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GEODESIC CIRCLES AND TOTAL CURVATURE

By

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0. Introduction.

Let M be a non-compact 2-dimensional complete Riemannian manifold diffeomorphic to an Euclidean plane E^2 and having a non-negative Gaussian curvature $K \geqq 0$ which is positive at some point of M. In this situation, we have showed the following facts in [6].

FACT 1. Let M be a manifold stated as above. Then the set P of points which are poles in M is compact.

Here a point $p \in M$ is called a pole if exponential mapping $\exp_p : T_p(M) \to M$ is of maximal rank on $T_p(M)$ (=tangent space of M at p).

And if M satisfy the further conditions, then we have

FACT 2. Let M be a manifold as in Fact 1. If the total curvature $\int_M K\, dv$ of M satisfy

$$\int_M K\, dv = 2\pi$$

and support of K (=closure of $\{p \in M \mid K(p) > 0\}$) is compact, then the number of elements of P is at most one, i.e. there exists at most one pole on M.

Here dv is the volume element of M.

Note that for a manifold M satisfying the assumption of Fact 1, the total curvature of M satisfies

$$0 \leqq \int_M K\, dv \leqq 2\pi .$$

This is a well known result by Cohn-Vossen, see [3]. And note that a manifold satisfying the assumption of Fact 2 is a flat half cylinder with non-negatively curved cap. Thus, the assumption of Fact 2 is considerably tight. And so the purpose of this paper is to relax the assumption in Fact 2 slightly.

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1. Geodesic circles

Let M be a complete non-compact Riemanian manifold and d the distance function on M induced from the Riemannian metric of M. For a point $p \in M$ and non-negative real number r, let $S_r(p)$ be the geodesic sphere (geodesic circle if $\dim M=2$) with radius r centered at p, i.e.

$$S_r(p)=\{q \in M \mid d(p, q)=r\}.$$

And let

$$\operatorname{Diam} S_r(p)=\sup\{d(q, q') \mid q, q' \in S_r(p)\}.$$

Now, for a fixed point $p \in M$, we consider the following condition (*);

(*) there exists a constant $L>0$ such that for all $r \geqq 0$,

$$\operatorname{Diam} S_r(p) \leqq L$$

Now, we have

Proposition 1. *Let M be a complete non-compact Riemannian manifold. Then, for all point $p \in M$, the condition (*) is satisfied or for all point $p \in M$, the condition (*) is not satisfied.*

Proof. It will be sufficient to prove that if there exist a point $p \in M$ satisfying the condition (*), then for any point $q \in M$, the condition (*) is satisfied.

For a fixed $r>0$, let $m, n \in S_r(q)$ be the points such that

$$d(m, n)=\operatorname{Diam} S_r(q).$$

Put $r_0 := d(p, q)$. Without loss of generality, we can assume that $d(p, n) \leqq d(p, m)$. Then, by triangle inequality

$$r' := d(p, m) \leqq d(q, m)+d(p, q)=r+r_0$$

$$r'' := d(p, n) \geqq d(n, q)-d(p, q)=r-r_0$$

So

$$r'-r'' \leqq r+r_0-(r-r_0)=2r_0.$$

Then, we can find a point $m' \in S_{r''}(p)$ such that

$$d(m, m')=r'-r''.$$

Thus

$$d(m, n) \leqq d(m, m')+d(m', n) \leqq 2r_0+L,$$

because $m', n \in S_{r''}(p)$. That is

$$\operatorname{Diam} S_r(q) \leqq 2r_0+L \quad \text{for all } r \geqq 0.$$

Q. E. D.

Thus, from Proposition 1, we know that the set of all non-compact complete

Riemannian manifolds will be divided into two classes A and B, i. e.

A consists of all manifolds which satisfies the condition (*) and B consists of all manifolds which does not satisfy the condition (*).

In the following, we will say that manifold M is of type A (resp. of type B) if M is an element of A (resp. B). Examples of manifolds which is of type A or type B can be constructed easily. Especially, a flat half cylinder with non-negatively curved cap is a manifold of type A. It should be noted that the total curvature of this example equals to 2π. This example suggests that there exists a relation between the total curvature and $\mathrm{Diam}\, S_r(p)$ when $\dim M=2$. We will show this in the following.

Let M be a complete non-compact Riemannian manifold with non-negative Gaussian curvature K diffeomorphic to an Euclidean plane E^2. Let $p\in M$ be a fixed point and $i(p)$ the injectivity radius of $\exp_p$. In [7], we have shown the following facts.

M is divided into at most countable mutually disjoint domains $\{D_\lambda\}_{\lambda\in\Lambda}$ such that

(I) $\bigcup_{\lambda\in\Lambda}\bar{D}_\lambda=M$.

(II) For each λ, ∂D_λ consists of images of two rays γ_λ^+ and γ_λ^- from p. Here a geodesic $\gamma:[0,\infty)\to M$ is, by definition, a ray if any subarc of γ is a shortest connection between its end points. And we assume that every geodesic has arclength as its parameter.

(III) Let $T_p^1(M)=\{v\in T_p(M) \mid \text{norm of } v=\|v\|=1\}$. Then $T_p^1(M)$ is a unit circle in $T_p(M)$ and $\dot\gamma_\lambda^\pm(0)\in T_p^1(M)$, $\lambda\in\Lambda$. The set $\{\dot\gamma_\lambda^\pm(0)\mid\lambda\in\Lambda\}$ divides $T_p^1(M)$ into at most countable open connected subarcs $\{E_\lambda\}_{\lambda\in\Lambda}$ such that $\partial E_\lambda=\{\dot\gamma_\lambda^+(0),\dot\gamma_\lambda^-(0)\}$ and $\{\exp_p tv\mid v\in E_\lambda,\ 0\leqq t\leqq i(p)\}\subset D_\lambda$. Then each D_λ has the following property (i) or (ii):

(i) For any $v\in E_\lambda$, the geodesic $\gamma:[0,\infty)\to M$ defined by $\gamma(t)=\exp_p tv$ is not a ray from p

(ii) For each $v\in E_\lambda$, the geodesic $\gamma:[0,\infty)\to M$ defined by $\gamma(t)=\exp_p tv$ is a ray from p.

(IV) For each domain D_λ, if D_λ is of type (i), then

$$\int_{\bar D} K\,dv=\sphericalangle(\dot\gamma_\lambda^+(0),\dot\gamma_\lambda^-(0))$$

where $\sphericalangle(\dot\gamma_\lambda^+(0),\dot\gamma_\lambda^-(0))$ is the angle between $\dot\gamma_\lambda^+(0)$ and $\dot\gamma_\lambda^-(0)$ measured on $\bar D_\lambda$. And, if D_λ is of type (ii), then for any open connected subarc E'_λ of E_λ,

$$\int_{\bar D'_\lambda} K\,dv\leqq\sphericalangle(\dot\eta_\lambda^+(0),\dot\eta_\lambda^-(0))$$

where $D'_\lambda=\{\exp_p tv\mid v\in E'_\lambda,\ 0<t\}$ and $\partial E'_\lambda=\{\dot\eta_\lambda^+(0),\dot\eta_\lambda^-(0)\}$. And in particular,

$$\int_{\bar D_\lambda} K\,dv\leqq\sphericalangle(\dot\gamma_\lambda^+(0),\dot\gamma_\lambda^-(0)).$$

From property (IV), we have

$$\int_M K\,dv=\sum_\lambda \int_{\bar{D}_\lambda} K\,dv \leqq \sum_\lambda \sphericalangle(\dot{\gamma}_\lambda^+(0), \dot{\gamma}_\lambda^-(0))$$
$$=2\pi.$$

This is an another proof of Cohn-Vossen's result.

Using these results, we can show the following.

Theorem 1. *Let M be a complete non-compact 2-dimensional Riemannian manifold diffeomorphic to an Euclidean plane E^2 and with non-negative Gaussian curvature $K\geqq 0$. If M is of type A, i.e. there exists a point $p\in M$ satisfying the condition* (*), *then its total curvature satisfies*

$$\int_M K\,dv=2\pi.$$

Proof. We will show that if $\int_M K\,dv<2\pi$, then for any point $p\in M$, the condition (*) is not satisfied. Let $p\in M$ be a fixed point and $\{D_\lambda\}_{\lambda\in\Lambda}$ be a family of domains in M mentioned as above. Then, from property (I)~(IV), we can find a domain D_λ such that

(1) D_λ satisfies the condition (ii)

(2) $\int_{\bar{D}_\lambda} K\,dv<\sphericalangle(\dot{\gamma}_\lambda^+(0), \dot{\gamma}_\lambda^-(0))$

(3) In view of (III)-(ii), D_λ is expressed as $D_\lambda=\{\exp_p tv\,|\,v\in E_\lambda,\ t>0\}$.

Let $v_0\in E_\lambda$ be the middle point of E_λ, i.e. $\dot{\gamma}_\lambda^+(0)+\dot{\gamma}_\lambda^-(0)=\alpha v_0$ for some constant α. v_0 divides E_λ into two open connected subarcs E_λ^+, E_λ^- in E_λ. And the subarcs E_λ^+, E_λ^- divides D_λ into two connected domains D_λ^+, D_λ^- in D_λ, i.e.

$$D_\lambda^\pm=\{\exp_p tv\mid v\in E_\lambda^\pm,\ t>0\}.$$

Then, it holds

$$\int_{\bar{D}_\lambda^+} K\,dv\leqq\sphericalangle(\dot{\gamma}_\lambda^+(0), v_0),$$

$$\int_{\bar{D}_\lambda^-} K\,dv\leqq\sphericalangle(\dot{\gamma}_\lambda^-(0), v^0)$$

where the sum of the left hand sides is less than that of the right hand side. Let $\gamma:[0,\infty)\to M$ be a ray from p tangent to v_0. We will show that $\{d(\gamma_\lambda^+(t), \gamma(t))\,|\,t\geqq 0\}$ is unbounded. If it were not so, then there exists a constant K_0 such that $d(\gamma_\lambda^+(t), \gamma(t))\leqq K_0$ for all $t\geqq 0$. For each $t>0$, let $\eta_t:[0, d(\gamma_\lambda^+(t), \gamma(t))]\to M$ be a shortest geodesic from $\gamma_\lambda^+(t)$ to $\gamma(t)$. For this η_t, we can consider the following two cases;

Case (A): there exists a subsequence $\{t_j\}$ of $\{t\}$ such that $t_j\uparrow\infty$ as $j\to\infty$ and

$$\eta_{t_j}((0, d(\gamma_\lambda^\dagger(t_j), \gamma(t_j)))\subset D_\lambda^\dagger, \quad j=1, 2, \cdots$$

Case (B): there exists a constant t_0 such that

$$\eta_t((0, d(\gamma_\lambda^\dagger(t), \gamma(t)))\subset M-D_\lambda^\dagger \quad \text{for all } t\geqq t_0.$$

In case (A), put $r_0:=\sphericalangle(\dot\gamma_\lambda^\dagger(0), v_0)-\int_{\bar D_\lambda^\dagger} K\,dv$ and let D_j be the domain surrounded by the geodesic triangle $\Delta_j(\gamma_\lambda^\dagger|[0, t_j], \gamma|[0, t_j], \eta_{t_j})$. Then

$$D_1\subset D_2\subset\cdots\subset D_j\subset D_{j+1}\subset\cdots\subset D_\lambda^\dagger$$

and

$$\bigcup_j D_j=D_\lambda^\dagger.$$

First assertion follows from the fact that $\gamma_\lambda^\dagger$, γ are rays and the fact that $\eta_{t_j}((0, d(\gamma_\lambda^\dagger(t_j), \gamma(t_j)))\cap\eta_{t_{j+1}}((0, d(\gamma_\lambda^\dagger(t_{j+1}), \gamma(t_{j+1}))))=\emptyset$ and $\eta_{t_j}((0, d(\gamma_\lambda^\dagger(t_j), \gamma(t_j)))\subset D_\lambda^\dagger$ for $j=1, 2, \cdots$. To see the second assertion, assume that $\bigcup_j D_j\subsetneqq D_\lambda^\dagger$. Then from topological observation, we see that $d(\gamma_\lambda^\dagger(t_j), \gamma(t_j))\to\infty$, $j\to\infty$. And this contradicts to the assumption (*). So $\int_{\bar D_\lambda} K\,dv\geqq\int_{\bar D_j} K\,dv$, $j=1, 2, \cdots$ and $\int_{\bar D_j} K\,dv\uparrow\int_{\bar D_\lambda} K\,dv$ as $j\to\infty$, because $K\geqq 0$. We apply Gauss-Bonnet's Theorem to the geodesic triangle Δ_j. Then we have

$$\begin{aligned}\sphericalangle(-\dot\gamma_\lambda^\dagger(t_j), \dot\eta_{t_j}(0))+\sphericalangle(\dot\gamma(t_j), \dot\eta_{t_j}(d(\gamma_\lambda^\dagger(t_j), \gamma(t_j)))&=\int_{\bar D_j} K\,dv+\pi-\sphericalangle(\dot\gamma_\lambda^\dagger(0), v_0)\\ &\leqq\pi+\int_{\bar D_\lambda^\dagger} K\,dv-\sphericalangle(\dot\gamma_\lambda^\dagger(0), v_0)\\ &=\pi-r_0.\end{aligned}$$

Thus

$$\sphericalangle(-\dot\gamma_\lambda^\dagger(t_j), \dot\eta_{t_j}(0))+\sphericalangle(\dot\gamma(t_j), \dot\eta_{t_j}(d(\gamma_\lambda^\dagger(t_j), \gamma(t_j)))\leqq\pi-r_0<\pi \quad \text{for } j=1, 2, \cdots.$$

Now, if we apply Toponogov's Comparison Theorem (see [4; pp. 183~]) to the geodesic triangle Δ_j, we have triangles $\Delta(A_j, B_j, C_j)$, $j=1, 2, \cdots$ in a flat Euclidean plane E^2 with the following properties;

(i) $A_jB_j=A_jC_j=t_j$, $B_jC_j=d(\gamma_\lambda^\dagger(t_j), \gamma(t_j))$

(ii) $\sphericalangle(\dot\gamma_\lambda^\dagger(0), v_0)\geqq\angle A_j$,
$\sphericalangle(-\dot\gamma_\lambda^\dagger(t_j), \dot\eta_{t_j}(0))\geqq\angle B_j$
$\sphericalangle(\dot\gamma(t_j), \dot\eta_{t_j}(d(\gamma_\lambda^\dagger(t_j), \gamma(t_j)))\geqq\angle C_j$.

Thus, by assumption,

$$B_jC_j\leqq K_0 \quad \text{for } j=1, 2, \cdots$$

and

$$A_jB_j=A_jC_j=t_j\uparrow\infty \quad \text{as } j\to\infty.$$

And hence, we have

$$\angle B_j + \angle C_j \to \pi \quad \text{as } j \to \infty.$$

But, for $j=1, 2, \cdots$,

$$\pi - r_0 \geqq \measuredangle(-\dot{\gamma}_{\lambda}^{+}(t_j), \dot{\eta}_{t_j}(0)) + \measuredangle(\dot{\gamma}(t_j), \dot{\eta}_{t_j}(d(\gamma_{\lambda}^{+}(t_j), \gamma(t_j)))) \geqq \angle B_j + \angle C_j.$$

Thus, letting $j \to \infty$, we have

$$\pi - r_0 \geqq \pi.$$

This is a contradiction. So case (A) does not occur.

If case (B) occur, then for each $t \geqq t_0$, there exists a number t'; $0 < t' < d(\gamma_{\lambda}^{+}(t), \gamma(t)))$ such that

$$\eta_t((t', d(\gamma_{\lambda}^{+}(t), \gamma(t))) \subset D_{\lambda}^{-}$$

and $\eta_t(t') = \gamma_{\lambda}^{-}(t'')$ for some $t'' > 0$. For each $t \geqq t_0$, let D_t be the domain surrounded by the geodesic triangle $\Delta_t(\gamma_{\lambda}^{-} | [0, t''], \gamma | [0, t], \eta_t | [t', d(\gamma_{\lambda}^{+}(t), \gamma(t))])$. Then, we have

$$D_{t_1} \subset D_{t_2} \quad \text{if } t_1 < t_2$$

because $\eta_{t_1}([t_1', d(\gamma_{\lambda}^{+}(t_1), \gamma(t_1))]) \cap \eta_{t_2}([t_2', d(\gamma_{\lambda}^{+}(t_2), \gamma(t_2))]) = \emptyset$. This is seen by topological observation and shortestness of η_{t_1} and η_{t_2}. And also we see that

$$\bigcup_{t \geqq t_0} D_t = D_{\lambda}^{-},$$

because $t' \geqq t - t_0$ and hence $t' \to \infty$ as $t \to \infty$. Then, by the same argument just as in case (A), we can derive a contradiction. So case (B) does not occur. Thus both cases do not occur. And hence our assertion is proved. Q. E. D.

2. Geodesic circles and poles.

In this section, we will show a slight generalization of Fact 2 mentioned in § 0. It is stated as follows.

Theorem 2. *Let M be a 2-dimensional complete Riemannian manifold diffeomorphic to an Euclidean plane E^2 and with non-negative Gaussian curvature K. If M satisfy the condition (*) for some point $p \in M$, then there exists at most one pole on M.*

Remark. As is stated in § 1, a manifold satisfying the assumption in Fact 2 also satisfies the assumption in Theorem 2. Thus, Theorem 2 is a generalization of Fact 2. And we can easily construct an example of positively curved manifold satisfying the assumptions in Theorem 2 which does not satisfy the assumptions in Fact 2.

Proof of Theorem 2. We will show that if $p\in M$ is a pole, then there exists no poles other than p. Since p is a pole, all geodesics starting from p are rays. Let $c:[0, \infty)\to M$ be a fixed ray from p. For each $t\geqq 0$ and $q\in M$, let $B_t(q):=\{q'\in M \mid d(q, q')\leqq t\}$ be the geodesic ball with radius t centered at q and for the fixed point p, let denote $B_t:=B_t(p)$ for the brevity. And for each $t\geqq 0$, let

$$C_t:=\bigcap_{s\geqq 0}\overline{(M-B_s(c(t+s))}=\overline{M-\bigcup_{s\geqq 0}B_s(c(t+s))}.$$

Lemma 1. *For each $t\geqq 0$,*

$$B_t\subset C_t.$$

Proof. If there exist a point $q\in B_t-c_t$, then $q\in(C_t)^C=\bigcup_{s\geqq 0}B_s(c(t+s))$. So $q\in B_s(c(t+s))$ for some $s>0$, i.e. $d(q, c(t+s))\leqq s$. On the other hand, $d(p, q)\leqq t$, because $q\in B_t$. Thus

$$t+s=d(p, c(t+s))\leqq d(p, q)+d(q, c(t+s))\leqq t+s.$$

From this, we see $q=c(t)\in C_t$, because c is a ray. This is a contradiction.

Q. E. D.

We now fix an arbitrary small number $\varepsilon>0$.

Lemma 2. *For the above $\varepsilon>0$, there exists a number $t_0=t_0(\varepsilon)$ such that*

$$2\pi-\varepsilon\leqq\int_{B_{t_0}}K\,dv\leqq 2\pi$$

$$\left(\Leftrightarrow\int_{\overline{M-B_{t_0}}}K\,dv\leqq\varepsilon\right).$$

Proof. This follows by the facts that

(i) $B_t\subset B_{t'}$ if $0\leqq t\leqq t'$

(ii) $\bigcup_{t\geqq 0}B_t=M$

(iii) $\int_M K\,dv=2\pi$ (by Theorem 1)

(iv) $K\geqq 0$. Q. E. D.

Lemma 3. *For the number $t_0>0$ obtained in Lemma 2, there exists a number $t_1=t_1(t_0)\geqq t_0$ such that for any $t\geqq t_1$ and for any points $x, y\in\partial B_t$, every shortest geodesic $\nu:[0, d(x, y)]\to M$ from x to y satisfies*

$$\nu([0, d(x, y)])\subset B_t-B_{t_0}.$$

Proof. If it were not so, then there exists a monotone divergent sequence $\{t_i\}$, $t_1\geqq t_0$, points $x_i, y_i\in\partial B_{t_i}$, $i=1, 2, \cdots$ and shortest geodesics $c_i:[0, d(x_i, y_i)]$

$\to M$ from x_i to y_i, $i=1, 2, \cdots$ such that

$$c_i([0, d(x_i, y_i)])\cap B_{t_0}\neq\emptyset, \quad i=1, 2, \cdots.$$

Let $s_i>0$, $i=1, 2, \cdots$ be numbers such that $c_i(s_i)\in B_{t_0}$. By the compactness of B_{t_0}, we can choose a convergent subsequence $\{\dot{c}_{i_j}(s_{i_j})\}\subset\{\dot{c}_i(s_i)\}$ and its limit vector v, i.e.

$$\dot{c}_{i_j}(s_{i_j})\to v \quad \text{as } j\to\infty.$$

Then, the geodesic $c_0:(-\infty, \infty)\to M$ defined by $c_0(t)=\exp tv$ is a line by definition. Here a geodesic $\gamma:(-\infty, \infty)\to M$ is, by definition, a line if any subarc of γ is a shortest connection between its end points. Then, by Toponogov's Splitting Theorem (see [2]), M is isometric to a flat Euclidean plane. And hence

$$\int_M K\, dv=0.$$

This is a contradiction. Q.E.D.

Lemma 4. *For the number t_1 obtained in Lemma 3, there exists a number $t_2=t_2(t_1)\geqq t_1$ such that for any point $q\in\partial B_{t_2}$ and for any ray $\eta:[0, \infty)\to M$ starting from q,*

$$\sphericalangle(\dot{\eta}(0), \dot{\gamma}_q(0))<\varepsilon,$$

where $\gamma_q:[0, \infty)\to M$ be the ray from p satisfying $\gamma_q(d(p, q))=q$.

Proof. We will show that there exists a number $t_2\geqq t_1$ satisfying the following property;
for any point $q\in\partial B_{t_2}$ and for any ray $\nu_q:[0, \infty)\to M$ starting from q,

$$\nu_q((0, \infty))\cap B_{t_1}=\emptyset.$$

If there exist no such t_2, then there exists a monotone divergent sequence $\{\tilde{t}_i\}$, points $q_i\in\partial B_{\tilde{t}_i}$, $i=1, 2, \cdots$ and rays $\nu_{q_i}:[0, \infty)\to M$ from q_i, $i=1, 2, \cdots$ such that

$$\nu_{q_i}((0, \infty))\cap B_{t_1}\neq\emptyset, \quad i=1, 2, \cdots.$$

Let s_i, $i=1, 2, \cdots$ be the numbers such that $\nu_{q_i}(s_i)\in B_{t_1}$. Then we can choose a convergent subsequence $\{\dot{\nu}_{q_{i_j}}(s_{i_j})\}\subset\{\dot{\nu}_{q_i}(s_i)\}$ and its limit vector w, because B_{t_1} is compact. Then, the geodesic $\rho:(-\infty, \infty)\to M$ defined by $\rho(s)=\exp sw$ is a line by definition. So, by the same reason mentioned in the proof of Lemma 3, we get a contradiction.

Now, the number t_2 satisfies our requirement. To see this, let $q\in\partial B_{t_2}$ and $\nu_q:[0, \infty)\to M$ be a ray from q. Then

$$\nu_q([0, \infty))\cap B_{t_1}=\emptyset.$$

Let D be the domain in M whose boundary is $\nu_q([0, \infty))\cup\gamma_q([d(p, q), \infty))$ and satisfy $B_{t_1}\subset M-D$. Then, by Lemma 2,

$$\int_{\bar{D}} K\, dv \leqq \int_{\overline{M-B_{t_1}}} K\, dv \leqq \int_{\overline{M-B_{t_0}}} K\, dv \leqq \varepsilon .$$

And

$$\int_{\bar{D}} K\, dv = \sphericalangle(\dot\nu_q(0), \dot\gamma_q(t_2)),$$

because $\int_M K\, dv = 2\pi$ by Theorem 1. Q. E. D.

Lemma 5. *There exists a constant* $t_3=t_3(t_2)\geqq t_2$ *such that for any point* $q\in\partial B_{t_2}$ *and for any* $t\geqq t_3$,

$$0\leqq d(q, c(t))-d(c(t_2), c(t))\leqq 8\varepsilon\cdot L .$$

Proof. Since c is a ray, for any $t\geqq t_2$ and any $q\in\partial B_t$,

$$d(p, c(t))=d(p, c(t_2))+d(c(t_2), c(t)) .$$

Form this and the triangle inequality

$$d(p, c(t))\leqq d(p, q)+d(q, c(t)),$$

we have

$$d(c(t_2), c(t))\leqq d(q, c(t)),$$

because $d(p, q)=d(p, c(t_2))=t_2$.

To prove the inequality

$$d(q, c(t))-d(c(t_2), c(t))\leqq 8\varepsilon\cdot L ,$$

at first, we prove the following;

there exists a constant $t_3=t_3(t_2)\geqq t_2$ such that for any $t\geqq t_3$ and $q\in\partial B_{t_2}$,

$$\sphericalangle(\dot\eta_t(0), \dot\gamma_q(d(p, q)))<2\varepsilon ,$$

where $\eta_t: [0, d(q, c(t))]\to M$ be a shortest geodesic from q to $c(t)$. If this were not true, then there exists a monotone increasing divergent sequence $\{t_i\}$, a sequence $\{q_i\}\subset\partial B_{t_2}$ and shortest geodesics $\eta_{t_i}: [0, d(q_i, c(t_i))]\to M$ from q_i to $c(t_i)$ such that

$$\sphericalangle(\dot\eta_{t_i}(0), \dot\gamma_{q_i}(t_2))\geqq 2\varepsilon , \quad i=1, 2, \cdots .$$

Since ∂B_{t_2} is compact, there exists a convergent subsequence $\{q_{i_j}\}\subset\{q_i\}$ and its limit point $q_0\in\partial B_{t_2}$, i.e. $q_{i_j}\to q_0\in\partial B_{t_2}$ as $j\to\infty$. Let $\eta_0: [0, \infty)\to M$ be the limit geodesic of $\{\eta_{t_j}\}$. Then η_0 is a shortest geodesic (=ray) starting from q_0 by definition and satisfies

$$\sphericalangle(\dot\eta_0(0), \dot\gamma_{q_0}(t_2))\geqq 2\varepsilon .$$

This is a contradiction by Lemma 4.

Now, for a fixed point $q\in\partial B_{t_2}$, we choose a number $t\geqq t_3$ and a shortest geodesic $\eta:[0, d(q, c(t))]\to M$ from q to $c(t)$ satisfying

$$\sphericalangle(\dot{\eta}(0), \dot{\gamma}_q(t_2))<2\varepsilon .$$

Then, by applying Gauss-Bonnet's Theorem to a geodesic triangle $\varDelta(c\,|\,[0, t], \eta, \gamma_q\,|\,[0, t_2])$,

$$\int_{\bar{D}} K\, dv=\sphericalangle(\dot{\gamma}_q(0), \dot{c}(0))+\sphericalangle(\dot{\eta}(0), -\dot{\gamma}_q(0))+\sphericalangle(\dot{c}(t), \dot{\eta}(d(q, c(t)))-\pi ,$$

where D is the domain surrounded by $\varDelta$. So, we have

$$\begin{aligned}\sphericalangle(\dot{c}(t), \dot{\eta}(d(q, c(t)))&\leqq\pi-\sphericalangle(\dot{\eta}(0), -\dot{\gamma}_q(0))\\ &<\pi-(\pi-2\varepsilon)\\ &=2\varepsilon ,\end{aligned}$$

because $\sphericalangle(\dot{\gamma}_q(0), \dot{c}(0))\geqq\int_{\bar{D}} K\, dv$ by Theorem A in [7; pp. 98~]. That is

$$\sphericalangle(\dot{c}(t), \dot{\eta}(d(q, c(t)))<2\varepsilon .$$

Let $\beta:[0, d(q, c(t_2))]\to M$ be a shortest geodesic from q to $c(t_2)$. Then, since B_{t_2} is totally convex,

$$\beta([0, d(q, c(t_2))])\subset B_{t_2} .$$

Here a set $C\subset M$ is called totally convex if for any points $x, y\in C$ and any geodesic $\alpha:[0, a]\to M$ from x to y, $\alpha([0, a])\subset C$. Since p is a pole, for each $t\geqq 0$, we have

$$B_t=\bigcap_{\eta}\overline{M-\bigcup_{s\geqq 0} B_s(\eta(t+s))} ,$$

where intersection is taken for all rays starting from p. And total convexity of the set $\bigcap_{\eta}\overline{M-\bigcup_{s\geqq 0} B_s(\eta(t+s))}$ is shown in [1]. By Gauss-Lemma (see [4; pp. 136~]), c and smooth curve ∂B_{t_2} (because p is a ray) meet orthogonally at $c(t_2)$. So

$$\sphericalangle(-\dot{\beta}(d(q, c(t_2)), c(t_2))\geqq\frac{\pi}{2} .$$

By the same reason, we have

$$\sphericalangle(\dot{\beta}(0), \dot{\gamma}_q(t_2))\geqq\frac{\pi}{2} .$$

From this inequality, we have

$$\sphericalangle(\dot{\beta}(0), \dot{\eta}(0))\geqq\frac{\pi}{2}-2\varepsilon .$$

Applying Gauss-Bonnet's Theorem to the geodesic triangle $\varDelta(\beta, c\,|\,[t_2, t], \eta)$, we have

$$(*)\qquad \int_{\bar{D}} K\,dv = \sphericalangle(\dot{\beta}(0),\ \dot{\gamma}(0)) + \sphericalangle(-\dot{\beta}(d(q,\ c(t_2))),\ \dot{c}(t_2)) + \sphericalangle(\dot{\gamma}(d(q,\ c(t)),\ \dot{c}(t)) - \pi\,,$$

where D is the domain surrounded by $\varDelta$. Since $\bar{D} \subset M - B_{t_0}$, by Lemma 2, it holds

$$\int_{\bar{D}} K\,dv < \varepsilon\,.$$

Thus, from (*), we have

$$\varepsilon > \sphericalangle(\dot{\beta}(0),\ \dot{\gamma}(0)) + \frac{\pi}{2} - \pi\,,$$

i. e.

$$\sphericalangle(\dot{\beta}(0),\ \dot{\gamma}(0)) < \frac{\pi}{2} + \varepsilon\,.$$

Hence, we have

$$\frac{\pi}{2} - 2\varepsilon < \sphericalangle(\dot{\beta}(0),\ \dot{\gamma}(0)) < \frac{\pi}{2} + \varepsilon\,.$$

Again, by (*),

$$\varepsilon > \frac{\pi}{2} - 2\varepsilon + \sphericalangle(-\dot{\beta}(d(q,\ c(t_2)),\ \dot{c}(t_2)) - \pi\,.$$

Thus, we have

$$\frac{\pi}{2} \leqq \sphericalangle(-\dot{\beta}(d(q,\ c(t_2)),\ \dot{c}(t_2)) < \frac{\pi}{2} + 3\varepsilon\,.$$

Now, we apply Toponogov's Comparison Theorem to the geodesic triangle $\varDelta$. Then, we have a triangle $\tilde{\varDelta}(\tilde{A},\ \tilde{B},\ \tilde{C})$ in a flat Euclidean plane E^2 satisfying the following;

$$\tilde{A}\tilde{B} = d(c(t_2),\ c(t))\,,$$
$$\tilde{B}\tilde{C} = d(c(t),\ q)\,,$$
$$\tilde{C}\tilde{A} = d(q,\ c(t_2))$$

and

$$\angle\tilde{A} \leqq \sphericalangle(-\dot{\beta}(d(q,\ c(t_2)),\ \dot{c}(t_2))\,,$$
$$\angle\tilde{B} \leqq \sphericalangle(\dot{c}(t),\ \dot{\gamma}(d(q,\ c(t)))\,,$$
$$\angle\tilde{C} \leqq \sphericalangle(\dot{\beta}(0),\ \dot{\gamma}(0))\,.$$

Thus

$$\angle\tilde{A} \leqq \frac{\pi}{2} + 3\varepsilon\,,$$
$$\angle\tilde{B} \leqq 2\varepsilon\,,$$
$$\angle\tilde{C} \leqq \frac{\pi}{2} + \varepsilon$$

and hence, we have

$$\frac{\pi}{2} - 3\varepsilon \leqq \angle\tilde{A} \leqq \frac{\pi}{2} + 3\varepsilon\,,$$

$$\frac{\pi}{2}-5\varepsilon \leqq \angle \tilde{C} \leqq \frac{\pi}{2}+\varepsilon,$$

because $\angle \hat{A}+\angle \tilde{B}+\angle \tilde{C}=\pi$. Thus we have

$$4\varepsilon \leqq \angle \tilde{A}-\angle \tilde{C} \leqq 8\varepsilon.$$

Now, by sine formula, we have

$$\begin{aligned} & d(q, c(t))-d(c(t_2), c(t)) \\ & =(\sin \tilde{A}-\sin \tilde{C}) \cdot d(q, c(t_2))/\sin \tilde{B} \\ & =2 \cdot \sin(\tilde{B}/2) \cdot \sin((\tilde{A}-\tilde{C})/2) \cdot d(q, c(t_2))/\sin B \\ & \leqq 2 \cdot \sin((\tilde{A}-\tilde{C})/2) \cdot d(q, c(t_2)) \\ & \leqq 2 \cdot \sin(4\varepsilon) \cdot d(q, c(t_2)) \\ & \leqq 8\varepsilon \cdot d(q, c(t_2)) \\ & \leqq 8\varepsilon \cdot L. \end{aligned}$$

Q. E. D.

Lemma 5 shows that for a fixed $t \geqq t_3$,

$$\partial B_{t_2} \subset B_{t-t_2+8\varepsilon \cdot L}(c(t))$$

i. e.

$$\partial B_{t_2} \cap (M-B_{t-t_2+8\varepsilon \cdot L}(c(t)) = \emptyset.$$

So, putting $C_t=\overline{M-\bigcup_{s \geqq 0} B_s(c(t+s))}$ for $t \geqq 0$,

$$\partial B_{t_2} \cap C_{t_2-8\varepsilon \cdot L} = \emptyset.$$

Thus, we have

$$C_{t_2-8\varepsilon \cdot L} \subset B_{t_2}.$$

Now, for any numbers t and t' such that $0 \leqq t \leqq t'$, it holds

$$C_t=\{q \in C_{t'} \mid d(q, \partial C_{t'}) \geqq t'-t\}.$$

This follows by the fact that for any point $q \in M$ and for any numbers t and t' such that $0 \leqq t \leqq t'$, it holds

$$B_{t'}(q)=\{q' \in M \mid d(q', B_t(q)) \leqq t'-t\}.$$

And for a point p which is a pole, it holds

$$B_t(p)=\{q' \in B_{t'}(p) \mid d(q', \partial B_{t'}(p)) \geqq t'-t\}$$

for all t and t' such that $0 \leqq t \leqq t'$, because all geodesics starting from p are rays. Thus, we have

$$C_0=\{q \in C_{t_2-8\varepsilon \cdot L} \mid d(q, \partial C_{t_2-8\varepsilon \cdot L}) \geqq t_2-8\varepsilon \cdot L\}$$

and

$$B_{8\varepsilon \cdot L}=\{q \in B_{t_2} \mid d(q, \partial B_{t_2}) \geqq t_2-8\varepsilon \cdot L\}.$$

From these facts, we have

$$C_0 \subset B_{8\varepsilon \cdot L},$$

because $C_{t_2-8\varepsilon \cdot L} \subset B_{t_2}$. So, since ε is any small positive number, we have

Proposition 2. *Under the assumption in Theorem 2, it holds*

$$C_0 = B_0 = \{p\}.$$

Now, we assume that there exist an another pole $p_1 \neq p$. Let $c:[0, \infty) \to M$ be a ray starting from p through p_1, i.e. $c(d(p, p_1)) = p_1$. Then, applying Proposition 2, we have

$$C_0 = \{p\}.$$

Since p_1 is also a pole, applying the same argument for the ray $\tilde{c}:[0, \infty) \to M$ defined by $\tilde{c}(t) = c(t + d(p, p_1))$, we have

$$C_{d(p,p_1)} = \{p_1\}.$$

But this is a contradiction, because $c([0, d(p, p_1)]) \subset C_{d(p,p_1)}$. Q.E.D.

It is an interesting problem whether Theorem 2 holds under the only assumption that

$$\int_M K\, dv = 2\pi.$$

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