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ON CURVES IN PSEUDO-RIEMANNIAN SUBMANIFOLDS

By

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§ 0. Introduction.

Let M_α be a pseudo-Riemannian submanifold of index α in a pseudo-Riemannian manifold $\tilde{M}_\beta$ of index β. We denote the metric by $\langle , \rangle$ and the covariant differentiation of M_α by ∇. In our previous papers [1] and [6], we defined circles and helices in pseudo-Riemannian manifolds and studied the submanifolds which satisfy the following conditions:

(A) every circle in M_α with $\langle X, X\rangle=\varepsilon_0$ and $\langle\nabla_X X, \nabla_X X\rangle=\varepsilon_1 k^2$ is a circle in $\tilde{M}_\beta$ ($\varepsilon_0=+1$ or -1, $\varepsilon_1=+1$, -1 or 0, $-2\alpha+2\leqq\varepsilon_0+\varepsilon_1\leqq 2n-2\alpha-2$),

(B) every geodesic in M_α with $\langle X, X\rangle=\varepsilon_0$ is a circle in $\tilde{M}_\beta$ ($\varepsilon_0=+1$ or -1, $-\alpha\leqq\varepsilon_0\leqq n-\alpha$),

(C) every helix in M_α with $\langle X, X\rangle=\varepsilon_0$, $\langle\nabla_X X, \nabla_X X\rangle=\varepsilon_1 k^2$ and $\langle\nabla_X X, \nabla_X X\rangle=\varepsilon_0 k^4+\varepsilon_2 k^2 l^2$ is a helix in $\tilde{M}_\beta$ (ε_0, $\varepsilon_1=+1$ or -1, $\varepsilon_2=+1$, -1 or 0, $-2\alpha+3\leqq\varepsilon_0+\varepsilon_1+\varepsilon_2\leqq 2n-2\alpha-3$),

where k and l are positive constants, X is the unit tangent vector field of the curve and ∇_X is the covariant derivative along the curve. For the case of Riemannian or Lorentzian submanifolds, these conditions have been treated in many papers (see [3], [4], [7] and [9]).

In this paper, we study the submanifolds which satisfy the following conditions:

(D) every circle in M_α with $\langle X, X\rangle=\varepsilon_0$ and $\langle\nabla_X X, \nabla_X X\rangle=\varepsilon_1 k^2$ is a helix in $\tilde{M}_\beta$ ($\varepsilon_0=+1$ or -1, $\varepsilon_1=+1$, -1 or 0, $-2\alpha+3\leqq\varepsilon_0\leqq 2n-2\alpha-3$),

(E) every geodesic in M_α with $\langle X, X\rangle=\varepsilon_0$ is a helix in $\tilde{M}_\beta$ ($\varepsilon_0=+1$ or -1, $-\alpha\leqq\varepsilon_0\leqq n-\alpha$).

Nakagawa [5] has investigated isotropic Riemannian submanifolds which satisfy (E). Instead, we deal with pseudo-Riemannian hypersurfaces which satisfy (E).

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§ 1. Prelimiuaries.

Let V_α be an n-dimensional real vector space equipped with an inner product $\langle , \rangle$ of index α. A nonzero vector x of V_α is said to be *null* if $\langle x, x\rangle=0$ and *unit* if $\langle x, x\rangle=+1$ or -1. Our main tools in this paper are the following lemmas of linear algebra [1]:

Lemma 1.1. *For any r-linear mapping T on V_α to a real vector space W and $\varepsilon_0=+1$ or -1 $(-\alpha\leqq\varepsilon_0\leqq n-\alpha)$, the following conditions are equivalent:*

(a) $T(x, \cdots, x)=0$ *for any* $x\in V_\alpha$ *such that* $\langle x, x\rangle=\varepsilon_0$,

(b) $T(x, \cdots, x)=0$ *for any* $x\in V_\alpha$.

Lemma 1.2. *For any $2r$-linear mapping T on V_α to a real vector space W and $\varepsilon_0=+1$ or -1, $\varepsilon_1=+1$, -1 or 0 $(2-2\alpha\leqq\varepsilon_0+\varepsilon_1\leqq 2n-2\alpha-2)$, the following conditions are equivalent:*

(a) $\sum_{i=1}^{2r}T(x, \cdots, x, \underset{i}{u}, x, \cdots, x)=0$ *for any orthogonal vectors* $x, u\in V_\alpha$ *such that* $\langle x, x\rangle=\varepsilon_0$ *and* $\langle u, u\rangle=\varepsilon_1$,

(b) *there exists* $w\in W$ *such that* $T(x, \cdots, x)=\langle x, x\rangle^r w$ *for any* $x\in V_\alpha$.

Next, we recall the general theory of pseudo-Riemannian submanifold to fix our notation. Let M_α be an n-dimensional pseudo-Riemannian manifold of index α $(0\leqq\alpha\leqq n)$ isometrically immersed into an m-dimensional pseudo-Riemannian manifold $\tilde{M}_\beta$ of index β. Then M_α is called a *pseudo-Riemannian submanifold* of $\tilde{M}_\beta$. We denote the metrics of M_α and $\tilde{M}_\beta$ by the symbol $\langle , \rangle$ and the covariant differentiation of M_α (resp. $\tilde{M}_\beta$) by ∇ (resp. $\tilde{\nabla}$). Gauss' formula is

$$\tilde{\nabla}_X Y=\nabla_X Y+B(X, Y),$$

where X and Y are tangent vector fields of M_α and B is the second fundamental form of M_α. Weingarten's formula is

$$\tilde{\nabla}_X\xi=-A_\xi X+\nabla^\perp_X\xi,$$

where X (resp. ξ) is a tangent (resp. normal) vector field of M_α, $\nabla^\perp$ is the covariant differentiation with respect to the induced connection in the normal bundle $N(M_\alpha)$ and A_ξ is the *shape operator* of M_α. We have the following relation:

$$\langle A_\xi X, Y\rangle=\langle B(X, Y), \xi\rangle.$$

For the second fundamental form and the shape operator, we define their covariant derivatives by

$$\bar{\nabla}B(X, Y, Z)=\nabla^\perp_Z(B(X, Y))-B(\nabla_Z X, Y)-B(X, \nabla_Z Y),$$

$$\bar{\nabla}^2 B(X, Y, Z, W) = \nabla^{\perp}_W(\bar{\nabla}B(X, Y, Z)) - \bar{\nabla}B(\nabla_W X, Y, Z) \\ - \bar{\nabla}B(X, \nabla_W Y, Z) - \bar{\nabla}B(X, Y, \nabla_W Z),$$

$$(\bar{\nabla}_Y A)_\xi X = \nabla_Y(A_\xi X) - A_{\nabla^{\perp}_Y \xi} X - A_\xi \nabla_Y X,$$

where X, Y, Z, W are tangent vector fields of M_α and ξ is a normal vector field of M_α. The *mean curvature vector field* H of M_α is defined by

$$H := (1/n)\sum_{i=1}^{n}\langle e_i, e_i\rangle B(e_i, e_i),$$

where $\{e_1, \cdots, e_n\}$ is an orthonormal frame of M_α. H is said to be parallel when $\nabla^{\perp}H=0$ holds. If the second fundamental form B satisfies

$$B(X, Y) = \langle X, Y\rangle H$$

for any tangent vector fields X, Y of M_α, then M_α is said to be *totally umbilic*. A totally umbilical submanifold with the parallel mean curvature vector field is called an *extrinsic sphere*. If the second fundamental form vanishes identically on M_α, then M_α is said to be *totally geodesic*.

By using Lemma 1.2, we proved the following lemma in [1]:

Lemma 1.3. *If $B(n, n)=0$ holds for any null vector n of M_α $(1 \leqq \alpha \leqq n-1)$, then M_α is totally umbilic.*

§ 2. Curves in a pseudo-Riemannian manifold.

Let $c=c(t)$ be a regular curve in a pseudo-Riemannian manifold M_α. We denote the tangent vector field $c'(t)$ by the letter X. When $\langle X, X\rangle=+1$ or -1, c is called a *unit speed curve*. In this paper, a unit speed curve c in M_α is said to be a *helix* if and only if there exist constants α, β and vector fields U, V of constant length along c such that X, U, V are orthogonal and the following equations hold:

$$\nabla_X X = U, \quad \nabla_X U = \alpha X + V, \quad \nabla_X V = \beta U,$$

where ∇_X is the covariant derivative along c. Especially, if $V=0$ in this equation, the curve is called a *circle*. Moreover, if $U=V=0$ in this equation, the curve is a geodesic. We have the following lemma [6]:

Lemma 2.1. *A unit speed curve c in M_α is a helix if and only if there exists a constant λ such that*

$$\nabla_X \nabla_X \nabla_X X = \lambda \nabla_X X,$$

where $X := c'(t)$.

Let M_α be a pseudo-Riemannian submanifold in a pseudo-Riemannian manifold $\tilde{M}_\beta$. By Gauss' formula we have

(2.1) $$\tilde{\nabla}_X X=\nabla_X X+B(X, X).$$

Differentiating with respect to X and using Gauss' formula and Weingarten's formula, we get

$$\tilde{\nabla}_X\tilde{\nabla}_X X=\nabla_X\nabla_X X-A_{B(X,X)}X+3B(X, \nabla_X X)+\bar{\nabla}B(X, X, X),$$

from which we obtain

(2.2) $$\begin{aligned}\tilde{\nabla}_X\tilde{\nabla}_X\tilde{\nabla}_X X=&\nabla_X\nabla_X\nabla_X X-2A_{\bar{\nabla}B(X,X,X)}X-5A_{B(X,\nabla_X X)}X\\&-(\bar{\nabla}_X A)_{B(X,X)}X-A_{B(X,X)}\nabla_X X\\&-B(X, A_{B(X,X)}X)+4B(X, \nabla_X\nabla_X X)+3B(\nabla_X X, \nabla_X X)\\&+5\bar{\nabla}B(X, \nabla_X X, X)+\bar{\nabla}B(X, X, \nabla_X X)+\bar{\nabla}^2 B(X, X, X, X).\end{aligned}$$

§ 3. The first main theorem.

Now we state our result concerning the condition (D).

Theorem 3.1. *Let M_α be a pseudo-Riemannian submanifold in a pseudo-Riemannian manifold $\tilde{M}_\beta$ and $\varepsilon_0=+1$ or -1 $(-2\alpha+3\leqq\varepsilon_0\leqq 2n-2\alpha-3)$, $\varepsilon_1=+1$, -1 or 0. For any positive constant k, the following conditions are equivalent:*
(a) *every circle in M_α with $\langle X, X\rangle=\varepsilon_0$ and $\langle\nabla_X X, \nabla_X X\rangle=\varepsilon_1 k^2$ is a helix in $\tilde{M}_\beta$,*
(b) *M_α is an extrinsic sphere.*

Proof. Suppose that (a) holds. Let x and u be any mutually orthogonal nonzero vectors at p such that

$$\langle x, x\rangle=\varepsilon_0 \quad\text{and}\quad \langle u, u\rangle=\varepsilon_1.$$

There exists a circle c of M_α such that

$$c(0)=p, \quad X(p)=x \quad\text{and}\quad (\nabla_X X)(p)=ku,$$

where $X:=c'(t)$. By the definition, there exists a constant α such that

$$\nabla_X\nabla_X X=\alpha X.$$

Since $\langle\nabla_X X, X\rangle=0$, α is calculated as

$$\begin{aligned}\alpha=\varepsilon_0\langle\alpha X, X\rangle(p)&=\varepsilon_0\langle\nabla_X\nabla_X X, X\rangle(p)\\&=-\varepsilon_0\langle\nabla_X X, \nabla_X X\rangle(p)=-\varepsilon_0\varepsilon_1 k^2,\end{aligned}$$

which means that

$$\nabla_X\nabla_X X=-\varepsilon_0\varepsilon_1 k^2 X.$$

Substituting this into (2.2), we have

$$\tilde{\nabla}_X\tilde{\nabla}_X\tilde{\nabla}_X X = -\varepsilon_0\varepsilon_1 k^2 \nabla_X X - 2A_{\bar{\nabla}B(X,X,X)}X - 5A_{B(X,\nabla_X X)}X \tag{3.1}$$
$$-(\bar{\nabla}_X A)_{B(X,X)}X - A_{B(X,X)}\nabla_X X$$
$$-B(X, A_{B(X,X)}X) - 4\varepsilon_0\varepsilon_1 k^2 B(X, X) + 3B(\nabla_X X, \nabla_X X)$$
$$+5\bar{\nabla}B(X, \nabla_X X, X) + \bar{\nabla}B(X, X, \nabla_X X) + \bar{\nabla}^2 B(X, X, X, X).$$

On the other hand, since c is a helix in $\tilde{M}_\beta$ by the assumption, there exists a constant $\tilde{\lambda}$ such that

$$\tilde{\nabla}_X\tilde{\nabla}_X\tilde{\nabla}_X X = \lambda\tilde{\nabla}_X X.$$

The constant $\tilde{\lambda}$ depends on the initial vectors x, u. So we rewrite the above equation as

$$\tilde{\nabla}_X\tilde{\nabla}_X\tilde{\nabla}_X X = \tilde{\lambda}(x, u)\tilde{\nabla}_X X.$$

If we substitute (2.1) and (3.1) into this equation and take the tangential part and the normal part at p respectively, then we obtain

$$\tilde{\lambda}(x, u)ku = -\varepsilon_0\varepsilon_1 k^3 u - 2A_{\bar{\nabla}B(x,x,x)}x - 5kA_{B(x,u)}x \tag{3.2}$$
$$-(\bar{\nabla}_x A)_{B(x,x)}x - kA_{B(x,x)}u,$$

$$\tilde{\lambda}(x, u)B(x, x) = -B(x, A_{B(x,x)}x) - 4\varepsilon_0\varepsilon_1 k^2 B(x, x) \tag{3.3}$$
$$+3k^2 B(u, u) + 5k\bar{\nabla}B(x, u, x) + k\bar{\nabla}B(x, x, u)$$
$$+\bar{\nabla}^2 B(x, x, x, x).$$

Adding (3.2) to the equation obtained by changing u into $-u$ in (3.2), we have

$$\{-\tilde{\lambda}(x, -u) + \tilde{\lambda}(x, u)\}ku = -4A_{\bar{\nabla}B(x,x,x)}x - 2(\bar{\nabla}_x A)_{B(x,x)}x, \tag{3.4}$$

By subtracting (3.2) from the equation obtained by changing u into $-u$ in (3.2), we get

$$\{-\tilde{\lambda}(x, -u) - \tilde{\lambda}(x, u)\}u = 2\varepsilon_0\varepsilon_1 k^2 u + 10A_{B(x,u)}x + 2A_{B(x,x)}u. \tag{3.5}$$

Next, subtracting (3.3) from the equation obtained by changing u into $-u$ in (3.3), we have

$$\{\tilde{\lambda}(x, -u) - \tilde{\lambda}(x, u)\}B(x, x) = -10k\bar{\nabla}B(x, u, x) - 2k\bar{\nabla}B(x, x, u), \tag{3.6}$$

By adding (3.3) to the equation obtained by changing u into $-u$ in (3.3), we get

$$\{\tilde{\lambda}(x, -u) + \tilde{\lambda}(x, u)\}B(x, x) = -2B(x, A_{B(x,x)}x) - 8\varepsilon_0\varepsilon_1 k^2 B(x, x) \tag{3.7}$$
$$+6k^2 B(u, u) + 2\bar{\nabla}^2 B(x, x, x, x).$$

Let w be any tangent vector of M_α at p which is linearly independent of u and satisfies

$$\langle w, w\rangle = \varepsilon_1 \quad \text{and} \quad \langle x, w\rangle = 0.$$

Subtracting (3.4) from the equation obtained by changing u into w in (3.4), we have

$$\{-\tilde{\lambda}(x, -w)+\tilde{\lambda}(x, w)\}kw-\{-\tilde{\lambda}(x, -u)+\tilde{\lambda}(x, u)\}ku=0,$$

from which we have

$$\tilde{\lambda}(x, -u)=\tilde{\lambda}(x, u).$$

Thus (3.5), (3.6) and (3.7) are reduced to

$$\tilde{\lambda}(x, u)u=-\varepsilon_0\varepsilon_1k^2u-5A_{B(x,u)}x-A_{B(x,x)}u, \tag{3.8}$$

$$5\bar{\nabla}B(x, u, x)+\bar{\nabla}B(x, x, u)=0, \tag{3.9}$$

$$\begin{aligned}\tilde{\lambda}(x, u)B(x, x)=&-B(x, A_{B(x,x)}x)-4\varepsilon_0\varepsilon_1k^2B(x, x)\\&+3k^2B(u, u)+\bar{\nabla}^2B(x, x, x, x).\end{aligned} \tag{3.10}$$

Here, we divide the situation into two cases where $\varepsilon_1=0$ (Case 1) and $\varepsilon_1=+1$ or -1 (Case 2).

Case 1. Note that (3.8) and (3.10) hold for any mutually orthogonal nonzero vectors $x, u\in T_p(M_\alpha)$ with $\langle x, x\rangle=\varepsilon_0$ and $\langle u, u\rangle=0$. Subtracting (3.8) from the equation obtained by changing u into $2u$ in (3.8) and dividing with 2, we can see that

$$\tilde{\lambda}(x, 2u)-\tilde{\lambda}(x, u)=0.$$

On the other hand, if we substract (3.10) from the equation obtained by changing u into $2u$ in (3.10), then we get

$$\{\tilde{\lambda}(x, 2u)-\tilde{\lambda}(x, u)\}B(x, x)=9k^2B(u, u).$$

Consequently, we find that $B(u, u)=0$. This equation holds for any null vector $u\in T_p(M_\alpha)$ because there exists $x\in T_p(M_\alpha)$ such that $\langle x, x\rangle=\varepsilon_0$ and $\langle x, u\rangle=0$.

Case 2. Taking the inner product with u in (3.8), we have

$$\tilde{\lambda}(x, u)=-\varepsilon_0\varepsilon_1k^2-\varepsilon_1(5\langle B(x, u), B(x, u)\rangle+\langle B(x, x), B(u, u)\rangle), \tag{3.11}$$

which, together with (3.8), yields that

$$\begin{aligned}&(5\langle B(x, u), B(x, u)\rangle+\langle B(x, x), B(u, u)\rangle)u\\&=\langle u, u\rangle(5A_{B(x,u)}x+A_{B(x,x)}u).\end{aligned}$$

By Lemma 1.1, this equation holds for any $u\in T_p(M_\alpha)$ which is orthogonal to x. Especially, for any null vector $n\in T_p(M_\alpha)$ such that $\langle x, n\rangle=0$, we have

$$5\langle B(x, n), B(x, n)\rangle+\langle B(x, x), B(n, n)\rangle=0. \tag{3.12}$$

On the other hand, by making use of (3.10) and (3.11), we get

$$\varepsilon_1(5\langle B(x, u), B(x, u)\rangle+\langle B(x, x), B(u, u)\rangle)B(x, x)$$
$$-\varepsilon_1\langle u, u\rangle B(x, A_{B(x,x)}x)+3k^2B(u, u)$$
$$-3\varepsilon_0\langle u, u\rangle k^2B(x, x)+\varepsilon_1\langle u, u\rangle\bar{\nabla}^2B(x, x, x, x)=0.$$

Since this equation also holds for any $u\in T_p(M_\alpha)$ such that $\langle x, u\rangle=0$ by means of Lemma 1.1, we have

$$\varepsilon_1(5\langle B(x, n), B(x, n)\rangle+\langle B(x, x), B(n, n)\rangle)B(x, x)+3k^2B(n, n)=0,$$

for any null vector $n\in T_p(M_\alpha)$ such that $\langle x, u\rangle=0$, which, together with (3.12), means that $B(n, n)=0$. This equation holds for any null vector $n\in T_p(M_\alpha)$ because there exists $x\in T_p(M_\alpha)$ such that $\langle x, x\rangle=\varepsilon_0$ and $\langle x, n\rangle=0$. Consequently, we get $B(n, n)=0$ for both cases. By Lemma 1.3, we see that M_α is totally umbilic. So we have $B(x, y)=\langle x, y\rangle H$ for any $x, y\in T_p(M_\alpha)$, from which we get $\bar{\nabla}B(x, y, z)=\langle x, y\rangle\nabla^\perp_z H$ for any $x, y, z\in T_p(M_\alpha)$, which implies that (3.9) is reduced to $\nabla^\perp_u H=0$. Note that this equation holds for any $u\in T_p(M_\alpha)$ such that $\langle u, u\rangle=\varepsilon_1$. Let $y\in T_p(M_\alpha)$ be a vector which is orthogonal to x and satisfies $\langle y, y\rangle=\varepsilon_1-\varepsilon_0$. Since $\langle x+y, x+y\rangle=\langle x-y, x-y\rangle=\varepsilon_1$, we have

$$\nabla^\perp_{x+y}H=\nabla^\perp_{x-y}H=0,$$

from which we get $\nabla^\perp_x H=0$. Applying Lemma 1.1 to this equation, we have $\nabla^\perp H=0$ and see that M_α is an extrinsic sphere.

Conversely, if M_α is an extrinsic sphere, every circle in M_α with $\langle X, X\rangle=\varepsilon_0$ and $\langle\nabla_X X, \nabla_X X\rangle=\varepsilon_1 k^2$ is a circle in $\tilde{M}_\beta$ (see [1]). Since a circle is a kind of a helix, we can say that (b) implies (a). Q. E. D.

§ 4. The second main theorem.

We prove the following theorem concerning the condition (E):

Theorem 4.1. *Let M_α be a pseudo-Riemannian hypersurface in a pseudo-Riemannian manifold $\tilde{M}_\beta$ and $\varepsilon_0=+1$ or -1 $(-\alpha\leqq\varepsilon_0\leqq n-\alpha)$. Then the following conditions are equivalent:*

(a) *every geodesic in M_α with $\langle X, X\rangle=\varepsilon_0$ is a helix in $\tilde{M}_\beta$,*
(b) $\bar{\nabla}B=0$.

Proof. For a geodesic, (2.1) and (2.2) are reduced to

$$\tilde{\nabla}_X X=B(X, X), \tag{4.1}$$

$$\tilde{\nabla}_X\tilde{\nabla}_X\tilde{\nabla}_X X=-2A_{\bar{\nabla}B(X,X,X)}X-(\bar{\nabla}_X A)_{B(X,X)}X -B(X, A_{B(X,X)}X)+\bar{\nabla}^2B(X, X, X, X) \tag{4.2}$$

because $\nabla_X X=0$. Assume that (a) holds. Let $x\in T_p(M_\alpha)$ be any vector with $\langle x, x\rangle=\varepsilon_0$. There exists a geodesic c of M_α such that

$$c(0)=p \quad \text{and} \quad X(p)=x,$$

where $X:=c'(t)$. Since c is a helix in $\tilde{M}_\beta$ by the assumption, there exists a constant $\tilde{\lambda}$ such that

$$\tilde{\nabla}_X\tilde{\nabla}_X\tilde{\nabla}_X X=\tilde{\lambda}\tilde{\nabla}_X X.$$

Substituting (4.1) and (4.2) into this equation and taking the tangential part, we obtain

$$2A_{\bar{\nabla}(X,X,X)}X+(\bar{\nabla}_X A)_{B(X,X)}X=0. \tag{4.3}$$

If we take the inner product with X, then we have

$$\langle\bar{\nabla}B(X, X, X), B(X, X)\rangle=0, \tag{4.4}$$

from which we find

$$X(\langle B(X, X), B(X, X)\rangle)=0,$$

which means that $\langle B(X, X), B(X, X)\rangle$ is constant along c. If $\langle B(X, X), B(X, X)\rangle$ is nonzero, $B(X, X)$ is nonzero at any point of c, so that (4.4) implies $\bar{\nabla}B(X, X, X)=0$, because M_α is a hypersurface. When $\langle B(X, X), B(X, X)\rangle=0$, we have $B(X, X)=0$. Thus we get $\bar{\nabla}B(X, X, X)=0$ for both cases. Taking the value at p, we have

$$\bar{\nabla}B(x, x, x)=0. \tag{4.5}$$

Let y be any vector of $T_p(M_\alpha)$. Applying Lemma 1.1 to the above equation, we get

$$\bar{\nabla}B(y, y, y)=0.$$

Changing y into $x+y$, we have

$$\begin{aligned}0&=\bar{\nabla}B(x+y, x+y, x+y)\\&=\bar{\nabla}B(x, x, y)+2\bar{\nabla}B(x, y, x)+\bar{\nabla}B(y, y, x)+2\bar{\nabla}B(y, x, y),\end{aligned}$$

Adding this equation to the equation obtained by changing y into $-y$ in this equation, we get

$$\bar{\nabla}B(y, y, x)+2\bar{\nabla}B(y, x, y)=0. \tag{4.6}$$

On the other hand, we have $(\bar{\nabla}_x A)_{B(x,x)}x=0$ by (4.3) and (4.5). Making use of Lemma 1.1, we obtain $(\bar{\nabla}_y A)_{B(y,y)}y=0$. Taking the inner product with x, we have $\langle\bar{\nabla}B(y, x, y), B(y, y)\rangle=0$. Combining this equation with (4.6), we get $\langle\bar{\nabla}B(y, y, x), B(y, y)\rangle=0$. Now we extend the vector y to the parallel local vector field Y along c. Since the above equation holds at any point of M_α, we have

$$\langle \bar{\nabla} B(Y, Y, X), B(Y, Y)\rangle = 0, \tag{4.7}$$

from which we find

$$X(\langle B(Y, Y), B(Y, Y)\rangle) = 0,$$

which means that $\langle B(Y, Y), B(Y, Y)\rangle$ is constant along c. If $\langle B(Y, Y), B(Y, Y)\rangle$ is nonzero, $B(Y, Y)$ is nonzero at any point of c, so that (4.7) implies $\bar{\nabla} B(Y, Y, X)=0$, because M_α is a hypersurface. When $\langle B(Y, Y), B(Y, Y)\rangle=0$, we have $B(Y, Y)=0$. Thus we get $\bar{\nabla} B(Y, Y, X)=0$ for both cases. Taking the values at p, we get $\bar{\nabla} B(y, y, x)=0$. Let v, w be any vectors of $T_p(M_\alpha)$. Applying Lemma 1.1 to the above equation, we get $\bar{\nabla} B(y, y, w)=0$. Changing y into $y+v$, we have

$$0=\bar{\nabla} B(y+v, y+v, w)=2\bar{\nabla} B(y, v, w),$$

which means that $\bar{\nabla} B=0$ holds.

Conversely, suppose that (b) holds. Let c be any geodesic in M_α with $\langle X, X\rangle=\varepsilon_0$ where $X:=c'(t)$. Since $\bar{\nabla} B=0$, $\langle B(X, X), B(X, X)\rangle$ is constant along c. If $\langle B(X, X), B(X, X)\rangle=0$, we have $B(X, X)=0$ and find $\tilde{\nabla}_X X=0$ by (4.1), which means that c is a geodesic in $\tilde{M}_\beta$. Next, we suppose $\langle B(X, X), B(X, X)\rangle$ is a nonzero constant. Since M_α is a hypersurface, there exists a scalar field $\tilde{\lambda}$ along c such that

$$B(X, A_{B(X,X)}X) = -\tilde{\lambda} B(X, X). \tag{4.8}$$

By taking the inner product with $B(X, X)$, we find that $\tilde{\lambda}$ is equal to

$$-\langle B(X, X), B(X, A_{B(X,X)}X)\rangle / \langle B(X, X), B(X, X)\rangle,$$

which is constant along c because of $\bar{\nabla} B=0$. On the other hand, (4.1), (4.2), (4.8) and $\bar{\nabla} B=0$ imply

$$\tilde{\nabla}_X\tilde{\nabla}_X\tilde{\nabla}_X X = -B(X, A_{B(X,X)}X) = \tilde{\lambda} B(X, X) = \tilde{\lambda}\tilde{\nabla}_X X.$$

Consequently, it follows that c is a helix in $\tilde{M}_\beta$ from Lemma 2.1. Q.E.D.

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