YOKOHAMA MATHEMATICAL
JOURNAL VOL, 36, 1988

A NOTE ON INTEGRABLE ACTIONS ON VON NEUMANN ALGEBRAS

By

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(Received June 21, 1988)

Abstract. In this note, we study (not necessarily ergodic) integrable systems on von Neumann algebras. As a generalization of A. Amann [1, Capter II, Theorem 2], we show that a W^*-dynamical system $(\mathcal{M}, G, \alpha)$ is integrable if and only if there is a normal covariant embedding of $L^{\infty}(G)$ into $(\mathcal{M}, G, \alpha)$.

Let $(\mathcal{M}, G, \alpha)$ be a W^*-dynamical system with a von Neumann algebra $\mathcal{M}$, a locally compact group G and a σ-weakly continious action α on $\mathcal{M}$. The notion of integrable actions were introduced and studied by A. Connes and M. Takesaki in [2]. In [5], Y. Nakagami showed a characterization of integrable actions when the fixed point algebra $\mathcal{M}^{\alpha}$ is properly infinite. Further, if G is separable abelian and if $(\mathcal{M}, G, \alpha)$ is ergodic, then D. De Schreye in [3] showed that $(\mathcal{M}, G, \alpha)$ is integrable if and only if there are unitary eigenoperators. Recently, if $(\mathcal{M}, G, \alpha)$ is ergodic, where G is not necessarily abelian, then A. Amann in [1] showed that $(\mathcal{M}, G, \alpha)$ is integrable if and only if there is a normal covariant embedding of $L^{\infty}(G)$ into $(\mathcal{M}, G, \alpha)$. Our aim in this note is to show a characterization of not necessarily ergodic integrable systems (Theorem 1).

Let $(\mathcal{M}, G, \alpha)$ be a W^*-dynamical system. Let $\mathfrak{q}_{\alpha}$ be the set of all $x\in\mathcal{M}$ such that there is some $y\in\mathcal{M}$ with $y=\int_G \alpha_s(x^*x)d\mu(s)$, where μ is the left Haar measure on G. Then $(\mathcal{M}, G, \alpha)$ is called an *integrable system* whenever $\mathfrak{p}_{\alpha}=$ the linear span of $\{y^*x\,;\, x, y\in\mathfrak{q}_{\alpha}\}$ is σ-weakly dense in M. Further, a unital positive linear mapping $\varphi: L^{\infty}(G)\rightarrow\mathcal{M}$ is called a *covariant embedding of* $L^{\infty}(G)$ *into* $(\mathcal{M}, G, \alpha)$ whenever $\varphi(\mathrm{Ad}(\lambda)_g)f)=\alpha_g(\varphi(f))$, for each $g\in G$ and $f\in L^{\infty}(G)$, where λ is the left regular representation of G. If φ is a normal mapping from $L^{\infty}(G)$ to M, then φ is called a *normal covariant embedding*. A covariant embedding φ is called a *covariant representation* whenever $\varphi(fg)=\varphi(f)\varphi(g)$ for each $f, g\in L^{\infty}(G)$.

1980 Mathematics Subject Classification (1985 Revision). Primary 46L10, 46L55.

Let $(\mathcal{M}, G, \alpha)$ be a W^*-dynamical system. Amann showed that an ergodic W^*-dynanical system is integrable if and only if there exists a normal covariant embedding of $L^\infty(G)$ into $(\mathcal{M}, G, \alpha)$ ([1, Chapter II, Theorem 2]). We attempt the generalization of this result thank to the duality theorem of crossed products (cf. [4, Theorem 3.1]). Then we have the following theorem.

Theorem 1. *Let $(\mathcal{M}, G, \alpha)$ be a W^*-dynamical system with a separable locally compact group G. Then $(\mathcal{M}, G, \alpha)$ is integrable if and only if there is a normal convariant embedding of $L^\infty(G)$ into $(\mathcal{M}, G, \alpha)$.*

Proof. Suppose that there is a normal covariant embedding of $L^\infty(G)$ into $(\mathcal{M}, G, \alpha)$. Take $k \in L^\infty(G)_+$ such that $\int_G k(g)\Delta(g^{-1})d\mu(g) < +\infty$, where μ is the left Haar meassure on G and Δ is the modular function of G. Let φ be a normal covariant embedding of $L^\infty(G)$ into $(\mathcal{M}, G, \alpha)$ and take an arbitrary element ψ of $\mathcal{M}_*$. Then

$$\int_G \psi(\alpha_g(\varphi(k)))d\mu(g) = \int \psi\circ\varphi(Ad(\lambda_g)k)d\mu(g) \cdots\cdots\cdots\cdots\cdots (\#)$$

Since φ is normal, $\psi\circ\varphi$ is a normal functional on $L^\infty(G)$. Since the predual of $L^\infty(G)$ is isomorphic to $L^1(G)$, we identify "normal functional" $\psi\circ\varphi$ with "L^1-function" $\psi\circ\varphi$. Then we have

$$\begin{aligned}(\#) &= \int_G\int_G (\psi\circ\varphi)(h)(\mathrm{Ad}(\lambda_g)k)(h)d\mu(h)d\mu(g)\\ &= \int_G\int_G (\psi\circ\varphi)(h)k(g^{-1}h)d\mu(h)d\mu(g)\\ &= \int_G\int_G (\psi\circ\varphi)(h)k(g^{-1})d\mu(g)d\mu(h)\\ &= \Big(\int_G (\psi\circ\varphi)(h)d\mu(h)\Big)\Big(\int_G k(g^{-1})d\mu(g)\Big)\\ &= (\psi\circ\varphi(1)\int_G k(g)\Delta(g^{-1})d\mu(g)\\ &= \int_G k(g)\Delta(g^{-1})d\mu(g) < +\infty\,.\end{aligned}$$

Hence this implies that $\varphi(k) \in \mathfrak{p}_\alpha^*$. We can take a net $\{k_\iota\}_{\iota\in I}$ in $L^\infty(G)_+$ such that

$$\int_G k_\iota(g)\Delta(g^{-1})d\mu(g) < +\infty \quad \text{and} \quad k_\iota \uparrow 1 \ (\sigma\text{-weak in } L^\infty(G)).$$

Then $\varphi(k_i)\in\mathfrak{p}_\alpha^*$ and $\varphi(k_i)$ converges to 1 with respect to σ-weak topology on $\mathcal{M}$ from the normality of φ. Hence this means that $\mathfrak{p}_\alpha$ has a σ-weakly approximate identity. Hence $(\mathcal{M}, G, \alpha)$ is integrable (cf. [6, 11.5]).

Conversely, suppose that $(\mathcal{M}, G, \alpha)$ is integrable. Let F_∞ be a type I_∞-factor. We define the W^*-dynamical system $(\bar{\mathcal{M}}, G, \bar{\alpha})$ by

$$\bar{\mathcal{M}}=\mathcal{M}\otimes F_\infty, \qquad \bar{\alpha}=\alpha\otimes id$$

By [2, Chapter 3, Lemma 2.10], $(\bar{\mathcal{M}}, G, \bar{\alpha})$ is also integrable and the fixed point algebra $\bar{\mathcal{M}}^{\bar{\alpha}}=\mathcal{M}^\alpha\otimes F_\infty$ is properly infinite. Hence by [5, Theorem 4.1], there is a projection p in $(\bar{\mathcal{M}}\otimes \boldsymbol{B}(L^2(G)))^{\bar{\alpha}\otimes \mathrm{Ad}(\lambda)}$ such that

$$\{\bar{\mathcal{M}}\otimes \boldsymbol{B}(L^2(G)), \bar{\alpha}\otimes\mathrm{id}\}\cong\{\bar{\mathcal{M}}\otimes \boldsymbol{B}(L^2(G)), \bar{\alpha}\otimes\mathrm{Ad}\,(\lambda)\}_p$$

From the duality theorem ([4, Theorem 3.1]), there is a projection q in $\bar{\mathcal{M}}\times_{\bar{\alpha}}G$ such that

$$\{\bar{\mathcal{M}}\otimes \boldsymbol{B}(L^2(G)), \bar{\alpha}\otimes\mathrm{id}\}\cong\{\bar{\mathcal{M}}\times_{\bar{\alpha}}G)\times_{\hat{\bar{\alpha}}}\hat{G}, \hat{\hat{\bar{\alpha}}}\}_q \quad\cdots\cdots\cdots\cdots\cdots(\flat)$$

So, there is a $*$-isomorphism $\pi: L^\infty(G)\to(\bar{\mathcal{M}}\times_{\bar{\alpha}}G)\times_{\hat{\bar{\alpha}}}\hat{G}$ such that

$$\pi(\mathrm{Ad}\,(\lambda_g)f)=\hat{\hat{\bar{\alpha}}}_g(\pi(f)) \qquad (f\in L^\infty(G),\ g\in G)$$

Let Φ be a reduction of $(\bar{\mathcal{M}}\times_{\bar{\alpha}}G)\times_{\hat{\bar{\alpha}}}\hat{G}$ by q. Since $q\in\bar{\mathcal{M}}\times_{\bar{\alpha}}G=((\bar{\mathcal{M}}\times_{\bar{\alpha}}G)\times_{\hat{\bar{\alpha}}}\hat{G})^{\hat{\hat{\bar{\alpha}}}}$, we have $\Phi\circ\hat{\hat{\bar{\alpha}}}=(\hat{\hat{\bar{\alpha}}})^q\circ\Phi$. Let J be a $*$-isomorphism in $(\flat)$. We define a normal covariant embedding ϕ of $L^\infty(G)$ into $(\bar{\mathcal{M}}\otimes \boldsymbol{B}(L^2(G)), G, \bar{\alpha}\otimes\mathrm{id})$ by $\phi=J\circ\Phi\circ\pi$. Let p_1 be a minimal projection in F_∞, let p_2 be a minimal projection in $\boldsymbol{B}(L^2(G))$ and let Ψ be a reduction of $\bar{\mathcal{M}}\otimes \boldsymbol{B}(L^2(G))$ by $1\otimes p_1\otimes p_2$. Then, for any $x\in\bar{\mathcal{M}}\otimes \boldsymbol{B}(L^2(G))$ and $g\in G$, we have $\Psi(\bar{\mathcal{M}}\otimes \boldsymbol{B}(L^2(G)))=\mathcal{M}$ and $\Psi((\bar{\alpha}_g\otimes\mathrm{id})(x))=\alpha_g(\Psi(x))$ (cf. [7, Chapter IV, Theorem 1.9]). Put $\varphi=\Psi\circ\phi$. Then it is clear to prove that φ is a desired normal covariant embedding. This completes the proof.

Next, we suppose that $\mathcal{M}$ is a σ-finite von Neumann algebra and that G is a separable locally compact abelian group. Then De Schreye in [3] investigated that, if $(\mathcal{M}, G, \alpha)$ is ergodic, the existence of unitary eigenoperators u_γ, $\gamma\in\hat{G}$ is equivarent to α being an integrable action of G. In the following proposition, we consider the result when $(\mathcal{M}, G, \alpha)$ is not necessarily ergodic.

Proposition 2. *Let $(\mathcal{M}, G, \alpha)$ be a W^*-dynamical system with a σ-finite von Neumann algebra $\mathcal{M}$, a separable locally compact abelian group G. If, for any $\gamma\in\hat{G}$, there is a non-zero unitary operation u_γ in $\mathcal{M}$ such that $\alpha_g(u_\gamma)=\langle\gamma, g\rangle u_\gamma$ $(g\in G)$, then $(\mathcal{M}, G, \alpha)$ is integrable.*

Proof. By the same arguments in [1, Chapter III, Theorem 6], there is a normal covariant embedding of $L^\infty(G)$ into $(\mathcal{M}, G, \alpha)$. Hence by Theorem 1, $(\mathcal{M}, G, \alpha)$ is integrable. This completes the proof.

However, the converse is not true without ergodicity. That, we have the following counterexample.

Example 3. Let T be a unit circle, that is, $T=\{z\in \boldsymbol{C}\,;\,|z|=1\}$. For $z\in \boldsymbol{T}$, we define a 2×2-unitary matrix u_z by

$$u_z=\begin{pmatrix}1 & 0\\ 0 & z\end{pmatrix}.$$

Then $z\rightarrow u_z$ is a unitary representation of $\boldsymbol{T}$ on $M_2(\boldsymbol{C})$. Since $\boldsymbol{T}$ is compact, the W^*-dynamical system $(M_2(\boldsymbol{C}), \boldsymbol{T}, \mathrm{Ad}(u))$ is integrable. On the other hand, it is clear to prove that there is no integer n $(\neq 0, \pm 1)$ such that there exists a nonzero element $x\in M_2(\boldsymbol{C})$ satisfying $u_z x u_z^* = \langle z, n\rangle x = z^n x$ for any $z\in \boldsymbol{T}$.

Acknowledgement The auther would like to thank Professor Kichi-Suke Saito for many valuable suggestions and comments.

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