YOKOHAMA MATHEMATICAL
JOURNAL VOL. 36, 1988

ASYMPTOTIC NORMALITY OF NEAREST NEIGHBOR REGRESSION FUNCTION ESTIMATES BASED ON SOME DEPENDENT OBSERVATIONS

(Dedicated to Professor Yukihiro Kodama on his 60th Birthday)

By

KEN-ICHI YOSHIHARA

(Received December 13, 1987)

1. Introduction. For a random vector (X, Y) in R^2 with finite expectation $E|Y|$, the regression function $m(x)=E(Y|X=x)$, $x\in R$, of Y on X exists and is (μ-almost surely in x) uniquely defined in view of the equation

$$m(X)=E(Y|X)$$

where R^2 is the 2-dimensional Euclidean space and μ is the distribution of X.

Let $\{(X_i, Y_i)\}$ be independent random observations each of which has the same distributions as (X, Y).

Stute (1984) considered a smoothed nearest neighborhood estimate

$$m_n(x_0)=(na_n)^{-1}\sum_{i=1}^{n} Y_i K\left(\frac{F(x_0)-F(X_i)}{a_n}\right) \tag{1.1}$$

of $m(x_0)$ and obtained the central limit theorem of $m_n(x_0)$ for μ-almost all $x_0\in R$.

In this paper, we show that Stute's result (1984) remains true for some weakly dependent observations if $E|Y|^{2+\delta}<\infty$ for some $\delta>0$.

2. Main results. Let $\{(X_n, Y_n),\ n=0, \pm1, \pm2, \cdots\}$ be a strictly stationary sequence of random vectors in the plane with distribution function (d.f.) $H(x, y)$. We consider the following two conditions:

(I) $\{(X_n, Y_n)\}$ satisfies the *-mixing condition, i.e.,

$$\text{(2.1)} \qquad \psi(n)=\sup_{A\in\mathcal{M}_{-\infty}^{0},\, B\in\mathcal{M}_{n}^{\infty}}\left|\frac{P(AB)-P(A)P(B)}{P(AB)}\right|\longrightarrow 0 \quad \text{as} \quad n\to\infty\,;$$

(II) $\{(X_n, Y_n)\}$ satisfies the ϕ-mixing condition, i.e.,

$$\text{(2.2)} \qquad \phi(n)=\sup_{A\in\mathcal{M}_{-\infty}^{0},\, B\in\mathcal{M}_{n}^{\infty}}\left|\frac{P(AB)-P(A)P(B)}{P(A)}\right|\longrightarrow 0 \quad \text{as} \quad n\to\infty.$$

Here, $\mathcal{M}_a^b$ denotes the σ-algebra generated by $(X_a, Y_a), \cdots, (X_b, Y_b)$ $(a\leq b)$.

Let H_n denote the (bivariate) empirical d.f. of the sample $\{(X_1, Y_1), \cdots,$

$(X_n, Y_n)\}$ and F_n the empirical d.f. of $\{X_1, \cdots, X_n\}$. Let K be a twice continuously differentiable kernel function vanishing outside some bounded interval. We assume without loss of generality that K vanishes outside $(-1, 1)$. Further, let $\{a_n\}$ be any bandsequence such that $a_n \downarrow 0$.

We consider the following nearest neighbor estimator of $m(x_0)$:

$$\begin{aligned} m_n(x_0) &= (n a_n)^{-1} \sum_{i=1}^{n} Y_i K\Big(\frac{F_n(x_0)-F_n(X_i)}{a_n}\Big) \\ &= a_n^{-1} \int y K\Big(\frac{F_n(x_0)-F_n(X_i)}{a_n}\Big) H_n(dx, dy). \end{aligned} \tag{2.3}$$

Now, we state our main results, which are extensions of Stute's result (1984).

Theorem 1. *Let $\{(X_i, Y_i)\}$ be a strictly stationary *-mixing sequence of random vectors satisfying the following conditions;*

(i) *each X_i has a continuous d.f. F and*

(ii) *there exists a positive number δ for which $E|Y_i|^{2+\delta}<\infty$ and $\sum(n+1)^2\phi^{1/4}(n)<\infty$. Let*

$$\sigma_1^2 = \mathrm{Var}(Y_1 | X_1 = x_0) \int_{-1}^{1} K^2(u) du > 0. \tag{2.4}$$

Suppose $n a_n^3 \to \infty$. Then, we have

$$(n a_n)^{1/2}\{m_n(x_0) - \bar{m}_n(x_0)\} \xrightarrow{D} N(0, \sigma_1^2) \tag{2.5}$$

for μ-almost all $x_0 \in R$ where

$$\bar{m}_n(x_0) = a_n^{-1} \int y K\Big(\frac{F(x_0)-F(x)}{a_n}\Big) H(dx, dy). \tag{2.6}$$

Theorem 2. *Let $\{(X_n, Y_n)\}$ be a strictly stationary ϕ-mixing sequence of random vectors satisfying the following conditions;*

(i) *each X_i has a continuous d.f. F and*

(ii) *there exists a positive number δ for which $E|Y_i|^{2+\delta}<\infty$ and $\sum_{n=1}^{\infty}(n+1)^2\phi^{1/4}(n)<\infty$.*

Suppose $n a_n^3 \to \infty$. Suppose further that

$$\sigma_0^2 = \lim_{n\to\infty}\Big(\frac{n}{a_n}\Big) E\Big[\int\{y - m(x_0)\} K\Big(\frac{F((x_0)-F(x_1)}{a_n}\Big) \times \{H_n(dx, dy) - H(dx, dy)\}\Big]^2 \tag{2.7}$$

exists and is positive. Then, the conclusion of Thearem 1 holds.

Next, let

$$m\circ F^{-1}(u)=E(Y_1|F(X_1)=u).$$

Corollary. *Let K be a twice continuously differentiable kernel function such that $K(u)=0$ for $|u|\geqq 1$, $\int K(u)du=1$ and $\int uK(u)du=0$. Suppose the conditions in Theorem 2 hold. Suppose further $m\circ F^{-1}$ is twice continuously differentiable in a neighborhood of x_0 $(0<F(x_0)<1)$ and $na_n^5\to 0$. Then*

$$(na_n)^{1/2}(\bar{m}_n(x_0)-m(x_0))\longrightarrow 0 \quad as \quad n\to\infty \tag{2.8}$$

holds. Hence, we have

$$(an_n)^{1/2}\{m_n(x_0)-m(x_0)\}\xrightarrow{D} N(0, \sigma_0^2). \tag{2.9}$$

3. Auxiliary results. Let $\{\xi_i\}$ be a strictly stationary ϕ-mixing sequence of d-dimensional random vectors.

Let $i_1<i_2<\cdots<i_k$ be arbitrary integers and put

$$\begin{aligned}&P_j^{(k)}(E^{(j)}\times E^{(k-j)})=P((\xi_{i_1},\cdots,\xi_{i_j})\in E^{(j)})P((\xi_{i_{j+1}},\cdots,\xi_{i_k})\in E^{(k-j)})\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (j=1,\cdots,k).\\ &P_0^{(k)}(E^{(k)})=P((\xi_{i_1},\cdots,\xi_{i_k})\in E^{(k)}).\end{aligned} \tag{3.1}$$

where $E^{(i)}$ is a Borel set in R^{id}. The following lemma is a special case of Lemma 1 in Yoshihara (1976) since a ϕ-mixing sequence is an absolutely regular sequence.

Lemma 3.1. *Let $h(x_1,\cdots,x_k)$ be a Borel function such that*

$$\int\cdots\int|h(x_1,\cdots,x_k)|^{1+\delta}dP_j^{(k)}\leqq M \qquad (0\leqq j\leqq k-1) \tag{3.2}$$

for some $\delta>0$. Then

$$\begin{aligned}&\left|\int\cdots\int h(x_1,\cdots,x_k)dP_0^{(k)}-\int\cdots\int h(x_1,\cdots,x_k)dP_j^{(k)}\right|\\ &\qquad\leqq 4M^{1/(1+\delta)}\beta^{\delta/(1+\delta)}(i_{j+1}-i_j).\end{aligned} \tag{3.3}$$

Next, let $\{X_n\}$ be a strictly stationary ϕ-mixing sequence, and put

$$\alpha_n(x)=n^{1/2}\{F_n(x)-F(x)\}, \qquad x\in R \tag{3.4}$$

which denotes the empirical process pertaining to $X_1,\cdots,X_n$.

In the following, we often use the well-known inequalities.

Lemma 3.2. *Let $\{\xi_i\}$ be *-mixing with mixing coefficient $\psi(k)$. Let η be*

$\mathcal{M}_{-\infty}^{k}$*-measurable and let* ζ *be* $\mathcal{M}_{k+n}^{\infty}$*-measurable. If* $E|\eta|<\infty$ *and* $E|\zeta|<\infty$*, then*

(3.5) $$|E\xi\eta-E\xi E\eta|\leqq\phi(n)E|\xi|E|\eta|.$$

(See, for example, Samur (1984)).

Lemma 3.3. *Let* $\{\xi_i\}$ *be* ϕ*-mixing with mixing coefficient* $\phi(k)$*. Let* η *be* $\mathcal{M}_{-\infty}^{k}$*-measurable and let* ζ *be* $\mathcal{M}_{k+n}^{\infty}$*-measurable. If* $E|\eta|^r<\infty$, $E|\zeta|^s<\infty$ *and* $r^{-1}+s^{-1}=1$ $(r, s>0)$*, then*

(3.6) $$|E\xi\eta-E\xi E\eta|\leqq 2\{\phi(n)\}^{1/r}\{E|\eta|^r\}^{1/r}\{E|\zeta|^s\}^{1/s}.$$

(See, for example, Billingsley (1968)).

Lemma 3.4. *Suppose* $\{\xi_i\}$ *is a strictly stationary* ϕ*-mixing sequence of zero-mean random variables such that* $M_\delta=E|\xi_i|^{2+\delta}$ *for some* δ (>0) *and*

$$c_0=\sum_{k=0}^{\infty}(k+1)^2\phi^{1/4}(k) \qquad (\phi(0)=1).$$

Then

(3.7) $$E\left|\sum_{i=1}^{n}\xi_i\right|^{2+\delta}\leqq c_0M_\delta n^{1+(\delta/2)}$$

(See, Corollary 2.1 in Utev (1984)).

Next, let $\{X_n\}$ be a strictly stationary ϕ-mixing sequence of random variables and put

(3.8) $$\alpha_n(x)=n^{1/2}\{F_n(x)-x\}, \qquad 0\leqq x\leqq 1,$$

which denotes the uniform empirical process pertaining to $X_1, \cdots, X_n$. In the following, we often use the well-known representation

$$\alpha_n(x)=\bar{\alpha}_n(F(x)), \qquad x\in R$$

of α_n in terms of a uniform empirical process α_n.

From now on, we shall agree to denote by the letter c, with or without subscript, some quantity bounded in absolute value.

Lemma 3.5. *Let* $\{X_n\}$ *be a strictly stationary* ϕ*-mixing sequence of random variables. If* $\sum_{n=1}^{\infty}n\phi^{1/2}(n)<\infty$*, then*

(3.9) $$P(\sup_{s\leq F(x)\leq s+C_1a_n}|\bar{\alpha}_n(F(x))-\bar{\alpha}_n(s)|\geqq\lambda a_n^{1/2})\leqq c\lambda^{-4}$$

for all n *sufficiently large.*

Proof. In Sen (1971) it was proved that for every ε $(0<\varepsilon<1)$, if $t-s>\varepsilon n^{-1}$, then for all n

$$E|\bar{\alpha}_n(t)--\bar{\alpha}_n(s)|^4 \leqq c\varepsilon^{-1}(t-s)^2.$$

Hence, by the method of the proof of (22.20) in Billingsley (1968) we have

$$P(\sup_{s\leq t\leq s+mp}|\bar{\alpha}_n(t)-\bar{\alpha}_n(s)|\geqq \lambda a_n^{1/2})\leqq c\varepsilon^{-1}\lambda^{-4}a_n^{-2}(mp)^2. \tag{1.10}$$

Thus, putting $mp=C_1a_n$, we have (3.9) from (3.10). □

Lemma 3..6 *Let $\varepsilon>0$ be given arbitrarily. Then, there is a constant C_1 such that, up to an event of probability less than or equal to ε, we have*

$$|F(x_0)-F(x)|\leqq C_1a_n \tag{3.11}$$

whenever

$$|F_n(x_0)-F_n(x)|\leqq a_n. \tag{3.12}$$

Poof. Since by Theorem 22.1 in Billingsley (1968) (cf. Sen (1971))

$$\sup_{x:\,0\leq F(x)\leq 1}\sqrt{n}\,|F_n(x)-F(x)| \xrightarrow{D} \sup_{0\leq t\leq 1}|Z(t)| \qquad (n\to\infty),$$

so for an arbitrary number $\varepsilon>0$ there exists some finite number C such that for all n sufficiently large

$$P(A_n(C))\geqq 1-\varepsilon$$

where

$$A_n(C)=\{\sup_{x:\,0\leq F(x)\leq 1}|F_n(x)-F(x)|\leqq Cn^{-1/2}\}.$$

Because of (3.12), on the set $A_n(C)$ we have that

$$\begin{aligned}|F(x_0)-F(x)| &\leqq |F(x_0)-F_n(x_0)|+|F_n(x_0)-F_n(x)|+|F_n(x)-F(x)|\\ &\leqq a_n+2Cn^{-1/2}\leqq C_1a_n\end{aligned}$$

for some $C_1<\infty$, and the proof is completed. □

4. Lemmas. In this and the following section, we always assume that conditions of Theorem 2 are satisfied. We note first that if K is twice differentiable, then by Taylor's expansion we have

$$\begin{aligned} m_n(x_0)=&a_n^{-1}\int yK\Big(\frac{F(x_0)-F(x)}{a_n}\Big)H_n(dx,\,dy)\\ &+a_n^{-2}\int y\{F_n(x_0)-F_n(x)-F(x_0)+F(x)\}K'\Big(\frac{F(x_0)-F(x)}{a_n}\Big)H_n(dx,\,dy) \end{aligned} \tag{4.1}$$

$$+a_n^{-3}\int y\{F_n(x_0)-F_n(x)-F(x_0)+F(x)\}^2\frac{K''(\Delta)}{2}H_n(dx,\ dy)$$
$$=I_1+I_2+I_3 \text{ (say)},$$

where Δ is some number between $a_n^{-1}\{F_n(x_0)-F(x)\}$ and $a_n^{-1}\{F(x_0)-F(x)\}$.

Lemma 4.1. *$(na_n)^{1/2}I^3\to 0$ in probability as $n\to\infty$.*

Proof. Since K vanishes outside $(-1, 1)$, the above expansion of $m_n(x_0)$ holds true with integration restricted to those x for which $|F_n(x)-F(x)|<a_n$. Let $b_n=(na_n^3)^{1/6}$. Then, by Lemma 3.5 and 3.7

$$P(\sup_{x:\ |F(x_0)-F(x)|\leq C_1a_n}(na_n^{-1})^{1/2}|F_n(x_0)-F_n(x)-F(x_0)+F(x)|\geqq b_n)$$
$$=P(\sup_{x:\ |F(x_0)-F(x)|\leq C_1a_n}|\bar{\alpha}_n(F(x_0))-\bar{\alpha}_n(F(x))|\geqq b_na_n^{1/2})$$
$$\leqq cb_n^{-4}=c(na_n^3)^{-2/3}$$

On the other hand, as $\{Y_i\}$ is strictly stationary and ϕ-mixing, and $E|Y_i|^{2+\delta}<\infty$, so by the pointwise ergodic theorem for stationary sequences

$$\limsup_{n\to\infty}\int|y|H_n(dx,\ dy)<\infty \quad \text{a.s.} \tag{4.2}$$

Hence, the assertion of the lemma follows from (4.2) and the fact that K'' is bounded, upon observing that $\varepsilon>0$ was arbitrary and $na_n^3\to\infty$. □

Next, to show that $(na_n)^{1/2}I_2$ is asymptotically equivalent to

$$-(na_n)^{1/2}a_n^{-1}m(x_0)\int K'\Big(\frac{F(x_0)-F(x)}{a_n}\Big)\{F_n(dx)-F(dx)\},$$

define $\{Z_n,\ n\geqq 1\}$ by

$$Z_n=n^{-1}a_n^{-3/2}\sum_{i=1}^n\{Y_i-m(X_i)\}\{\alpha_n(x_0)-\alpha_n(X_i)\}K'\Big(\frac{F(x_0)-F(X_i)}{a_n}\Big). \tag{4.3}$$

Lemma 4.2. *$Z_n\to 0$ in probability as $n\to\infty$.*

Proof. We rewrite Z_n as follows;

$$Z_n=n^{-3/2}a_n^{-3/2}\sum_{i=1}^n\sum_{j=1}^n\{Y_i-m(X_i)\}g(X_j;\ x_0,\ X_i)K'\Big(\frac{F(x_0)-F(X_i)}{a_n}\Big), \tag{4.4}$$

where

$$g(X_j;\ x_0,\ X_i)=\{I(X_j<x_0)-F(x_0)\}-\{I(X_j<X_i)-F(X_i)\} \tag{4.5}$$

and

$$I(y<x)=I_{(-\infty,\ x)}(y)-F(x). \tag{4.6}$$

We note that if $\{(\tilde{X}_i, \tilde{Y}_i),\ i=1, 2, \cdots, n\}$ are i.i.d. random vectors with $d.f.$ $H(x, y)$, then

(4.7) $$E\{\tilde{Y}_i - m(\tilde{X}_i) \mid \tilde{X}_i\}=0 \quad \text{a.s.}$$

and

(4.8) $$E\{g(\tilde{X}_j;\ x_0, \tilde{X}_i) \mid \tilde{X}_i\}=0 \quad \text{a.s.}$$

Put

(4.9) $$G((x_1, y_1);\ x_0, x_2)=(y_1-m(x_1))g(x_2;\ x_0, x_1)K'\Big(\frac{F(x_0)-F(x_1)}{a_n}\Big)$$

and let $P(x_{i_1}, y_{i_1}, \cdots, x_{i_k}, y_{i_k})$ be the d.f. of $(X_{i_1}, Y_{i_1}, \cdots, X_{i_k}, Y_{i_k})$. Then it is obvious that

(4.10) $$\int G((x_1, y_1);\ x_0, x_2)dP(x_2)=0$$

and

(4.11) $$\int G((x_1, y_1);\ x_0, x_2)dP(x_1, y_1)=0.$$

For brevity, let

(4.12) $$\theta(i;\ j)=G((X_i, Y_i);\ x_0, X_j).$$

To prove $\lim_{n\to\infty} EZ_n^2=0$, we consider the following quantity;

(4.13) $$n^3a_n^3EZ_n^2=\sum_{i=1}^{n}\sum_{j=1}^{n}E\theta^2(i;\ j)+2\sum_{i=1}\sum_{1\le j_1<j_2\le n}E\theta(i;\ j_1)\theta(i;\ j_2)$$
$$+2\sum_{1\le i_1<i_2\le n}\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}E\theta(i_1;\ j_1)\theta(i_2;\ j_2)$$
$$=J_1+2J_2+2J_3,\ \text{(say)}.$$

Since K' and g are bouded and

$$E|Y_i-m(X_i)|^{2+\delta}\leqq cE|Y_i|^{2+\delta}<\infty,$$

so

(4.14) $$E\theta^2(i;\ j)\leqq c \quad \text{for all } i, j.$$

Thus, we have

(4.15) $$|J_1|\leqq cn^2.$$

Next, if $i<j_1$ and $j_1-i>j_2-j_1$, then by Lemma 3.1 and (4.10)

$$|E\{\theta(i;j_1)\theta(i;j_2)\}|$$
$$\leqq\left|\int\cdots\int G((x_i,y_i);x_0,x_{j_1})G((x_i,y_i);x_0,x_{j_2})dP(x_i,y_i,x_{j_1})dP(x_{j_2})\right.$$
$$+c\phi^{2/(2+\delta)}(j_2-j_1)$$
$$\leqq c\phi^{2/(2+\delta)}(j_2-j_1).$$

Hence, we have

$$\sum_{1\leq i<j_1<j_2\leq n}|E\theta(i;j_1)\theta(i;j_2)| \leqq \sum_{1\leq i<j_1<j_2\leq n}c\phi^{2/(2+\delta)}(j_2-j_1)\leqq cn^2. \tag{4.16}$$

Similarly, we have

$$\sum_{1\leq j_1<j_2<i\leq n}|E\{\theta(i;j_1)\theta(i;j_2)\}|\leqq c\sum_{1\leq j_1<j_2<i\leq n}\phi^{2/(2+\delta)}(j_2-j_1)\leqq cn^2 \tag{4.17}$$

and

$$\sum_{1\leq j_1<i\leq j_2\leq n}|E\{\theta(i;j_1)\theta(i;j_2)\}|\leqq c\sum_{1\leq j_1<i<j_2\leq n}\phi^{2/(2+\delta)}(j_2-i)\leqq cn^2. \tag{4.18}$$

Further, by (4.15)

$$\Big\{\sum_{1\leq i=j_1<j_2\leq n}+\sum_{1\leq j_1<j_2=i\leq n}\Big\}|E\{\theta(i;j_1)\theta(i;j_2)\}|\leqq cn^2 \tag{4.19}$$

Thus, from (4.16)-(4.19) we obtain

$$|J_2|\leqq cn^2. \tag{4.20}$$

It remains to prove $|J_3|\leqq cn^2$. We use the method of the proof of Lemma 2 in Yoshihara (1976)

By Lemma 3.1, (4.10) and (4.11), we have the following inequalities:

If $1\leqq i_1<i_2\leqq j_1<j_2\leqq n$ and $j_2-j_1\geqq i_2-i_1$, then

$$|E\{\theta(i_1;j_1)\theta(i_2;j_2)\}|\leqq c\phi^{2/(2+\delta)}(j_2-j_1) \tag{4.21}$$

and similarly, if $1\leqq i_1<i_2\leqq j_1<j_2\leqq n$ and $i_2-i_1\geqq j_2-j_1$, then

$$|E\{\theta(i_1;j_1)\theta(i_2;j_2)\}|\leqq c\phi^{2/(2+\delta)}(i_2-i_1). \tag{4.22}$$

Thus, from (4.21) and (4.22) we obtain

$$\begin{aligned}&\Big|\sum_{1\leq i_1<i_2\leq j_1<j_2\leq n}E\{\theta(i_1;j_1)\theta(i_2;j_2)\}\Big|\\ &\leqq\Big\{\sum_{\substack{1\leq i_1<i_2\leq j_1<j_2\leq n\\ i_2-i_1\geq j_2-j_1}}+\sum_{\substack{1\leq i_1<i_2\leq j_1<j_2\leq n\\ i_2-i_1\leq j_2-j_1}}\Big\}|E\theta(i_1;j_1)\theta(i_2;j_2)|\\ &\leqq cn^2\sum_{k=1}^{n}(k+1)\phi^{2/(2+\delta)}(k)\leqq cn^2.\end{aligned} \tag{4.23}$$

(ii) Similarly, we have

$$\begin{aligned}&\Big|\sum_{1\le i_1<j_1\le i_2<j_2\le n} E\{\theta(i_1;j_1)\theta(i_2;j_2)\}\Big|\\ &\le\Big\{\sum_{\substack{1\le i_1<j_1\le i_2<j_2\le n\\ j_1-i_1\ge j_2-i_2}}+\sum_{\substack{1\le i_1<j_1\le i_2<j_2\le n\\ j_1-i_1\le j_2-i_2}}\Big\}|E\theta(i_1;j_1)\theta(i_2;j_2)|\\ &\le cn^2,\end{aligned}\tag{4.24}$$

$$\begin{aligned}&\Big|\sum_{1\le i_1<j_1<j_2<i_2\le n}|E\theta(i_1;j_1)\theta(i_2;j_2)|\Big|\\ &\le\Big\{\sum_{\substack{1\le i_1<j_1<j_2<i_2\le n\\ j_1-i_1\ge i_2-j_2}}+\sum_{\substack{1\le i_1<j_1<j_2<i_2\le n\\ j-i_1\ge i_2-i_2}}\Big\}|E\theta(i_1;j_1)\theta(i_2;j_2)|\\ &\le cn^2,\end{aligned}\tag{4.25}$$

$$\begin{aligned}&\Big|\sum_{1\le i_1,j_1\le n}\sum_{i_2=1}^{n}E\theta(i_1;j_1)\theta(i_2;j_1)\Big|\\ &\le\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}|E\theta(i_1;j_1)\theta(i_2;i_1)|\\ &\quad+2\sum_{1\le i_1<j_1\le n}\sum_{i_2=1}^{n}|E\theta(i_1;j_1)\theta(i_2;j_1)|\\ &\le cn^2\Big(1+\sum_{k=1}^{n}\phi^{2/(2+\delta)}(k)\Big)\le cn^2\end{aligned}\tag{4.26}$$

$$\Big|\sum_{1\le i_2,j_2\le n}\sum_{i_1=1}^{n}E\theta(i_1;j_1)\theta(i_2;j_2)\Big|\le cn^2\sum_{k=1}^{n}\phi^{2/(2+\delta)}(k)\le cn^2.\tag{4.27}$$

Hence, from (4.24)-(4.26) we have

$$|J_3|\le cn^2.\tag{4.28}$$

Now, from (4.15), (4.20), (4.28) and the fact that $na_n^3\to\infty$ we have $\lim EZ_n^2=0$, which completes the proof. □

Next, consider the function

$$k(x,y)=m(x)K'\Big(\frac{F(x_0)-F(x)}{a_n}\Big)\{I_{(-\infty,x_0]}(y)-I_{(-\infty,x]}(y)\}\tag{4.29}$$

with corresponding "von Mises" statistic

$$\begin{aligned}T_n&=n\int k(x,y)\{F_n(dy)-F(dy)\}\{F_n(dx)-F(dx)\}\\ &=\int k(x,y)\alpha_n(dy)\alpha_n(dx).\end{aligned}$$

Then, using the method of the proof of Lemma 2 in Yoshihara and noting $\sum(k+1)^2\phi^{1/4}(k)<\infty$, we have

$$ET_n^2=O(1) \quad \text{as} \quad n\to\infty,$$

which implies

$$a_n^{-3/2}\int k(x,\ y)\alpha_n(dy)\{F_n(dx)-F(dx)\} \longrightarrow 0 \quad \text{in probability}$$

since $na_n^3\to\infty$.

Thus, we obtain that $(na)^{1/2}I_2$ is asymptotically equivalent to

$$a_n^{-3/2}\int m(x)\{\alpha_n(x_0)-\alpha_n(x)\}K'\Big(\frac{F(x_0)-F(x)}{a_n}\Big)F(dx). \tag{4.30}$$

Lemma 4.3. *If conditions of Theorem 2 hold, then*

$$I_4=a_n^{-3/2}\int|m(x)-m(x_0)||\alpha_n(x_0)-\alpha_n(x)||K'\Big(\frac{F(x_0)-F(x)}{a_n}\Big)|F(dx)\longrightarrow 0 \tag{4.31}$$

in probability

for μ-almost all $x_0\in R$.

Proof. In the proof of Lemma 3 in Stute (1984), it was proved that $F^{-1}(F(x_0))=x_0$ fo μ-almost all x_0 and that

$$\lim_{n\to\infty}\int_{-1}^{1}|m(F^{-1}(s-ua_n))-m(F^{-1}(s))|\,|K'(u)|\,du=0$$

for μ-almost all s, say, for all $s\in A$, if F is continuous. Hence

$$\mu(\{x_0:F(x_0)\in A,\ F^{-1}(F(x_0))=x_0\})=1.$$

Let $F(x_0)\in A$ and put

$$e_n(x_0)=\Big\{a_n^{-1}\int_0^1|m(F^{-1}(u))-m(x_0)|\,\Big|K'\Big(\frac{F(x_0)-u}{a_n}\Big)\Big|\,du\Big\}^{-1/6}$$

Then, by Lemma 3.5 we have that for any $\varepsilon>0$

$$\begin{aligned}
&P(I_4>\varepsilon)\\
&\leqq P(a_n^{-3/2}\sup_{u:\,|F(x_0)-u|\leqq a_n}|\bar{\alpha}_n(F(x_0))-\bar{\alpha}_n(u)|\\
&\qquad\times\int_0^1|m(F^{-1}(u))-m(x_0)|\,\Big|K'\Big(\frac{F(x_0)-u}{a_n}\Big)\Big|\,du>\varepsilon)\\
&\leqq P(\sup_{u:\,|F(x_0)-u|\leqq a_n}|\bar{\alpha}_n(F(x_0))-\bar{\alpha}_n(u)|\geqq e_n(x_0)a_n^{1/2})\\
&\leqq ce_n^{-4}(x_0)=o(1)
\end{aligned}$$

as $n\to\infty$. Hence, (4.31) is obtained. □

Lemma 4.4. $(na_n)^{1/2}I_2$ *is asymptotically equivalent to*

$$-a_n^{-1/2}m(x_0)\int K\Big(\frac{F(x_0)-F(x)}{a_n}\Big)\alpha_n(dx). \tag{4.32}$$

Proof. From Lemma 4.3 it follows that $(na_n)^{1/2}I_2$ is asymptotically equivalent to

$$\begin{aligned}
&a_n^{-3/2}m(x_0)\int[\alpha_n(x_0)-\alpha_n(x)]K'\Big(\frac{F(x_0)-F(x)}{a_n}\Big)F(dx)\\
&=-a_n^{-3/2}m(x_0)\int\alpha_n(x)K'\Big(\frac{F(x_0)-F(x)}{a_n}\Big)F(dx)\\
&\doteq-a_n^{-3/2}m(x_0)\int K\Big(\frac{F(x_0)-F(x)}{a_n}\Big)\alpha_n(dx),
\end{aligned}$$

which completes the proof. □

5. Proofs. Let

$$\begin{aligned}
W_{ni}=\,&a_n^{-1/2}(Y_i-m(x_0))K\Big(\frac{F(x_0)-F(X_i)}{a_n}\Big)\\
&-a_n^{-1/2}E\Big\{(Y_i-m(x_0))K\Big(\frac{F(x_0)-F(X_i)}{a_n}\Big)\Big\}
\end{aligned} \tag{5.1}$$

$$(i=1,\cdots,n).$$

Then, it is clear that $W_{n1},\cdots,W_{nn}$ are identically distributed and satisfy the ϕ-mixing condition with

$$\sum(k+1)^2\phi^{1/4}(k)<\infty.$$

Lemma 5.1. *For any* δ' $(0\leqq\delta'<1)$

$$E\Big|\sum_{i=1}^{n}W_{ni}\Big|^{2+\delta'}ck^{(2+\delta')/2}a_n^{-\delta'/2}. \tag{5.2}$$

Proof. By Lemma 3.3

$$E\Big|\sum_{i=1}^{k}W_{ni}\Big|^{2+\delta'}\leqq ck^{(2+\delta')}E|W_{nl}|^{2+\delta'}$$

Since K vanishes outside $(-1, 1)$ and is bounded, so for all n sufficiently large we have

$$\begin{aligned}
E|W_{n1}|&\leqq ca_n^{-(2+\delta')/2}\int|y-m(x_0)|^{2+\delta'}\Big|K\Big(\frac{F(x_0)-F(x)}{a_n}\Big)\Big|^{2+\delta'}H(dx,dy)\\
&=ca_n^{-(2+\delta')/2}\int h(x)\Big|K\Big(\frac{F(x_0)-F(x)}{a_n}\Big)\Big|^{2+\delta'}F(dx)\\
&\leqq ca_n^{-\delta'/2}\int_{-1}^{1}h(F^{-1}(F(x_0)-a_nu))|K(u)|^{2+\delta'}du
\end{aligned}$$

$$\leqq c a_n^{-\delta'/2}$$

where

$$h(x)=E\{|Y_1-m(x_0)|^{2+\delta}|X=x\}.$$

Thus, (5.2) is obtained. □

Proof of Theorem 2. According to Lemma 4.4, to prove Theorem 2 it remains to show

$$(5.3)\qquad I_5=\left(\frac{n}{a_n}\right)^{1/2}\int\{y-m(x_0)\}K\left(\frac{F(x_0)-F(x)}{a_n}\right)\{H_n(dx, dy)-H(dx, dy)\}$$

$$\xrightarrow{D} N(0, \sigma_0^2).$$

Let $p=[n^{2/3}]$, $q=[n^{1/3}]$ and $k=[n/(p+q)]$ where $[s]$ denotes the largest integer j such that $j\leqq s$. Put

$$\eta_j=\sum_{i=(j-1)(p+q)+1}^{(j-1)(p+q)+p} W_{ni} \qquad (j=1, \cdots, k),$$

$$\zeta_j=\sum_{i=(j-1)(p+q)+p+1}^{j(p+q)} W_{ni} \quad (j=1, \cdots, k),$$

$$\zeta_{k+1}=\sum_{i=k(p+q)+1}^{n} W_{ni}.$$

We rewrite I_5 as follows:

$$I_5=\frac{1}{\sqrt{n}}\sum_{i=1}^{n} W_{ni}=\frac{1}{\sqrt{n}}\sum_{j=1}^{k}\eta_j+\frac{1}{\sqrt{n}}\sum_{j=1}^{k+1}\zeta_j.$$

By Lemma 5.1 and definitions of q and k we can easily show that

$$\frac{1}{\sqrt{n}}\sum_{j=1}^{k+1}\zeta_j \longrightarrow 0 \quad \text{in probability}.$$

So, by Lemma 18. 4.1 in Ibragimov and Linnik (1971) to show (5.3) it suffices to prove

$$\frac{1}{\sqrt{n}}\sum_{j=1}^{k}\eta_j \xrightarrow{D} N(0, \sigma_0^2).$$

Since $\{\eta_j\}$ is ϕ-mixing with $\sum(j+1)^2\phi^{1/4}(j)<\infty$, so

$$\left|E\left\{\exp\left(it\frac{1}{\sqrt{n}}\sum_{j=1}^{k}\eta_j\right)\right\}-\left[E\left\{\exp\left(it\frac{\eta_i}{\sqrt{n}}\right)\right\}\right]^k\right|\leqq ck\phi(q)=o(1)$$

(cf. Lemma 2 in Billingsley (1968) page 171).

Hence, to show (5.3), it is enough to prove

$$\left[E\left\{\exp\left(it\frac{\eta_j}{\sqrt{n}}\right)\right\}\right]^k \longrightarrow e^{-t^2/2} \tag{5.4}$$

By Lemma 5.1 and the definition of p

$$E\left\{\exp\left(it\frac{\eta_j}{\sqrt{n}}\right)\right\}=1-\frac{t^2}{2n}E\eta_j^2+O\left(\frac{|t|^{2+\delta}}{n^{(2+\delta)/2}}E|\eta_j|^{2+\delta}\right)$$

$$=1-\frac{t^2}{2n}E\eta_j^2+o(|t|^{2+\delta}n^{-(2+\delta)/6}a_n^{-\delta/2}).$$

So, using the facts that $na_n^3\to\infty$ and

$$\lim_{n\to\infty}\frac{kE\eta_j^2}{n}=\sigma_0^2,$$

we have (5.4). Thus, the proof is completed. □

Proof of Theorem 1. Since any *-mixing sequence is ϕ-mixing, so by Theorem 2 to prove Theorem 1 it is enough to show

$$\lim_{n\to\infty}\frac{1}{n}E\left(\sum_{i=1}^n W_{ni}\right)^2=\mathrm{Var}(Y_1|X=x_0)\int_{-1}^1 K^2(u)du=\sigma_1^2.$$

Now, we evaluate

$$E\left(\sum_{i=1}^n W_{ni}\right)^2=\sum_{i=1}^n EW_{ni}^2+2\sum_{1\le i<j\le n}EW_{ni}W_{nj}.$$

By Lemma 3.4 and the fact that $EW_{ni}=0$, we have

$$|EW_{ni}W_{nj}|\le c\phi(|i-j|)E|W_{ni}|E|W_{nj}|.$$

As

$$E|W_{ni}|\le ca_n^{-1/2}\left\{\int|y-m(x_0)|\left|K\left(\frac{F(x_0)-F(x)}{a_n}\right)\right|H(dx, dy)\right.$$

$$\le ca_n^{1/2}\int_{-1}^1|m(F^{-1}(F(x_0)-ua_n))-m(x_0)||K(u)|du$$

$$\le ca_n^{1/2}$$

for all n sufficiently large, so using the fact that $\sum\phi(k)<\infty$

$$\left|\frac{1}{n}E\left|\sum_{i=1}^n W_{ni}\right|^2-EW_{n1}^2\right|\le ca_n.$$

On the other hand, by the method of the proof of Theorem 2

$$EW_{n1}^2 \longrightarrow \mathrm{Var}(Y|X=x_0)\int_{-1}^1 K^2(u)du.$$

Henc, we have

$$\lim_{n\to\infty}\frac{1}{n}E\Big(\sum_{i=1}^{n}W_{ni}\Big)^2=\mathrm{Var}(Y\mid X=x_0)\int_{-1}^{1}K^2(u)du.$$

Thus, the proof of Theorem 1 is completed. □

Proof of Corollary. The proof is identical to that of Corollary in Stute (1984) and so is omitted.

References

[1] Billingsley, P.: *Convergence of probability measures,* New York, Wiley, 1968.

[2] Ibragimov, I. and Linnik, Yu. V.: *Independent and stationary sequence of random variables,* Groningen, Wolters-Noordhoff Publishing, 1971.

[3] Samur, J. D.: *Convergence of sums of mixing triangular arrays of random vectors with stationary rows,* Ann. Probab. 12 (1984), 390-426.

[4] Stute, W.: *Asymptotic normality on nearest neighbor regression function estimates,* Ann. Statist. 12 (1984), 917-926.

[5] Utev, S. A.: *Inequalities for sums of weakly dependent random variables and estimates of convergence rate in the invariance principle,* Limit theorems for sums of random variables, Trudy Inst. Mat. Tom. 3, 50-76 Akad. Nauk. SSSR. Sibir Otdel (In Russian), 1984.

[6] Yoshihara, K.: *Limiting behavior of U-statistics for stationary, absolutely regular processes,* Z. Wahrscheinlichkeits-theorie verw. Gebiete. 35 (1976), 237-252.

Yokohama National University
Department of Mathematics
Faculty of Engineering
156, Tokiwadai, Hodogaya,
Yokohama, Japan