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WEAK CONVERGENCE TO BROWNIAN EXCURSION

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ABSTRACT. We show that Brownian motion conditioned to be positive and tied-down is Brownian excursion. For a random walk defined by weakly dependent random variables, a conditional limit theorem is proved under some conditions on the path.

1. Introduction. Our purpose in this paper is to investigate relationship between Brownian motion and Brownian excursion.

Let $\{W(t): t\geqq 0\}$ be a standard Brownian motion. Let

$$n_s(x)=(2\pi s)^{-1/2}\exp(-x^2/2s),\quad N_s(a, b)=\int_a^b n_s(x)dx,\quad \text{and}$$

$$g(t, x, y)=n_t(y-x)-n_t(y+x).$$

Brownian excursion, $W_0^+=\{W_0^+(t): 0\leqq t\leqq 1\}$ is a continuous, nonhomogeneous Markov process. The transition density is given by

$$P[W_0^+(t)\in dy]=p_0^+(0, 0, t, y)=\frac{2y^2\exp(-y^2/2t(1-t))}{(2\pi t^3(1-t)^3)^{1/2}}dy$$

for $0<t\leqq 1$; for $0<s<t<1$ and $x, y>0$,

$$P[W_0^+(t)\in dy \mid W_0^+(s)=x]=p_0^+(s, x, t, y)dy$$

$$=g(t-s, x, y)\{(1-s)/(1-t)\}\frac{y\exp(-y^2/2(1-t))}{x\exp(-x^2/2(1-s))}dy.$$

Let $\{W^+(t): 0\leqq t\leqq 1\}$ be Brownian meander.

According to Durret-Iglehart-Miller [2] we introduce some notation, for $0\leqq t\leqq 1$, let $m(t)=\inf\{W(s); 0\leqq s\leqq t\}$ and $M(t)=\sup\{W(s); 0\leqq s\leqq t\}$. Furthermore let $C=C[0, 1]$ be the space of continuous functions on $[0, 1]$ and $\mathcal{C}$ be the topological σ-field of C induced by the supremum metric $d(\ ,\ )$. For a random function Y in $(C, \mathcal{C})$ let Q be the probability measure of Y on $(C, \mathcal{C})$ and Q_Λ the probability measure restricted by a Borel set $\Lambda\in\mathcal{C}$ with $Q(\Lambda)>0$ which is defined by $Q_\Lambda(A)=Q(A)/Q(\Lambda)$ for $A\in\Lambda\cap\mathcal{C}$. Then we can define a random function $Y|\Lambda$ with the probability measure Q_Λ as the restriction of Y to $Y^{-1}(\Lambda)$. We now define the following conditioned random function of $(C, \mathcal{C})$, for any $\varepsilon>0$,

$$\overline{W}_\varepsilon = W \mid \{m(1) > -\varepsilon,\ W(1) < \varepsilon\}.$$

This process is Markov by virtue of Lemma (1.5) in [2].

2. Convergence of conditioned Brownian motion to Brownian excursion.

In this section we prove the following.

Theorem (2.1). $\overline{W}_\varepsilon \Rightarrow W_0^+$ *as* $\varepsilon \to 0$.

Proof. We first show that the finite dimensional distributions converge. Since $\overline{W}_\varepsilon$ and W_0^+ are Markov, it suffices to show that the probability transition densities converge. For $0<t<1$ and $y>0$, we have

$$\begin{aligned}\bar{p}_\varepsilon(0, 0, t, y) &= P[W(t)\in dy \mid m(1)>-\varepsilon,\ W(1)<\varepsilon] \\ &= \frac{P[W(t)\in dy,\ m(t)>-\varepsilon]P^y[m(1-t)>-\varepsilon,\ W(1-t)<\varepsilon]}{P[m(1)>-\varepsilon,\ W(1)<\varepsilon]} \\ &= \frac{g(t, \varepsilon, y+\varepsilon)\int_{-\varepsilon}^{\varepsilon} g(1-t, y+\varepsilon, u+\varepsilon)du\,dy}{\int_{-\varepsilon}^{\varepsilon} g(1, \varepsilon, u+\varepsilon)du}.\end{aligned}$$

Using L'Hospital's rule twice on the ratio above gives

$$\lim_{\varepsilon\to\infty} \bar{p}_\varepsilon(0, 0, t, y) = \frac{2y^2\exp(-y^2/2t(1-t))}{(2\pi t^3(1-t)^3)^{1/2}} = \bar{p}_0(0, 0, t, y).$$

For $0<s<t\leqq 1$ and $x, y>0$

$$\begin{aligned}\bar{p}_\varepsilon(s, x, t, y)dy &= P[\overline{W}_\varepsilon(t)\in dy \mid \overline{W}_\varepsilon(s)=x] \\ &= \{P[W(s)\in dx,\ m(s)>-\varepsilon]P^x[m(t-s)>-\varepsilon,\ W(t-s)\in dy] \\ &\quad \times P^y[m(1-t)>-\varepsilon,\ W(1-t)>\varepsilon]\}/\{P[W(s)\in dx,\ m(s)>-\varepsilon] \\ &\quad \times P^x[m(1-s)>-\varepsilon,\ W(1-s)<\varepsilon]\} \\ &= \frac{g(t-s, x+\varepsilon, y+\varepsilon)\int_{-\varepsilon}^{\varepsilon} g(1-t, y+\varepsilon, u+\varepsilon)du\,dy}{\int_{-\varepsilon}^{\varepsilon} g(1-s, x+\varepsilon, u+\varepsilon)du}.\end{aligned}$$

Divide numerator and denominator by ε^2 and use the same application of l'Hospital's rule as in above we get

$$\lim_{\varepsilon\to\infty} \bar{p}_\varepsilon(s, x, t, y) = p_\varepsilon^+(s, x, t, y),$$

which implies the convergence of the finite dimensional distributions.

The tightness of $\overline{W}_\varepsilon$ can be shown by a slight modification of the proof of Theorem (4.1) in [2].

3. Conditioned limit theorem for weakly dependent random variables. Let $\{X_i, i\in Z\}$ be a sequence of strictly stationary random variables. Let $\mathcal{F}_n^m$ denote the σ-field generated by random variables X_i, $i=m, m+1, \cdots, n$. Suppose that the sequence $\{X_i, i\in Z\}$ satisfies the strong mixing condition, that is

$$\alpha(n)=\sup_{A\in\mathcal{F}_{-\infty}^0,\, B\in\mathcal{F}_n^\infty}|P[A\cap B]-P[A]P[B]|\to 0 \quad \text{as } n\to\infty .$$

If the following limit exists, $\sigma^2\equiv\lim_{n\to\infty} n^{-1}E\left\{\sum_{i=1}^n X_i\right\}^2>0$, then define a random element Y_n on $C[0, 1]$ by

$$Y_n(t)=\begin{cases}(nt)X_1/(\sigma n^{1/2}), & \text{for } t\in[0, 1/n]\\ \sum_{i=1}^k X_i/(\sigma n^{1/2})+(nt-k)X_{k+1}/(\sigma n^{1/2}), & \text{for } t\in(k/n, (k+1)/n],\\ & k=1, \cdots, n-1.\end{cases}$$

Let P_n be the distribution of $\{Y_n(t), 0\leqq t\leqq 1\}$ and Q the Wiener measure on $(C, \mathcal{C})$. The Prokhorov-Lévy metric $\rho(\ ,\ )$ on the space of probability measures on $(C, \mathcal{C})$ is defined by

$$\rho(R, P)=\inf\{\varepsilon>0;\ R[B]\leqq\varepsilon+P[y;\ d(x, y)<\varepsilon,\ x\in B],$$
$$P[B]\leqq\varepsilon+R[y;\ d(x, y)<\varepsilon,\ x\in B] \text{ for all } B\in\mathcal{C}\},$$

where R and P are probability measures on $(C, \mathcal{C})$.
S. Kanagawa ([3] page 104) proved the following:

Theorem (1 [3] page 104). *Let $\{X_i, i\in Z\}$ be a sequence of strictly stationary random variables with $EX_1=0$ and $E|X_1|^r<\infty$ for some $r>2$. Suppose that the sequence $\{X_i, i\in Z\}$ satisfies the strong mixing condition with coefficient $\alpha(n)$ and that there exists s with $2<s<r$ such that*

$$\sum_{i=1}^\infty (\alpha(i))^{(r-s)/rs}<\infty . \tag{3.1}$$

If $s\leqq 4$, then for any $\delta<s(s-2)/\{4(s-1)(s+1)\}$, we have

$$\rho(P_n, Q)=o(n^{-\delta}), \quad \text{as } n\to\infty.$$

If $r>4$ and (3.1) holds for some s with $4<s<r$, then for any $k<(s-4)/30(s+1)$, we have

$$\rho(P_n, Q)=o(n^{-2/15-k}), \quad \text{as } n\to\infty.$$

Define $m_n=\inf_{0\leq t\leq 1} Y_n(t)$ and put $\overline{Y}_n=(Y_n|m_n>\varepsilon_n, Y_n(1)<\varepsilon_n)$, $n\geqq 1$, where $\{\varepsilon_n, n\geqq 1\}$ is a sequence of positive constants.

Now we prove the following theorem.

Theorem (3.1). *Under the assumption of Theorem* 1 ([3] page 104) $\overline{Y}_n \Rightarrow W_0^+$, $n \to \infty$ *provided* $\{\varepsilon_n, n \geqq 1\}$ *is a sequence of positive numbers such that, in the case* $s \leqq 4$, *for every* $\delta < s(s-2)/\{4(s-1)(s+1)\}$, $(n^\delta \varepsilon_n^2) = O(1)$ *as* $n \to \infty$,

and, in the case $4 < s < r$, *for every* $k < (s-4)/30(s+1)$, $(n^{2/15+k} \varepsilon_n^2) = O(1)$ *as* $n \to \infty$.

Proof. Let A be a set such that $P[W_0^+ \in \partial A] = 0$. We are going to show that

$$\lim_{n \to \infty} P[\overline{Y}_n \in A] = P[W_0^+ \in A]. \tag{3.2}$$

Indeed, we have

$$\begin{aligned} P[\overline{Y}_n \in A] = \{ & P[Y_n \in A, m_n > -\varepsilon_n, Y(1) < \varepsilon_n] - P[W \in A, m(1) > -\varepsilon_n, W(1) < \varepsilon_n] \\ & + P[W \in A, m(1) > -\varepsilon_n, W(1) < \varepsilon_n]\} / \{P[m_n > -\varepsilon_n, Y_n(1) < \varepsilon_n] \\ & - P[m(1) > -\varepsilon_n, W(1) < \varepsilon_n] + P[m(1) > -\varepsilon_n, W(1) < \varepsilon_n]\} \\ = \{ & I_1^{(n)} + P[W \in A, m(1) > -\varepsilon_n, W(1) > \varepsilon_n]\} / \{I_2^{(n)} + P[m(1) > -\varepsilon_n, \\ & W(1) < \varepsilon_n]\}. \end{aligned}$$

By our assumptions and Theorem 1 ([3] page 104)

$$I_1^{(n)} / \varepsilon_n^2 \to 0, \quad I_2^{(n)} / \varepsilon_n^2 \to 0, \quad \text{as } n \to \infty.$$

Furthermore, one can verify that

$$P[m(1) > -\varepsilon_n, W(1) < \varepsilon_n] / \varepsilon_n^2 \to \sigma / (2\pi)^{1/2},$$

and by Theorem (2.1) we have

$$\lim_{n \to \infty} P[W \in A \mid m(1) > -\varepsilon_n, W(1) < \varepsilon_n] = P[W_0^+ \in A],$$

so that we get (3.2). This fact proves Theorem (3.1).

References

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