1.2),

$$R(JN, N)N=2JN. \tag{1.4}$$

We show that this relation imposes a condition on geodesic spheres. First we make some remarks on the notation used in the remainder of this section. For a spherical normal neighbourhood B of a point m on a Riemannian manifold we denote by N the unit vector field on $B\setminus\{m\}$ which is tangential to geodesic rays from m. Then we define $A=-\nabla N$. For any geodesic sphere S in B with centre m the restriction of A to tangent vectors to S is just the Weingarten map on S with respect to N as unit normal vector field. We call A the spherical Weingarten map and note also that $AN=0$. Furthermore, if γ is a geodesic in B through m then on $\gamma\setminus\{m\}$ the curvature tensor R satisfies

$$R(N, X)N=A^2X-(\nabla_N A)X \tag{1.5}$$

for any vector field X along γ. These results are well known (see [1] for example.)

Next we require the following result again from [1].

Lemma 1.2. *Let m be a point in a Riemannian locally symmetric space M of dimension >2. Then m has a normal neighbourhood B such that, for each unit vector $N_m \in M_m$ and corresponding geodesic γ, the parallel translate of an eigenspace of the linear map $R(N_m-)N_m$ along γ is contained in an eigenspace of the spherical Weingarten map A.*

As an immediate consequence of this lemma and (1.4) we have the following.

Proposition 1.3. *Let $m \in SO(2n)/U(n)$ and choose a normal neighbourhood B of m as in Lemma 1.2. Then the spherical Weingarten map A satisfies the relation*

$$\mathrm{P}_5: \ AJN = fJN$$

for some real-valued function f on $B \setminus \{m\}$ and for N satisfying $T(N, N, N)=2N$ and of unit length.

We remark that, from the definition of A, f is smooth along $\gamma \setminus \{m\}$ for any geodesic γ in B through m.

Our main theorem can now be stated.

Theorem 1.4. *Let M be a non-flat complete simply connected Kaehler manifold of dimension $n(n-1)>2$ with metric g and complex structure J, and let T be a parallel tensor field of type $(1, 3)$ on M satisfying $\mathrm{P}_1-\mathrm{P}_4$. Suppose each $m \in M$ has a normal neighbourhood B on which P_5 is satisfied. Then M is homothetic to the Riemannian symmetric space $SO(2n)/U(n)$ or its non-compact dual.*

2. A characterisation of T.

The proof of the main theorem depends largely on a characterisation of the tangent space structure at a point of $SO(2n)/U(n)$ as described earlier. For this purpose, we require the following result.

Proposition 2.1. *Let Λ be a real vector space of dimension $n(n-1)>2$ with complex structure J and Hermitian inner product $\langle , \rangle$, and let T be a tensor of type $(1, 3)$ on Λ satisfying $\mathrm{P}_1-\mathrm{P}_4$ with $\langle , \rangle$ replacing g. Then there is a complex linear isomorphism of Λ onto the real vector space $S(n)$ of all complex skew-symmetric matrices such that, under identification, $JX=iX$, $T(XYZ)=$ $X\bar{Y}^t X + Z\bar{Y}^t X$* and $\langle X, X \rangle = 1/2 \operatorname{tr} X\bar{X}$.

The proof of the proposition requires several lemmas all relating to Λ under the above assumptions. First, we derive some consequences of property P_3.

Lemma 2.2. (i) *For all* $X\in\Lambda$, T_{XX} *and* T^X_X *are self-adjoint endomorphisms;*
(ii) *for all* $X, Y, Z, U, V\in\Lambda$,

$$\langle T(X, Y, Z), U\rangle=\langle T(Y, X, U), Z\rangle=\langle T(Z, U, X), Y\rangle \tag{2.1}$$

and

$$T_{XY}T(Z, U, V)=T(T_{XY}Z, U, V)-T(Z, T_{YX}U, V)+T(Z, U, T_{XY}V). \tag{2.2}$$

Proof. (i) For T_{XX}, this is an immediate consequence of P_2 and P_3(i). For T^X_X the self-adjoint property follows using (ii).

(ii) Use (i) and linearisation in T_{XX} to obtain

$$\langle T_{XY}Z+T_{YX}Z, U\rangle=\langle Z, T_{XY}U+T_{YX}U\rangle,$$

then replace Y, Z by JY, JZ and use P_2 to get

$$\langle T_{XY}Z-T_{YX}Z, U\rangle=\langle Z, T_{YX}U-T_{XY}U\rangle.$$

Addition gives $\langle T_{XY}Z, U\rangle=\langle Z, T_{YX}U\rangle$, as required and the last part of (2.1) follows using P_1. This also proves the self-adjoint property of T^X_X. Next, from P_3(ii) and P_2,

$$T_{XX}T(U, V, W)=T(T_{XX}U, V, W)-T(U, T_{XX}V, W)+T(U, V, T_{XX}W),$$

and linearisation gives

$$\begin{aligned} T_{XY}T(U, V, W)+T_{YX}T(U, V, W)=&T(T_{XY}U, V, W)+T(T_{YX}U, V, W)\\ &-T(U, T_{XY}V, W)-T(U, T_{YX}V, W)\\ &+T(U, V, T_{XY}W)+T(U, V, T_{YX}W). \end{aligned}$$

Replace X by JX and use P_2 to get

$$\begin{aligned} T_{XY}T(U, V, W)-T_{YX}T(U, V, W)=&T(T_{XY}U, V, W)-T(T_{YX}U, V, W)\\ &+T(U, T_{XY}V, W)-T(U, T_{YX}V, W)\\ &+T(U, V, T_{XY}W)-T(U, V, T_{YX}W). \end{aligned}$$

Then addition gives (2.2) which completes the proof.

We require some further general properties of T.

Lemma 2.3. (i) *For* $X, Y, Z, W\in\Lambda$

$$T(Y, X, T(Z, X, Z))=T(Y, T(X, Z, X), Z), \tag{2.3}$$

$$2T^2_{XX}Y=T^{X^2}_XY+T(T(X, X, X), X, Y), \tag{2.4}$$

$$T(X, Y, T(X, Z, X))=T(X, Z, T(X, Y, X))=T(X, T(Y, X, Z), X), \tag{2.5}$$

$$T^X_XT_{XX}=T_{XX}T^X_X, \tag{2.6}$$

$$T(T(X, Y, X), Z, T(X, W, X))=T(X, T(Y, T(X, Z, X), W), X). \tag{2.7}$$

(ii) *if* $X\neq 0$ *then* $T(X, X, X)\neq 0$.

Proof. (i) Equations (2.3) and (2.4) follow by considering $T_{ZX}T(Z, X, Y)$ and $T_{YX}T(X, X, X)$ then applying (2.2). Again, (2.5) follows by applying (2.2) to $T_{XY}T(X, Z, X)-T_{XZ}T(X, Y, X)$, and (2.6) is a special case of (2.5).

To prove (2.7), we use (2.2) and (2.5) to obtain

$$\begin{aligned}T(T(X, Y, X), Z, T(X, W, X)) &= 2T(T(T(X, Y, X), Z, X), W, X)\\ &\quad -T(X, T(Z, T(X, Y, X), W), X)\\ &= 2T(T(X, T(Y, X, Z), X), W, X)\\ &\quad -T(X, T(Z, T(X, Y, X), W), X)\\ &= 2T(X, T(T(Y, X, Z), X, W), X)\\ &\quad -T(X, T(Z, T(X, Y, X), W), X).\end{aligned}$$

Now (2.7) follows from this equation by noting that linearisation of Z in (2.3) implies

$$2T(T(Y, X, Z), X, W)=T(Z, T(X, Y, X), W)+T(Y, T(X, Z, X), W).$$

(ii) Suppose $X\neq 0$ and $T(X, X, X)=0$. Then (2.4) implies $2T^2_{XX}=T^{X^2}_X$. Also, writing $Z=X$ in (2.5) gives $T_{XX}T^X_X=0$, so $2T^4_{XX}=T^2_{XX}T^{X^2}_X=0$. Since T_{XX} is self-adjoint, we have $T_{XX}=0$ which contradicts P_4(i) and proves (ii).

From now on, for any non-zero $X\in\Lambda$ we write im $T_{XX}=\Lambda_X$, and im $T^X_X=\Lambda^X$; also, for $\lambda\in\boldsymbol{R}$ we write Λ^λ_X (resp. Λ^X_λ) for the corresponding eigenspace of T_{XX} (resp. T^X_X) with the convention that $\Lambda^\lambda_X=\{0\}$ (resp. $\Lambda^X_\lambda=\{0\}$) if λ is not an eigenvalue.

Lemma 2.4. *Suppose X is a unit vector such that $T(X, X, X)=\lambda X$ for some $\lambda\in\boldsymbol{R}$. Then the eigenspaces of T_{XX} and T^X_X are Λ^λ_X, $\Lambda^{\lambda/2}_X$, Λ^0_X, and Λ^X_0, Λ^X_λ, $\Lambda^X_{-\lambda}$ respectively, where $\Lambda^X_0=\Lambda^0_X\oplus\Lambda^{\lambda/2}_X$, $\Lambda^X_\lambda=J\Lambda^X_{-\lambda}$ and $\Lambda^\lambda_X=\Lambda^X_\lambda\oplus\Lambda^X_{-\lambda}=\Lambda^X$. Moreover, for some integer $k>1$, $\lambda=4/k$,* dim *$\Lambda^\lambda_X=k(k-1)$ and* dim *$\Lambda^{\lambda/2}_X=2k(n-k)$, with the convention that $\Lambda^{\lambda/2}_X=\{0\}$ when $k=n$.*

Proof. If $T(X, X, Z)=\theta Z$ then (2.4) and (2.5) imply

$$2\theta^2 Z = T_X^{X^2} Z + \lambda\theta Z\,,$$

$$\theta T_X^X Z = \lambda T_X^X Z\,.$$

Consequently,

$$\theta \neq \lambda \Longrightarrow T_X^X Z = 0 \Longrightarrow \theta = 0 \quad \text{or} \quad \frac{\lambda}{2}\,,$$

$$\theta = \lambda \Longrightarrow T_X^{X^2} Z = \lambda^2 Z \Longrightarrow T_X^X Z = \pm\lambda Z\,,$$

and the relations between eigenspaces of T_{XX} and T_X^X follow immediately where we must allow for the possibility that $\Lambda_X^0 = \{0\}$ or $\Lambda_X^{\lambda/2} = \{0\}$, and note, in particular, that $\Lambda_{-\lambda}^X = J\Lambda_\lambda^X$ because of P_2. Next, suppose $\dim \Lambda_X^\lambda = r$ and $\dim \Lambda_X^{\lambda/2} = s$. Then from P_4,

$$r\lambda + s\frac{\lambda}{2} = 4(n-1)\,,$$

$$r\lambda^2 = 16 - 4\lambda\,.$$

Elimination of λ gives

$$\frac{2r+s}{n-1} = 1 + \sqrt{(1+4r)}$$

from which $1+4r = (2k-1)^2$ for some positive integer k, so $r = k(k-1)$, $s = 2k(n-k)$, and $\lambda = 4/k$ as required.

The next lemma proves the existence of a vector X as above. We first remark that a subspace A of Λ will be called J (resp. T)-invariant if JX (resp. $T(X, Y, Z)$ belong to A whenever $X, Y, Z \in A$.

Lemma 2.5. (i) *If $X \in \Lambda$ and $Y \in \Lambda^X$ then $\Lambda^Y \subset \Lambda^X$.*

(ii) *If A is a non-zero subspace of Λ which is J and T-invariant there exists a unit vector $Y \in A$ such that $T(Y, Y, Y) = \lambda Y$ for some $\lambda \in \mathbf{R}$ and $\Lambda^Y \cap A$ is generated by $\{Y, JY\}$. Moreover, if $\Lambda^X \subset A$ for each $X \in A$ then there exists a unit vector $Y \in A$ such that $T(Y, Y, Y) = Y$; in particular, such a vector Y exists in Λ.*

Proof. (i) If $Y = T(X, W, X)$ and $Z = T(Y, V, Y)$ then, from (2.7), $Z = T(X, T(W, T(X, V, X), W), X) \in \Lambda^X$.

(ii) Choose a non-zero vector $X \in A$ such that $\dim(\Lambda^X \cap A) \leq \dim(\Lambda^Z \cap A)$ for all non-zero $Z \in A$. Note that $\dim(\Lambda^X \cap A) > 0$ since, by (ii) of Lemma 2.3, $T(X, X, X) \neq 0$. Choose any $Y \in \Lambda^X \cap A$ which is non-zero. Then by (i), $\Lambda^Y \cap A = \Lambda^X \cap A$ so T_Y^Y is a non-singular endomorphism of $\Lambda^X \cap A$. Let $T_{YY} Z = \theta Z$ and $T_{YY} W = \varphi W$ where Z, W are non-zero vectors in $\Lambda^X \cap A$. Then using (2.2), $T_{YY} T(Z, W, Z) = (2\theta - \varphi) T(Z, W, Z)$ so $2\theta - \varphi$ is an eigenvalue of $T_{YY} | \Lambda^X \cap A$, noting that $T(Z, W, Z) \neq 0$. If follows that the eigenvalues of T_{YY} are unbounded unless $T_{YY}|\Lambda^X \cap A$ is just a multiple of the identity map I. Thus there is a real valued function f on $\Lambda^X \cap A$ such that if Z is any non-zero vector in $\Lambda^X \cap A$ then $T_{ZZ}|\Lambda^X \cap A = f(Z)\langle Z, Z\rangle I$ and (2.1) shows $f=\lambda$ for some $\lambda \in \boldsymbol{R}$. Next we prove that $\dim \Lambda^X \cap A = 2$. For all $Y, Z, W \in \Lambda^X \cap A$,

$$T(Y, W, Z)+T(W, Y, Z)=2\lambda\langle Y, W\rangle Z$$

hence

$$T(Y, W, Y)=2\lambda\langle Y, W\rangle Y-\lambda\langle Y, Y\rangle W .$$

Now suppose Y is a unit vector orthogonal to W and JW, noting that $\Lambda^X \cap A$ is J-invariant. Then

$$\lambda W=-T(Y, W, Y)=-JT(Y, JW, Y)=\lambda JW ,$$

so $T(Y, W, Y)=0$ which is impossible unless $W=0$. Thus $\Lambda^X \cap A$ is generated by $\{Y, JY\}$ and $T\{Y, Y, Y)=\lambda Y$. In particular, when $\Lambda^Y \subset A$ we see that $\dim \Lambda^Y=2$ so, from Lemma 2.4, $T(Y, Y, Y)=2Y$. Finally, by choosing $A=\Lambda$ we see that this relation must hold for some unit vector $Y \in \Lambda$ and the proof is complete.

From now on we denote by D the subset of vectors X in Λ satisfying $T(X, X, X)=2\langle X, X\rangle X$.

Corollary 2.6. (i) *For any $X \in \Lambda$ let θ be an eigenvalue of T_{XX}. Then Λ_X^θ is J and T-invariant.*

(ii) *Suppose X is a unit vector such that $T(X, X, X)=\lambda X$ for some $\lambda \in \boldsymbol{R}$. Then for each $Y \in \Lambda_X^0$, $T_Y^Y(\Lambda_X^\lambda)=T_Y^Y(\Lambda_X^{\lambda/2})=\{0\}$.*

(iii) *For X as in* (ii), *$\Lambda_X^\lambda \cap D \neq \{0\}$ and $\Lambda_X^0 \cap D \neq \{0\}$.*

Proof. (i) Λ_X^θ is J and T-invariant because of P_2 and (2.2).

(ii) If $Z \in \Lambda_X^\theta$ where $\theta=\lambda$ or $\lambda/2$ then from (2.2), $T(X, X, T(Y, Z, Y))= -\theta T(Y, Z, Y)$ so $T(Y, Z, Y)=0$ since T_{XX} has only λ, $\lambda/2$ as non-zero eigenvalues.

(iii) By Lemma 2.4, $\Lambda_X^\lambda=\Lambda^X$ and $\Lambda^X \cap D \neq \{0\}$ because of Lemma 2.5. Again, suppose $Y \in \Lambda_X^0$. Since Λ has a direct sum decomposition into the 0, λ, $\lambda/2$ eigenspaces of T_{XX} we see from (i) and (ii) that $\Lambda_X^0 \cap D \neq \{0\}$.

Lemma 2.7. *Let X be a unit vector in D. Then*
(i) *$T(X, U, V) \in \Lambda_X$ for all $U, V \in \Lambda$ and $T_{XW}=T_{WX}=0$ for all $W \in \Lambda_X^0$;*
(ii) *$T(X, Y, Z)=\langle Y, Z\rangle X+\langle JY, Z\rangle JX$ for all $Y, Z \in \Lambda_X^1$.*

Proof. (i) For $U, V \in \Lambda$,

$$T(X, U, V)=T_{VU}T(X, X, X)$$
$$=2T(T(V, U, X), X, X)-T(X, T(U, V, X), X)$$

so $T(X, U, V)\in \Lambda_X$. Then for $W\in \Lambda_X^0$,

$$\langle T(W, X, U), V\rangle=\langle T(X, W, V), U\rangle=\langle T(X, U, V), W\rangle=0$$

and (i) follows.

(ii) This follows by noting that, from (2.2), $T(X, Y, Z)\in \Lambda^X$ and then taking inner products.

Lemma 2.8. *Suppose X_1, X_2 are unit vectors in D such that $X_2\in \Lambda_{X_1}^0$; write $X=(1/\sqrt{2})(X_1+X_2)$ and $A=\Lambda_{X_1}^1\cap \Lambda_{X_2}^1$.*

(i) *$\Lambda^X=\Lambda^{X_1}\oplus\Lambda^{X_2}\oplus A$ and $\dim \Lambda^X=12$.*

(ii) *$T_{X_2}^{X_1}(\Lambda)\subset A$ and $T_{X_2}^{X_1}$ restricts to an orthogonal bijection of A of order 2.*

(iii) *If $Y\in A\cap D$ is non-zero then $T_{X_2}^{X_1}(Y)\in \Lambda_Y^0\cap D$, $\dim \Lambda_Y^0\cap A=2$ and $\dim \Lambda_Y^1\cap A=4$.*

(iv) *There exist unit vectors $Y_1, Y_2\in A\cap D$ such that $Y_2\in \Lambda_{Y_1}^1$. Write $Y_3=-T(X_1, Y_2, X_2)$, $Y_4=T(X_1, Y_1, X_2)$. Then A has an orthonormal basis*

$$\{Y_1, Y_2, Y_3, Y_4, JY_1, JY_2, JY_3, JY_4\}$$

of vectors in D.

(v) *If X_3 is a unit vector in $\Lambda_{X_1}^0\cap \Lambda_{X_2}^0\cap D$ then $A\subset \Lambda_{X_3}^0$; if $Y\in A$ then $\Lambda_Y\subset \Lambda_{X_1}\cup \Lambda_{X_2}$.*

Proof. (i) From (i) of Lemma 2.7, $T(X, X, X)=X$. Also, Lemma 2.4 shows that $\Lambda^X=\Lambda_X^1$ with $\dim \Lambda^X=12$. Now if $Z\in \Lambda$ then

$$2T(X, X, Z)=T(X_1, X_1, Z)+T(X_2, X_2, Z)$$

and the direct sum decomposition follows by considering the components of Z in $\Lambda_{X_1}^1$, $\Lambda_{X_1}^0$ and noting that these subspaces are invariant by $T_{X_2X_2}$.

(ii) Lemma 2.7 shows that $T_{X_2}^{X_1}(\Lambda)\subset A$. Next, if $Z\in A$ then

$$0=T_{ZX_2}T(X_1, X_1, X_2)=-T(X_1, T(X_2, Z, X_1), X_2)$$
$$+T(X_1, X_1, T(Z, X_2, X_2))$$

so $T_{X_2}^{X_1}(T_{X_2}^{X_1}Z)=Z$. The restriction of $T_{X_2}^{X_1}$ to A is seen to be orthogonal by taking inner products.

(iii) if Y is a unit vector in $A\cap D$ then from (2.2) and Lemma 2.7, $T_{YX_1}T(X_1, Y, X_2)=X_2$ and $T(X_1, Y, T(X_1, Y, X_2))=0$ since $T(X_1, Y, X_2)\in \Lambda_Y^0$. Then by considering $T_{X_1Y}T(X_2, T(X_1, Y, X_2), T(X_1, Y, X_2))$ we obtain $T(X_1, Y, X_2)\in D$. Next we note that $X_1\in \Lambda_Y^1$ and $\Lambda^{Y+Z}=\Lambda^X$ where $Z=T(X_1, Y, X_2)$. It follows that $\dim(\Lambda_Y^0\cap \Lambda^X)=\dim(\Lambda_{X_1}^0\cap \Lambda^X)=2$ and $\dim(\Lambda_Y^1\cap \Lambda^X)=\dim(\Lambda_{X_1}^1\cap \Lambda^X)$

$=8$; the last part of (iii) is then an easy consequence.

(iv) From Lemma 2.5, there exists a unit vector $Y_1 \in A$ such that, for some $\lambda \in \boldsymbol{R}$, $T(Y_1, Y_1, Y_1)=\lambda Y_1$ and $\dim(\Lambda^Y \cap A)=2$. Now $T(Y_1, Y_1, X_1)=X_1$ so from Lemma 2.4, $\lambda=1$ or 2. If $\lambda=1$ then $\dim \Lambda^{Y_1}=12$ and, from (i) of Lemma 2.5, $\Lambda^{Y_1}=\Lambda^X$ which contradicts $\dim(\Lambda^{Y_1} \cap A)=2$, $\dim A>2$. Thus $\lambda=2$ and $Y_1 \in D$ as required. Since $\dim \Lambda_Y^1 \cap A=4$, the same proof shows the existence of a unit vector $Y_2 \in \Lambda_{Y_1}^1 \cap A \cap D$. The orthonormal basis is now obtained using (ii) and (iii).

(v) Clearly $T_{X_3 X_3}(A) \subset A$ so suppose $Z \in A$ is non-zero and $T_{X_3 X_3} Z=Z$. Then $T(X_3, X_3, T(X_1, Z, X_2))=-T(X_1, Z, X_2)$ which implies $T(X_1, Z, X_2)=0$; but this is impossible from (ii) so $A \subset \Lambda_{X_3}^0$. Next if $Y \in A$ then $Y=T(X_1, U, X_2)$ for some $U \in A$ so for each $Z \in A$

$$\begin{aligned} T(Y, Y, Z) &= T(T(X_1, U, X_2), Y, Z) \\ &= T(X_1, U, T(X_2, Y, Z)-T(X_2, T(U, X_1, Y), Z) \\ &\quad +T(X_2, Y, T(X_1, U, Z)). \end{aligned}$$

Thus from (i) of Lemma 2.7 $\Lambda_Y \subset \Lambda_{X_1}^1 \cup \Lambda_{X_2}^1$ and the proof is complete.

We remark that the minus sign in the definition of Y_3 is chosen for convenience later.

Lemma 2.9. *There exists an orthonormal set of vectors* $\{U_1, U_2, \cdots, U_m\} \subset D$ *where* $m=[n/2]$ *such that* $T(U_i, U_j, U_k)=2\delta_j^i \delta_k^i U_i$.

Proof. Suppose, for some p, $\{U_1, \cdots, U_p\}$ is an orthonormal set of vectors in D satisfying $T(U_i, U_j, U_k)=2\delta_j^i \delta_k^i U_i$. Such a set exists when $p=1$. Write $X=(1/\sqrt{p})(U_1+U_2+\cdots+U_p)$. Then $T(X, X, X)=(1/p)X$. If $p<[n/2]$ then from Lemma 2.4 $\dim \Lambda_X^0=(n-2p)(n-2p-1)>0$, so from (iii) of Corollary 2.6 there exists a unit vector $U_{p+1} \in \Lambda_X^0 \cap D$. Thus $0=T(X, X, U_{p+1})=\sum_{i=1}^{p} T(U_i, U_i, U_{p+1})$ which implies $T(U_i, U_i, U_{p+1})=0$ for $i=1, \cdots, p$ since each $T_{U_i U_i}$ is positive semi-definite. Clearly U_{p+1} is orthogonal to each U_i and the lemma follows using Lemma 2.7 and induction on p.

We now use $\{U_1, \cdots, U_m\}$ to obtain vectors in D which form an orthonormal basis for $\bigoplus_{i=2}^{m} (\Lambda_{U_1}^1 \cap \Lambda_{U_i}^1)$.

Lemma 2.10. *For* $i=2, \cdots, m$, *each* $\Lambda_{U_1}^1 \cap \Lambda_{U_i}^1$ *has an orthonormal basis* $\{e_{\alpha\beta}, Je_{\alpha\beta} : 1 \leq \alpha \leq 2$ *and* $2i-1 \leq \beta \leq 2i\}$ *of vectors in* D *such that for* $1 \leq \alpha, \beta \leq 2$, $2 \leq i \leq m$, $2 \leq j, k, l \leq 2m$,

(i) $e_{2\,2i}=T(U_1, e_{1\,2i-1}, U_i)$, $e_{2\,2i-1}=-T(U_1, e_{1\,2i}, U_i)$;

(ii) $T(e_{\alpha j}, e_{\alpha k}, e_{\alpha l})=e_{\alpha j}\delta_{kl}+e_{\alpha l}\delta_{jk}$;

(iii) $T(e_{\alpha j}, e_{\alpha k}, e_{\beta k})=e_{\beta j}$ *if* $\alpha\neq\beta$;

(iv) $T(e_{\alpha j}, e_{\beta k}, X)=0$ *for all* $X\in\Lambda$ *if* $\alpha\neq\beta$ *and* $j\neq k$.

Proof. (i) First apply Lemma 2.8 with U_1, U_2 replacing X_1, X_2 and e_{13} replacing Y_1. If $m=2$ define $e_{14}=Y_2$, $e_{23}=Y_3$ and $e_{24}=Y_4$ as in the same lemma. Next assume $m>2$. Then

$$T_{e_{13}e_{13}}(\Lambda^1_{U_1}\cap\Lambda^1_{U_i})\subset\Lambda^1_{U_1}\cap\Lambda^1_{U_i} \qquad \text{for } i=3, \cdots, m$$

and the relation

$$T_{e_{13}e_{13}}T(U_1, X, U_i)=T(U_1, X, U_i)-T(U_1, T(e_{13}, e_{13}, X), U_i)$$

shows that $T^{U_1}_{U_i}$ restricts to a bijection of $\Lambda^1_{U_1}\cap\Lambda^1_{U_i}\cap\Lambda^1_{e_{13}}$ onto $\Lambda^1_{U_1}\cap\Lambda^1_{U_i}\cap\Lambda^0_{e_{13}}$. Hence $\dim(\Lambda^1_{U_1}\cap\Lambda^1_{U_i}\cap\Lambda^1_{e_{13}})=4$ and, by a proof similar to that for (iv) of Lemma 2.8, there exists a unit vector $e_{1\,2i-1}\in\Lambda^1_{U_1}\cap\Lambda^1_{U_i}\cap\Lambda^1_{e_{13}}\cap D$. Define $e_{2\,2i}=T(U_1, e_{1\,2i-1}, U_i)$ for $i=2, \cdots, m$ and note that $e_{2\,2i}\in\Lambda^1_{U_1}\cap\Lambda^1_{U_i}\cap\Lambda^0_{e_{13}}\cap\Lambda^0_{e_{1\,2i-1}}\cap D$ and is a unit vector.

Now for $i\geq3$ choose a unit vector $Y\in\Lambda^1_{U_1}\cap\Lambda^1_{U_i}\cap\Lambda^1_{e_{13}}$ orthogonal to $\{e_{1\,2i-1}, Je_{1\,2i-1}\}$. Since $\Lambda^1_{U_1}\cap\Lambda^1_{U_i}\cap\Lambda^1_{e_{13}}$ is J and T-invariant it follows by taking inner products with $e_{1\,2i-1}$ that $\{Y, JY\}\subset D$. Since $\{e_{2\,2i}, Je_{2\,2i}\}\subset\Lambda^0_{1\,2i-1}$ we see that $Y\in\Lambda^1_{e_{1\,2i-1}}$ and then $\Lambda^1_{U_1}\cap\Lambda^1_{U_i}\cap\Lambda^1_{e_{13}}\subset D$. Next, for $i, j\geq3$ suppose $X\in\Lambda^1_{U_1}\cap\Lambda^1_{U_i}\cap\Lambda^1_{e_{13}}$ and $Y\in\Lambda^1_{U_1}\cap\Lambda^1_{U_j}\cap\Lambda^1_{e_{13}}$ are unit vectors with Y orthogonal to $\{X, JX\}$. If $T(X, X, Y)=0$ then from Lemma 2.8, $T(X, e_{13}, Y)\neq0$. But $T(X, e_{13}, Y)\in\Lambda^1_{\bar{U}_2}=\{0\}$ so $T(X, X, Y)\neq0$. Since T_{XX} restricts to an endomorphism of $\Lambda^1_{U_1}\cap\Lambda^1_{U_j}\cap\Lambda^1_{e_{13}}$ it follows that $T(X, X, Y)=Y$. Next we note that $\Lambda^1_{U_1}\cap\Lambda^1_{U_2}\cap\Lambda^1_{e_{13}}\cap\Lambda^1_{e_{15}}$ has dimension 2 and choose a unit vector e_{14} in this subspace. It follows that $e_{14}\in\Lambda^1_{e_{24}}\cap D$ and $e_{23}=-T(U_1, e_{14}, U_2)\in\Lambda^1_{e_{13}}\cap D$. For any unit vector $X\in\Lambda^1_{U_1}\cap\Lambda^1_{U_i}\cap\Lambda^1_{e_{13}}$, $i>3$, we have $X\in D$. If $T(X, X, e_{14})=0$ then $T(e_{14}, e_{15}, X)\neq0$. But $T(e_{14}, e_{15}, X)\in\Lambda^1_{\bar{U}_3}=\{0\}$ and it follows as before that, necessarily, $T(X, X, e_{14})=e_{14}$. Finally, defining $e_{16}=T(e_{13}, e_{23}, e_{26})$, it follows that $e_{16}\in\Lambda^1_{e_{14}}$, also $e_{16}\in\Lambda^0_{e_{13}}\cap\Lambda^1_{e_{23}}$ so is a unit vector by Lemma 2.8. Then, as an immediate consequence of the above relations, we see that if $i\geq3$ and $X, Y\in\Lambda^1_{U_1}\cap\Lambda^1_{U_i}\cap\Lambda^1_{e_{13}}$ are unit vectors with Y orthogonal to $\mathrm{span}\{X, JX\}$ then $T(X, X, Y)=Y$. Similarly, for the same Y and for $X\in\mathrm{span}\{e_{13}, e_{14}, Je_{13}, Je_{14}\}$ we again have $T(X, X, Y)=Y$. Next, for $i=3, \cdots, m$, define $e_{1\,2i}=T(e_{13}, e_{23}, e_{2\,2i})\in\Lambda^1_{U_1}\cap\Lambda^1_{U_i}\cap\Lambda^1_{e_{13}}\cap D$ and $e_{2\,2i-1}=-T(U_1, e_{1\,2i}, U_i)\in\Lambda^1_{U_1}\cap\Lambda^1_{U_i}\cap\Lambda^0_{e_{13}}\cap D$. Then it is immediate from Lemma 2.8 that for $i=2, \cdots, m$ the set $\{e_{\alpha\beta}, Je_{\alpha\beta}: 1\leq\alpha\leq2$ and $2i-1\leq\beta\leq2i\}$ forms an orthonormal basis for $\Lambda^1_{U_1}\cap\Lambda^1_{U_i}$. Also (i) is satisfied by definition.

(ii) As shown above, $T(e_{1i}, e_{1i}, e_{1j})=e_{1j}$ for $i, j\geq2$ and $i\neq j$. The relations

$T(e_{1i}, e_{1i}, e_{2j})=0$ for $i\neq j$ and then $T(e_{2i}, e_{2i}, e_{2j})=e_{2j}$ for $i, j>2$ and $i\neq j$ follow easily. Consequently, (ii) follows from (ii) of Lemma 2.7.

(iii) Clearly this is true when $j=k$ so assume $j\neq k$. There are several cases to consider. Thus

$$\begin{aligned}
&T(e_{1\,2i-1}, e_{1\,2j-1}, e_{2\,2j-i})\\
&\quad=-T(e_{1\,2i-1}, e_{1\,2j-1}, T(U_1, T(e_{13}, e_{23}, T(U_1, e_{1\,2j-1}, U_j)), U_j)\\
&\quad=T(e_{23}, e_{13}, e_{1\,2i-1})\\
&\quad=T_{e_{23}e_{13}}T(U_1, T(U_1, e_{1\,2i-1}, U_i), U_i)\\
&\quad=-T(U_1, (T(e_{13}, e_{23}, T(U_1, e_{1\,2i-1}, U_i)), U_i)\\
&\quad=e_{2\,2i-1}\,.
\end{aligned}$$

Again

$$\begin{aligned}
T(e_{1\,2i-1}, e_{1\,2j}, e_{2\,2j})&=T(T(U_1, e_{2\,2i}, U_i), T(e_{13}, e_{23}, e_{2\,2j}), e_{2\,2j})\\
&=-T(T(U_1, T(e_{13}, e_{23}, e_{2\,2i}), U_i), e_{2\,2j}, e_{2\,2j})\\
&=T(e_{2\,2i-1}, e_{2\,2j}, e_{2\,2j})\\
&=e_{2\,2i-1}\,.
\end{aligned}$$

The same method of proof shows that $T(e_{1\,2i}, e_{1j}, e_{2j})=e_{2\,2i}$ for $j\neq 2i$. The remaining four cases follow by taking inner products and (iv) is proved.

(iv) As a consequence of (ii) and (iii) $T(e_{1j}, e_{1i}, e_{2j})=0$ if $i\neq j$. Then (iv) follows from (i) of Lemma 2.7.

For some positive integer $r\leq n$ let $A=\{\varepsilon_{ij}: 1\leq i<j\leq r\}$ be an orthonormal subset of $\varLambda$, define $\varepsilon_{ji}=-\varepsilon_{ij}$ for $1\leq i<j\leq r$ and write $\tilde{A}=\varepsilon_{ji}: 1\leq i<j\leq r\}$. We call such a subset A regular if the following three relations hold for all vectors in $A\cup\tilde{A}$:

$$R_1: T(\varepsilon_{ij}, \varepsilon_{kl}, X)=0 \qquad \text{for all } X\in\varLambda \text{ if } i, j, k, l \text{ are distinct;}$$

$$R_2: T(\varepsilon_{ij}, \varepsilon_{ik}, \varepsilon_{il})=\varepsilon_{ij}\delta_{kl}+\varepsilon_{il}\delta_{jk}\,;$$

$$R_3: T(\varepsilon_{lk}, \varepsilon_{lj}, \varepsilon_{ij})=\varepsilon_{ik} \qquad \text{if } i\neq l \text{ and } j\neq k\,;$$

We write $JA=\{J\varepsilon_{ij}: \varepsilon_{ij}\in A\}$ and call $A\cup JA$ a regular basis if it is a basis and A is regular.

Next, for $i=1, 2$ define e_{ij} as in Lemma 2.10 and $e_{12}=U_1$; also write $e_{ji}=-e_{ij}$ for $i=1, 2$ and $j=1, \cdots, 2m$. Then define

$$e_{ij}=T(e_{1i}, e_{12}, e_{2j})$$

for $2\leq i\leq 2m$ and $2<j\leq 2m$, noting the consistency when $i=2$, and write $B=\{e_{ij}: 1\leq i<j\leq 2m\}$. Finally, define $e=(1/\sqrt{m})(U_1, +\cdots+U_m)$; thus e is a unit

vector satisfying $T(e, e, e)=(2/m)e$ and dim $\Lambda^e=2m(2m-1)$ from Lemma 2.4.

Proof of Proposition 2.1. We first prove that Λ admits a regular basis $C\cup JC$ by showing that $B\cup JB$, for n even, or a suitable extension of $B\cup JB$, for n odd, forms such a basis. The proof is completed by considering the action of T on $C\cup\tilde{C}$.

To prove that B is regular we begin by showing that B is orthonormal with $e_{ij}=-e_{ji}$ for $1\leq i,\ j\leq 2m$ and that $U_i=e_{2i-1\,2i}$ for $i=1, \cdots, 2m$. Thus, for $2<i,\ j,\ k,\ l\leq 2m$,

$$\begin{aligned}\langle e_{ij}, e_{kl}\rangle &=\langle T(e_{1i}, e_{12}, e_{2j}), T(e_{1k}, e_{12}, e_{2l})\rangle\\ &=\langle T(e_{12}, e_{1i}, T(e_{1k}, e_{12}, e_{2l})), e_{2j}\rangle\\ &=\delta_{ik}\langle e_{2l}, e_{2j}\rangle-\delta_{il}\langle e_{2k}, e_{2j}\rangle\\ &=\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}\,.\end{aligned}$$

Also, for $2<i<j$, $e_{2j}\in\Lambda^0_{e_{1i}}$ so from Lemma 2.8, e_{ij} is a unit vector. This proves the orthogonality of B and the relation $e_{ij}=-e_{ji}$. In particular, $e_{ii}=0$ for $i=1, \cdots, 2m$. Next, for $i>1$,

$$\begin{aligned}e_{2i-1\,2} &=T(e_{1\,2i-1}, e_{12}, e_{2\,2i})\\ &=T_{e_{1\,2i-1}e_{12}}T(U_1, e_{1\,2i-1}, U_i)\\ &=U_i\end{aligned}$$

as required.

Next we prove the relations

$$T(U_i, e_{2i-1\,2j-1}U_j)=e_{2i\,2j},\ T(U_i, e_{2i-1\,2j}, U_j)=-e_{2i\,2j-1}\,. \tag{2.8}$$

We may assume $i>1$. Then

$$\begin{aligned}0 &=T_{e_{12}e_{2\,2j-1}}T(U_i, e_{1\,2i-1}, U_j)\\ &=-T(U_i, e_{2i-1\,2j-1}, U_j)+T(U_i, e_{1\,2i-1}, e_{1\,2j})\,.\end{aligned}$$

Now $e_{1\,2j}\in\Lambda^0_{U_i}$ so by considering $\langle T(U_i, e_{1\,2i-1}, e_{1\,2j}), e_{2i\,2j}\rangle$ and using Lemma 2.8 it follows that $T(U_i, e_{1\,2i-1}, e_{1\,2j})=e_{2i\,2j}$ which proves the first part of (2.8); the second part is proved similarly. We now prove B regular by considering R_1, R_2, R_3.

R_1: This is immediate using R_2 above, (i) of Lemma 2.7 and (v) of Lemma 2.8.

R_2: We first note as a consequence of Lemma 2.8 and 2.9 that Λ^e is the orthogonal direct sum of subspaces $\Lambda^2_{U_i}$ and $\Lambda^1_{U_i}\cap\Lambda^1_{U_j}$, $i\neq j$, for $i, j=1, 2, \cdots, m$. Then, from (2.8) $B\subset\Lambda^e$. Now, for $1\leq i<j\leq m$ and $X\in B\cap\Lambda^1_{U_i}\cap\Lambda^1_{U_j}$, the orthonormal set $\{U_1, \cdots, \hat{U}_i, \cdots, \hat{U}_j, \cdots, U_m, X, T(U_i, X, U_j)\}$ satisfies the same rela-

tions as those of $\{U_1, \cdots, U_m\}$ in Lemma 2.9. Hence $\dim(\Lambda_X^1 \cap \Lambda^e) = \dim(\Lambda_{U_1}^1 \cap \Lambda^e) = \dim \bigoplus_{i=2}^m (\Lambda_{U_1}^1 \cap \Lambda_{U_i}^1) = 8(m-1)$ and it follows using R_1 that if $e_{ij} \in B$ then $\Lambda_{e_{ij}}^1 \cap \Lambda^e$ is generated by

$$\{e_{lm}, J_{e_{lm}} : l=i \text{ or } j\} \setminus \{e_{ij}, e_{ji}, J_{e_{ij}}, J_{e_{ji}}\}.$$

This result together with (ii) of Lemma 2.7 gives R_2.

R_3: Suppose $2 \leq i,\ l \leq 2m$ and $2 < j,\ k \leq 2m$. Then it is easy to verify that $e_{lj}e_{lk}, e_{ij} \in B \cup B'$ and $i \neq l,\ j \neq k$

$$\begin{aligned}
&\Longrightarrow T(e_{2j}, e_{2k}, e_{lk}) = e_{lj} \\
&\Longrightarrow \langle T(e_{2j}, e_{lj}, e_{lk}), e_{2k} \rangle = 1 \\
&\Longrightarrow T(e_{2j}, e_{lj}, e_{lk}) = e_{2k} \\
&\Longrightarrow T(e_{1i}, e_{12}, T(e_{2j}, e_{lj}, e_{lk})) = e_{ik} \\
&\Longrightarrow T(e_{ij}, e_{lj}, e_{lk}) = e_{ik}\,.
\end{aligned}$$

The same method also establishes that for $i=1$ and R_3 is then proved.

Finally, we note from Lemma 2.8 that each $\Lambda_{U_i}^1 \cap \Lambda_{U_j}^1$, $i \neq j$, has an orthonormal basis $\{e_{\alpha\beta}, Je_{\alpha\beta} : 2i-1 \leq \alpha \leq 2i \text{ and } 2j-1 \leq \beta \leq 2j\}$. It then follows immediately from the proof of R_4 that

$$B \cup JB = \{e_{ij}, Je_{ij} : 1 \leq i < j \leq 2m\}$$

is an orthonormal basis for Λ^e, hence for Λ if $n=2m$. Next we assume that, in Proposition 2.1, $n=2m+1$ where $m \geq 1$. We wish to extend the above basis for Λ^e to a basis for Λ. In using the previous notation we will often replace $e_{2i-1\,2i}$ by the less cumbersome U_i.

First we show that $\Lambda_e^{1/m} = \bigoplus_{i=1}^m (\Lambda_e^{1/m} \cap \Lambda_{U_i}^1)$. Thus, it is clear that since each $T_{U_jU_j}$ is positive semi-definite then, for $1 \leq i,\ j \leq m$ and $i \neq j$, $(\Lambda_e^{1/m} \cap \Lambda_{U_i}^1) \subset \Lambda_{U_j}^0$. Now $\Lambda = \Lambda^e \oplus \Lambda_e^{1/m}$ since Λ^e and $\Lambda_e^{1/m}$ are orthonormal and $\dim \Lambda^e = 2m(2m-1)$, $\dim \Lambda_e^{1/m} = 4m$. Also for each U_i $\dim \Lambda_{U_i}^1 = 4(2m-1)$ and $\dim(\Lambda^e \cap \Lambda_{U_1}^1) = 8(m-1)$ hence $\dim(\Lambda_e^{1/m} \cap \Lambda_{U_i}^1) = 4$. The required direct sum decomposition follows immediately and is clearly orthogonal. Next we show that $\Lambda_e^{1/m} \cap \Lambda_{U_i}^1 \subset D$ for each i and $\Lambda_e^{1/m} = \Lambda^X \oplus (\Lambda_X^1 \cap \Lambda_e^{1/m})$ for each unit vector $X \in \Lambda_e^{1/m} \cap \Lambda_{U_i}^1$. Clearly, it is sufficient to consider the case when $i=1$. Thus for each unit vector $X \in \Lambda_e^{1/m} \cap \Lambda_{U_1}^1$ we have $T_X^X(\Lambda^e) = \{0\}$ since $\Lambda_e^0 = \{0\}$, also $T_X^X(\Lambda_{U_i}^1) = \{0\}$ for $i \neq 1$ since $X \in \Lambda_{U_i}^0$. Hence $\Lambda^X \subset \Lambda_e^{1/m} \cap \Lambda_U^1$ and from (ii) of Lemma 2.5, $\Lambda^X \cap D \neq \{0\}$, so we may assume $X \in D$. Again, if $Y \in \Lambda_e^{1/m} \cap \Lambda_{U_1}^1$ and is orthogonal to X and JX then it follows by taking inner products of $T(Y, Y, Y)$ with X and JX that $Y \in D$. Hence, since $Y \in \Lambda_X^1$ we see that $\Lambda_e^{1/m} \cap \Lambda_{U_1}^1 \subset D$. Now for $i \neq 1$ T_{XX}

restricts to an endomorphism of $\Lambda^1_{U_1} \cap \Lambda^1_{U_i}$ and for each $Y \in \Lambda^1_{U_1} \cap \Lambda^1_{U_i}$,

$$T_{XX}T(U_1, Y, U_2)=T(U_1, Y, U_i)-T(U_1, T(X, X, Y), U_i)$$

which together with Lemma 2.8 implies $\dim(\Lambda^1_X \cap \Lambda^1_{U_1} \cap \Lambda^1_{U_i})=4$. Hence $\dim(\Lambda^1_X \cap \Lambda^1_{U_1} \cap \Lambda^e)=4(m-1)+2=4m-2$. Also $X \in \Lambda^0_{U_i}$ for $i \neq 1$ so from (v) of Lemma 2.8 $\Lambda^e \cap \Lambda^0_{U_1} \subset \Lambda^0_X$. We now have $\dim \Lambda_X = 8m-4$ and $\dim(\Lambda^e \cap \Lambda^1_X)= 4m-2$. Hence $\dim(\Lambda^{1/m}_e \cap \Lambda^1_X)=4m-2$ from which $\Lambda^{1/m}_e = \Lambda^X \oplus (\Lambda^1_X \cap \Lambda^{1/m}_e)$ as required.

In order to obtain a regular basis for Λ we first assume $n \geq 5$ and choose a unit vector $e_{1n} \in \Lambda^{1/m}_e \cap \Lambda^1_{U_1}$. Then $\Lambda^1_{U_1} \cap \Lambda^1_{U_2} \cap \Lambda^1_{e_{1n}} \subset D$ and has dimension 4. Choose unit vectors $e_{13}, e_{14} \in \Lambda^1_{U_1} \cap \Lambda^1_{U_2} \cap \Lambda^1_{e_{1n}}$ with e_{14} orthogonal to $\{e_{13}, Je_{13}\}$; thus $e_{14} \in \Lambda^1_{e_{13}}$. Now if $3 \leq i \leq m$ then $\Lambda^1_{U_1} \cap \Lambda^1_{U_i} \cap \Lambda^1_{e_{13}} \subset \Lambda^1_{e_{1n}}$. For let $X \in \Lambda^1_{U_1} \cap \Lambda^1_{U_i} \cap \Lambda^1_{e_{13}}$ be a unit vector. Then $X \in D$ and if $X \in \Lambda^0_{e_{1n}}$ we have $T(X, e_{13} e_{1n}) \neq 0$; but $T(X, e_{13}, e_{1n}) \in \Lambda^{-1}_{\bar{U}_2}=\{0\}$ and it follows that $X \in \Lambda^1_{e_{1n}}$ as required. Then define e_{23}, e_{24}, and $e_{1\,2i-1} \in \Lambda^1_{U_1} \cap \Lambda^1_{U_i} \cap \Lambda^1_{e_{13}}$, $e_{2\,2i}=T(U_1, e_{1\,2i-1}, U_i) \in \Lambda^0_{e_{11}}$ as before. Finally, by considering $T(e_{15}, e_{1n}, e_{14})$ we see that $e_{14} \in \Lambda^1_{e_{15}}$ and the previous proof can now be applied to obtain the sets B, $\tilde{B}$ as above. Next define $e_{2n}= T(e_{23}, e_{13}, e_{1n})$. Clearly $e_{2n} \in \Lambda^{1/n}_e \cap \Lambda^1_{U_1}$ and is a unit vector orthogonal to $\{e_{1n}, Je_{1n}\}$. Define $e_{n1}=-e_{1n}$, $e_{n2}=-e_{2n}$, $e_{nn}=0$, $e_{1n}=T(e_{1i}, e_{12}, e_{2n})$ and $e_{ni}= T(e_{1n}, e_{12}, e_{2i})$ for $3 \leq i \leq 2m$. We now show that relations R_1, R_2, R_3 above extend to include vectors e_{ij} where $i=n$ or $j=n$. As the method of proof is unchanged we give only brief details. Clearly the set $A=\{e_{ij}: 1 \leq i<j \leq n\}$ is orthonormal, and by considering the inner product $\langle e_{in}, e_{ni} \rangle$ we see that $e_{ni}= -e_{in}$. Define $\tilde{A}=\{e_{ji}: 1 \leq i<j \leq n\}$ as before. Now (v) of Lemma 2.8 shows that if i, j, k, n are distinct then $T(e_{ij}, e_{kn}, X)=T(e_{kn}, e_{ij}, X)=0$ for all $X \in \Lambda$. Thus R_1 is established for $A \cap \tilde{A}$. Next we note that $e_{in} \in \Lambda^1_{e_{ij}}$ hence it follows from Lemma 2.7 that R_2 extends to all $e_{ij} \in A \cap \tilde{A}$. Finally for R_3 the relation $T(e_{\alpha k}, e_{\alpha n}, e_{\beta n})=e_{\beta k}$ is easily proved when $1 \leq \alpha$, $\beta \leq 2$ and $\alpha \neq \beta$. From this it follows that $T(e_{\alpha k}, e_{\beta k}, e_{\beta n})=e_{\alpha n}$ since $T(e_{\alpha k}, e_{\beta k}, e_{\beta n}) \in \operatorname{span}\{e_{\alpha n}, Je_{\alpha n}\}$ and $\langle T(e_{\alpha k}, e_{\beta k}, e_{\beta n}), e_{\alpha n} \rangle=1$. Similarly, we obtain $T(e_{ij}, e_{\alpha j}, e_{\alpha n})=e_{in}$, $T(e_{\alpha j}, e_{ij}, e_{in}) =e_{\alpha n}$ and finally $T(e_{ij}, e_{kj}, e_{kn})=e_{in}$ for all vectors $A \cup \tilde{A}$ where $i \neq k$ and $i, j, k <n$. Also, by taking inner products, as before, we have $T(e_{ij}, e_{in}, e_{kn})=e_{kj}$ which proves the extension of R_3 to $A \cup \tilde{A}$. The remaining case corresponding to $n=3$ follows trivially by choosing unit vectors $e_{13}, e_{23} \in \Lambda^1_{U_1}$ with e_{23} orthogonal to $\{e_{13}, Je_{13}\}$. Next, we note that in all cases Je_{ij} and e_{lm} are orthogonal, for

$$\langle Je_{ij}, e_{lm} \rangle = -\frac{1}{2} \langle T(e_{ij}, Je_{ij}, e_{ij}), e_{lm} \rangle$$

$$=-\frac{1}{2}\langle T(e_{ij}, e_{lm}, e_{ij}), Je_{ij}\rangle$$

$$=0.$$

Thus A is the required extension of B so we have proved that if $\dim \varLambda = n\geq 3$ then $\varLambda$ contains a regular basis, we write this as $C\cup JC=\{e_{ij}, Je_{ij}: 1\leq i<j\leq n\}$ for all $n\geq 3$.

To obtain the required isomorphism we defined an inner product $\langle\ ,\ \rangle'$ on $S(n)$ by $\langle X, X\rangle'=1/2\operatorname{tr} X\bar{X}$ and choose an orthonormal basis $C'=\{e'_{jk}, ie'_{jk}: 1\leq j<k\leq n\}$ for $S(n)$ where $e'_{jk}=(a_{pq})=(\delta_{pj}\delta_{qk}-\delta_{qj}\delta_{pk})$. Also define $e'_{kj}=-e'_{jk}$ for $j<k$. Let $\varphi: \varLambda\to S(n)$ be the isomorphism defined by $\varphi(e_{jk})=e'_{jk}$, $\varphi(Je_{jk})=ie'_{jk}$, thus φ preserve inner products and is a complex linear isomorphism with respect to the complex structures J and i on $\varLambda$ and $S(n)$ respectively. Define the tensor T' of type (1, 3) on $S(n)$ by $T'(X, Y, Z)=X\bar{Y}^tZ+Z\bar{Y}^tX$. Then, as in § 1, P_1, P_2, P_3, P_4 are satisfied by T'. Also the relations R_1, R_2, R_3 are satisfied by T' and the basis $C'\cup\tilde{C}'$ where $\tilde{C}'=\{e'_{kj}: 1\leq j<k\leq n\}$. It is also clear from these relations that $\varphi T(U, V, W)=T'(U', V', W')$ for any basis vectors $U, V, W\in C$ with images $U', V', W'\in C'$. This completes the proof of Proposition 2.1.

3. Proof of main theorem

We first prove two lemmas for which we use the notation and the regular basis $C\cup JC$ from the proof of Proposition 2.1.

Lemma 3.1. *Let R be a tensor of type* (1, 3) *on $\varLambda$ with the symmetry properties of a Riemannian curvature tensor and satisfying $\langle R(JX, JY)Z, W\rangle=\langle R(X, Y)Z, W\rangle$ on $\varLambda$, where we use the notation from* [3] *for R. Suppose for each $X\in D$ and $Y\in\varLambda$ orthogonal to X, $\langle R(X, JX)X, JY\rangle=0$. Then the holomorphic sectional curvature determined by R is constant on $D\setminus\{0\}$.*

Proof. Write $K(X)$ for the holomorphic sectional curvature for any nonzero vector $X\in\varLambda$. Let $\Omega\subset D$ be a J-invariant subspace of $\varLambda$ of dimension ≥ 3 and let X, Y be orthogonal unit vectors in Ω. Then, by hypothesis, $\langle R(X+Y), J(X+Y))(X+Y), J(X-Y)\rangle=0$ and it follows easily that $K(X)=K(Y)$. Next, if X, Y are arbitrary unit vectors in Ω choose a unit vector $Z\in\Omega$ orthogonal to X and Y to obtain $K(X)=K(Z)=K(Y)$. Thus K is constant on $\Omega\setminus\{0\}$. Next, from Lemma 1.1 and the isomorphism $\varphi: \varLambda\to S(n)$,

$$D=\{\sum_{1\leq j<k\leq n}(a_j+b_jJ)(c_k+d_kJ)e_{jk}: a_j, b_j, c_k, d_k\in\boldsymbol{R}\}.$$

Hence if $X=\sum_{1\le j<k\le n}(a_j+b_jJ)(c_k+d_kJ)e_{jk}$ then $X\in\text{span}\{X_j, JX_j: j=1, \cdots, n-1\}$ $\subset D$ where $X_j=\sum_{j<k\le n}(c_k+d_kJ)e_{jk}$ so $K(X)=K(\sum_{k=2}^{n}(c_k+d_kJ)e_{1k})$. Again, $\sum_{k=1}^{n}(c_k+d_kJ)e_{1k}\in\text{span}\{e_{1k}, Je_{1k}: k=2, \cdots, n\}\subset D$ so $K(X)=K(e_{12})$ which proves K is constant on $D\setminus\{0\}$.

Lemma 3.2. *With R defined as in the previous lemma, suppose $R(X, JX)X=0$ and $R(X, Y)T=0$ for all $X, Y\in D$. Then $R=0$ on Λ.*

Proof. We first show that if Ω is any J-invariant subspace of Λ contained in D then

(3.1) $$R(\Omega, \Omega)\Omega=0\,.$$

Thus, by linearising the equation $R(X, JX)X=0$ we obtain

$$R(X, JX)Y+2R(X, JY)X=0$$

for all $X, Y\in\Omega$. This together with the Bianchi identity and the J-invariance of Ω implies $R(X, Y)X=0$. On replacing X by $X+Z$ in this equation, (3.1) follows. In particular, for each $j=1, \cdots, n$ we write $\Omega_1=\text{span}\{e_{ij}, Je_{ij}: i=1, \cdots, n\}$. Clearly Ω_j is a J-invariant subspace of Λ contained in D so

(3.2) $$R(\Omega_j, \Omega_j)\Omega_j=0\,.$$

Next, the condition $R(X, Y)T=0$ implies that for any $U\in D$ and $X, Y\in\Lambda$,

$$\begin{aligned}R(X, Y)U&=R(X, Y)T(U, U, U)\\&=2T(R(X, Y)U, U, U)+T(U, R(X, Y)U, U)\end{aligned}$$

so from Lemma 2.7. $R(X, Y)U\in\Lambda_U$. Hence, as a consequence of R_1, R_2, R_3,

(3.3) $$R(X, Y)e_{ij}\in\Omega_i+\Omega_j\,.$$

Then it follows from (3.2) and (3.3) that on $C\cup\tilde{C}$,

(3.4) $$\begin{aligned}R(e_{ij}, e_{kl})e_{ij}&=R(e_{ij}, Je_{kl})e_{ij}\\&=R(e_{ij}, e_{kl})e_{ik}=0\,.\end{aligned}$$

We next prove that each $R(\Omega_i, \Lambda)\Omega_i=0$ by first noting that the vectors $e_{ij}+e_{ik}$, $e_{pj}+e_{pk}$ generate a subspace of Λ contained in D. Hence from (3.1) and (3.4)

$$\begin{aligned}0&=R(e_{ij}+e_{ik}, e_{pj}+e_{pk})(e_{ij}+e_{ik})\\&=R(e_{ij}, e_{pj})e_{ik}+R(e_{ik}, e_{pk})e_{ij}\,.\end{aligned}$$

From (3.2), (3.3) and (3.4) $R(e_{ij}, e_{pj})e_{ik}\in\Lambda_k\cap\Lambda_j^{\perp}$ and $R(e_{ik}, e_{pk})e_{ij}\in\Lambda_j\cap\Lambda_k^{\perp}$

where $\Lambda_j^{\perp}$, $\Lambda_k^{\perp}$ denote orthogonal complements of Λ_j, Λ_k. Hence

(3.5) $$R(e_{ij}, e_{pj})e_{ik}=0\,.$$

By applying the same proof to $J(e_{ij}+e_{ik})$ and $e_{pj}+e_{pk}$ we obtain

(3.6) $$R(e_{ij}, Je_{pj})e_{ik}=0\,.$$

Again, if i, j, p, k are distinct then (3.3) implies that for all $X, Y\in\Lambda$,

$$\begin{aligned}\langle R(e_{ij}, e_{pk})X, Y\rangle &= \langle R(Y, X)e_{pk}, e_{ij}\rangle\\ &=0\,.\end{aligned}$$

Thus for i, j, p, k distinct,

(3.7) $$R(e_{ij}, e_{pk})=0$$

and similarly,

(3.8) $$R(e_{ij}, Je_{pk})=0\,.$$

Then as a consequence of (3.2), (3.5), (3.6), (3.7) and (3.8) each $R(\Omega_i, \Lambda)\Omega_i=0$. Also, the Bianchi identity shows that $R(\Omega_i, \Omega_i)\Lambda=0$, and this equation together with (3.7) and (3.8) proves that $R(X, Y)Z=0$ for all $X, Y, Z\in C$. Since C is a basis then $R=0$ on Λ as required.

Proof of Theorem 1.4. Under the conditions of the theorem, suppose the unit vector $N_m\in M_m$ satisfies $T(N_m, N_m, N_m)=2N_m$ and let N be the unit tangent field to geodesic γ through m with initial tangent vector N_m. Then $T(N, N, N)=2N$ along γ and, from P_5, $AJN=fJN$ along $\gamma\setminus\{m\}$. From this and (1.5) it follows immediately that if Y is any parallel vector field along γ orthogonal to N then $g(R(N, JN)N, JY)=0$ on $\gamma\setminus\{m\}$ and hence at m by continuity.

Now consider M_m as the vector space Λ in Proposition 2.1. The tensor T at m satisfies $\mathrm{P}_1-\mathrm{P}_4$ and, as just shown, for each $X\in D$ and Y orthogonal to X, $g(R(X, JX)X, JY)=0$. Hence, from Lemma 3.1 the holomorphic sectional curvature on $D\setminus\{0\}$ is constant, say c, and then for all unit vectors $X\in D$, $R(X, JX)X=-cJX$. Next it is clear from Proposition 2.1 and equation (1.2) that a second curvature tensor R' is defined on M_m by

(3.9) $$R'(X, Y)Z=T(X, Y, Z)-T(Y, X, Z)$$

and R' also satisfies the conditions of Lemma 3.1 with respect to the given complex structure J on M restricted to M_m. Moreover, from (1.4), $R'(X, JX)X=-2JX$ for any unit vector $X\in D$.

The tensor $R-(c/2)R'$ then satisfies the conditions of Lemma 3.1 and Lemma 3.2, where we note that $R(X, Y)T=0$ since T is parallel on M and

$R'(X, Y)T=0$ is the corresponding property on $SO(2n)/U(n)$. Thus from Lemma 3.2,

$$R=\frac{c}{2}R' \tag{3.10}$$

on M_m. Also the Ricci tensor S' corresponding to R' and the metric g at m is a multiple of g as can be seen either by direct computation or by noting that $SO(2n)/U(n)$ is an Einstein space. Since m is arbitrary then, defining R' on M by (3.9), we see that on M

$$R=FR' \tag{3.11}$$

for some function F. Similarly, S' extends to a parallel tensor field on M which is a multiple of g. Hence, from (3.11), (M, g) is an Einstein space and $F=c/2$ on M. Then $\nabla R=F\nabla R'=0$ so (M, g) is a symmetric space where we assume, as in the theorem, that (M, g) is complete, simply connected and non-flat, that is $c\neq 0$.

It remains only to obtain (1.2) for a metric $\bar{g}$ on M homothetic to g. Define $\bar{g}=|c/2|g$ and $\bar{T}(X, Y, Z)=|c/2|T(X, Y, Z)$ on M. Then P_1-P_5 are satisfied by $\bar{g}$ and $\bar{T}$. Thus the conditions of the theorem still apply and since the curvature tensor corresponding to $\bar{g}$ is still R, we have

$$R(X, Y)Z=\frac{c}{|c|}(\bar{T}(X, Y, Z)-\bar{T}(Y, X, Z)) \tag{3.12}$$

for all vector fields X, Y, Z on M. Now assume $c>0$. Then it is immediate from Proposition 2.1 and equations (1.1), (1.2) and (3.12) that the tangent spaces to $SO(2n)/U(n)$ and M are related by a linear isomorphism which preserves inner products and the curvature tensors. Hence $SO(2n)/U(n)$ and M are isometric since each is complete and simply connected [3]. When $c<0$ we have the corresponding result for the non-compact dual and the proof is complete.

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