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L^2-TRANSVERSE CONFORMAL AND KILLING FIELDS ON COMPLETE FOLIATED RIEMANNIAN MANIFOLDS

By

TOSHIHIKO AOKI and SHINSUKE YOROZU

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0. The study of transverse fields on compact foliated Riemannian manifolds has been done in [4], [9], [11] and others. In the case of foliations by points, the results are well-known ones ([8], [17]). Our main aim is to study transverse fields on complete (non-compact) foliated Riemannian manifolds. To do this, we have to define the notion of "L^2-transverse fields", that is, transverse fields with finite global norms. L^2-transverse Killing fields are already studyed in [21] and [22].

In this paper, we discuss L^2-transverse conformal and Killing fields on complete foliated Riemannian manifolds such that the foliation is minimal and the metric is bundle-like with respect to the foliation.

We shall be in C^∞-category and deal only with connected and orientable manifolds without boundary. We use the following convention on the range of indices: $1 \leqq i,\ j \leqq p$ and $p+1 \leqq a, b, c, d \leqq p+q$. The Einstein summation convention will be used.

Our results are as follows:

Theorem A. *Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensional Riemannian manifold with an oriented foliation $\mathcal{F}$ of codimension q and a complete bundle-like metric g_M with respect to $\mathcal{F}$. Supposet that $\mathcal{F}$ is minimal and $q \geqq 3$. Let $s \in \tilde{V}(\mathcal{F})$ be an L^2-transverse field of $\mathcal{F}$. Then s is a transverse conformal field* (t. c. f.) *of $\mathcal{F}$ if and only if*

$$\Delta_D s = \rho_D(s) + \left(1 - \frac{2}{q}\right) \mathrm{grad}_D \mathrm{div}_D s.$$

Theorem B. *Let $(M, g_M, \mathcal{F})$ be as Theorem A. Suppose that $\mathcal{F}$ is minimal. Let $s \in \tilde{V}(\mathcal{F})$ be an L^2-transverse field of $\mathcal{F}$. Then s is a transverse Killing field* (t. K. f.) *of $\mathcal{F}$ if and only if*

$$\Delta_D s = \rho_D(s) \quad \text{and} \quad \mathrm{div}_D s = 0.$$

Theorem C. *Let $(M, g_M, \mathcal{F})$ be as Theorem A. Suppose that $\mathcal{F}$ is minimal and $q \geqq 3$. Let s be an L^2-*t. c. f. *of $\mathcal{F}$. Jf ρ_D is non-positive everywhere on M,* *then s is D-parallel. If ρ_D is non-positive everywhere and negative for at least one point of M, then s=0.*

Theorem D ([22]). *Let $(M, g_M, \mathcal{F})$ be as Theorem* A. *Suppose that $\mathcal{F}$ is minimal. Let s be an L^2-t.K.f. of $\mathcal{F}$. If ρ_D is non-positive everywhere on M, then s is D-parallel. If ρ_D is non-positive everywhere and negative for at least one point of M, then s=0.*

The compact versions of the above results are given in [11].

If $s\in\hat{V}(\mathcal{F})$ is D-parallel, then $|s|^2=g_Q(s, s)=$constant. Thus, by Theorem D, we have

Theorem E. *Let $(M, g_M, \mathcal{F})$ be as Theorem A. Suppose that $\mathcal{F}$ is minimal and $q\geqq 3$, and ρ_D is non-positive everywhere on M. Let $s\in\hat{V}(\mathcal{F})$ be an L^2-t.c.f. of $\mathcal{F}$. If M has infinite volume, then s=0.*

Theorem F. *Let $(M, g_M, \mathcal{F})$ be as Theorem* A. *Suppose that $\mathcal{F}$ is minimal and ρ_D is non-positive everywhere on M. Let $s\in\hat{V}(\mathcal{F})$ be an L^2-t.K.f. of $\mathcal{F}$. If M has infinite volume, then s=0.*

If $\mathcal{F}$ is a foliation by points, Theorem E and Theorem F have been given in [20].

1. Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensional Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a complete bundle-like metric g_M with respect to $\mathcal{F}$ ([13]). We assume that $\mathcal{F}$ is an oriented foliation ([14]). Let ∇ be the Levi-Civita connection with respect to g_M. Then the tangent bundle TM over M has an integrable subbundle E which is given by $\mathcal{F}$. The normal bundle Q of $\mathcal{F}$ is defined by $Q=TM/E$. We have a splitting σ of the exact sequence

$$0\longrightarrow E\longrightarrow TM \underset{\sigma}{\overset{\pi}{\rightleftarrows}} Q\longrightarrow 0. \tag{1.1}$$

where $\sigma(Q)$ is the orthogonal complement bundle $E^{\perp}$ of E in TM ([3]). Then g_M induces a metric g_Q on Q:

$$g_Q(t, u)=g_M(\sigma(t), \sigma(u)) \tag{1.2}$$

for any $t, u\in\Gamma(Q)$.

In a flat chart $U(x^i, x^a)$ with respect to $\mathcal{F}$ ([13]), a local frame $\{X_i, X_a\}=\{\partial/\partial x^i, \partial/\partial x^a-A_a^j\partial/\partial x^j\}$ is called the basic adapted frame to $\mathcal{F}$ ([9], [12], [16]). Here A_a^j are functions on U with $g_M(X_i, X_a)=0$. It is trivial that $\{X_i\}$ (resp. $\{X_a\}$) spans $\Gamma(E|_U)$ (resp. $\Gamma(E^{\perp}|_U)$). From now on, we omit "$|_U$" for simplicity. We set

$$
(1.3)\qquad
\begin{aligned}
&g_{ij}=g_M(X_i, X_j), \quad && g_{ab}=g_M(X_a, X_b)\\
&(g^{ij})=(g_{ij})^{-1}, && (g^{ab})=(g_{ab})^{-1}\\
&t_a=\pi(X_a).
\end{aligned}
$$

We remark that $g_Q(t_a, t_b)=g_{ab}$.

A connection D in Q is defined by

$$
(1.4)\qquad
\begin{aligned}
&D_X t=\pi([X, Y]) && \text{if } X\in\Gamma(E) \text{ and } t\in\Gamma(Q)\\
& && \text{with } \pi(Y)=t\\
&D_X t=\pi(\nabla_X Y_t) && \text{if } X\in\Gamma(E^{\perp}) \text{ and } t\in\Gamma(Q)\\
& && \text{with } Y_t=\sigma(t)
\end{aligned}
$$

([3]). Then we have

Proposition 1.1 ([3]). *The connection D in Q is torsion free and metrical with respect to g_Q.*

Let Q^* be the dual bundle of Q. The dual connection of D in Q^* is denoted by D^*. Q^* has the metric induced from g_Q.

Definition 1.2 ([3]). Let $\tau=g^{ij}\pi(\nabla_{X_i}X_j)$. Then τ is called the *tension field* of $\mathcal{F}$. The foliation $\mathcal{F}$ is *minimal* if $\tau=0$.

Let $V(\mathcal{F})$ be the space of all vector fields Y on M satisfying

$$
(1.5)\qquad [Y, Z]\in\Gamma(E)
$$

for any $Z\in\Gamma(E)$. An element of $V(\mathcal{F})$ is called an infinitesimal automorphism of $\mathcal{F}$ ([4], [10]). We set

$$
(1.6)\qquad \tilde{V}(\mathcal{F})=\{t\in\Gamma(Q)\,|\,t=\pi(Y),\ Y\in V(\mathcal{F})\}.
$$

It is trivial that $t\in\tilde{V}(\mathcal{F})$ satisfies $D_X t=0$ for any $X\in\Gamma(E)$.

Let $\wedge^r(M)$ be the space of all r-forms on M. We have the decompositions of $\wedge^r(M)$ and the exterior derivative d with respect to $\mathcal{F}$:

$$
(1.7)\qquad \wedge^r(M)=\sum_{w+z=r}\wedge^{w,z}(M),
$$

$$
(1.8)\qquad d=d'+d''+d'''
$$

([6], [13], [16], [18]). Let $\Delta^r(M)$ be a subspace of $\wedge^{0,r}(M)$ composed of d'-closed $(0, r)$-forms, that is, the space of all basic $(0, r)$-forms on M ([6], [13]). An operator δ: $\wedge^r(M)\to\wedge^{r-1}(M)$ is defined by $\delta=(-1)^{(p+q)(r+)+1}*d*$, where $*$ denotes the Hodge star operator. Then δ has a decomposition: $\delta=\delta'+\delta''+\delta'''$. The operator δ'' is define by

$$\delta''=(-1)^{(p+q)(r+1)+1}*d''* \tag{1.9}$$

on $\wedge^r(M)$ ([16], [18]).

Let $\chi_{\mathcal{F}}$ be the characteristic form of $\mathcal{F}$ given by Rummler ([5], [14]). Then we have

Proposition 1.3 ([5], [14]). *It holds that*

$$d\chi_{\mathcal{F}}\ (=d''\chi_{\mathcal{F}})=(-1)^{p+1}\chi_{\mathcal{F}}\wedge\kappa$$

where κ *denotes the mean curvature form of* $\mathcal{F}$.

The mean curvature form κ of $\mathcal{F}$ is parallel along the leaves of $\mathcal{F}$ if $\mathcal{L}_X\kappa=0$ for any $X\in\Gamma(E)$, where $\mathcal{L}_X$ denotes the Lie derivative operator with respect to X ([5]).

Proposition 1.4 ([5], [14]).

(i) $\mathcal{F}$ *is minimal if and only if* $\kappa=0$.

(ii) κ *is parallel along the leaves of* $\mathcal{F}$ *if and only if* $d'\kappa=0$.

An operator $*'': \Delta^r(M)\rightarrow\Delta^{q-r}(M)$ is defined by

$$*''\phi=*(\chi_{\mathcal{F}}\wedge\phi) \tag{1.10}$$

for any $\phi\in\Delta^r(M)$. Then we have

$$*\phi=(-1)^{pr}\chi_{\mathcal{F}}\wedge*''\phi \tag{1.11}$$

for any $\phi\in\Delta^r(M)$ ([5]). We define an operator $\delta_b'': \Delta^r(M)\rightarrow\Delta^{r-1}(M)$ by

$$\delta_b''=(-1)^{q(r+1)+1}*''d''*''. \tag{1.12}$$

([5], [15]).

Proposition 1.5 *If the mean curvature form* κ *of* $\mathcal{F}$ *is parallel along the leaves of* $\mathcal{F}$, *then*

$$\delta''\phi=\delta_b''\phi+(-1)^{q(r+1)}*''(\kappa\wedge*''\phi)$$

for any $\phi\in\Delta^r(M)$.

Corollary 1.6 *If* $\mathcal{F}$ *is minimal, then*

$$\delta''\phi=\delta_b''\phi$$

for any $\phi\in\Delta^r(M)$.

Let $\langle\ ,\ \rangle$ be the local scalar product on $\Gamma(Q)$ or $\Gamma(Q^*)$. The local scalar product may be exteded on $\Gamma(\otimes^{r_1}Q\otimes^{r_2}Q^*)$. Let $\Gamma_0(Q)$ (resp. $\Gamma_0(Q^*)$) be the space of all sections of Q (resp. Q^*) with compact supports. Let $\langle\langle\ ,\ \rangle\rangle$ be the global scalar product on $\Gamma_0(Q)$ or $\Gamma_0(Q^*)$, and $\|\cdot\|=\langle\langle\cdot,\ \cdot\rangle\rangle^{1/2}$. The global scalar product may be also extended on $\Gamma_0(\otimes^{r_1}Q\otimes^{r_2}Q^*)$.

On $\wedge^r(M)$ and $\wedge_0^r(M)$, we have also the local scalar product $\langle\ ,\ \rangle$ and the global scalar product $\langle\langle\ ,\ \rangle\rangle$ that are defined by the natural way.

Let $L^2(Q)$ (resp. $L^2(Q^*)$) be the completion of $\Gamma_0(Q)$ (resp. $\Gamma_0(Q^*)$) with respec to the global scalar product $\langle\langle\ ,\ \rangle\rangle$.

Definition 1.7 ([19], [21]). An element $s\in L^2(Q)\cap\Gamma(Q)$ is called an *L^2-transverse field of $\mathcal{F}$.*

If t is an L^2-transverse field of $\mathcal{F}$, then the dual $\tilde{t}$ of t, that is $\tilde{t}(\cdot)=g_Q(t, \cdot)$, belongs to $L^2(Q^*)\cap\Gamma(Q^*)$.

Definition 1.8 ([13], [16]). A function f on M is called a *foliated function* if $Xf=0$ for any $X\in\Gamma(E)$, that is, $d'f=0$.

We remark that $\Delta^0(M)$ is the space of all foliated functions on M. Let $C^\infty(M)$ be the space of all functions on M.

Definition 1.9 ([22]). An operator $\mathrm{div}_D\colon\Gamma(Q)\to C^\infty(M)$ defined by $\mathrm{div}_D t=g^{ab}g_Q(D_{X_a}t, \pi(X_b))$ is called the *transverse divergence operator with respect to D.*

We remark that if $t\in\tilde{V}(\mathcal{F})$ then $\mathrm{div}_D t$ is a foliated function on M.

Definition 1.10 ([23]). The *transverse gradient* $\mathrm{grad}_D f$ *of a function f with respect to D* is defined by $\mathrm{grad}_D f=g^{ab}X_a(f)\pi(X_b)$.

We remark that if f is a foliated function on M then $\mathrm{grad}_D f\in\tilde{V}(\mathcal{F})$.

Definition 1.11 ([5]). The *transverse Lie derivative $\Theta(Y)$ with respect to* $Y\in V(\mathcal{F})$ is defined by $\Theta(Y)t=\pi([Y, Y_t])$ for any $t\in\Gamma(Q)$ with $\pi(Y_t)=t$.

Definition 1.12 ([8], [11]). If $X\in V(\mathcal{F})$ satisfies $\Theta(X)g_Q=2\lambda\cdot g_Q$, then $s=\pi(X)$ is called a *transverse conformal field* (t. c. f.) *of $\mathcal{F}$.* Here λ is a function on M.

Proposition 1.13 ([11]). *If $s=\pi(X)\in\tilde{V}(\mathcal{F})$ is a* t. c. f. *of $\mathcal{F}$ with $\Theta(X)g_Q=2\lambda\cdot g_Q$, then $\lambda=(1/q)\mathrm{div}_D s$ is a foliated function on M.*

Definition 1.14 ([4], [9]). If $X\in V(\mathcal{F})$ satisfies $\Theta(X)g_Q=0$, then $s=\pi(X)$ is called a *transverse Killing field* (t. K. f.) *of $\mathcal{F}$.*

Let R_D be the curvature of D. The curvature R_D of D satisfies $i(X)R_D=0$ for any $X\in\Gamma(E)$, where i denotes the interior product ([3]).

Definition 1.15 ([4[). The *Ricci operator* $\rho_D\colon\Gamma(Q)\to\Gamma(Q)$ *of $\mathcal{F}$* is defined by

$$\rho_D(t)=g^{ab}R_D(\sigma(t), X_a)\pi(X_b)$$

for any $t\in\Gamma(Q)$.

Definition 1.16 ([4]). An operator $\Delta_D\colon \Gamma(Q)\to\Gamma(Q)$ is defined by $\Delta_D s=-g^{ab}\{D_{X_a}D_{X_b}s-D_{\nabla_{X_a}X_b}s\}-g^{ij}\{D_{X_i}D_{X_j}s-D_{\nabla_{X_i}X_j}s\}$ for any $s\in\Gamma(Q)$.

We remark that the original definition of Δ_D acting on $\Gamma^r(M, \Omega)$ is given by $\Delta_D=d_D^*d_D+d_Dd_D^*$ (for the definitions of $\Gamma^r(M, Q)$, d_D and d_D^*, see section 3).

Proposition 1.17 ([11]). *If* $s\in\tilde{V}(\mathcal{F})$ *is a* t.c.f. *of* $\mathcal{F}$, *then it halds that*

$$\Delta_D s=D_{\sigma(\tau)}s+\rho_D(s)+\left(1-\frac{2}{q}\right)\mathrm{grad}_D\,\mathrm{div}_D s\,.$$

Proposition 1.18 ([4], [9], [11], [21], [22], [23]). *If* $s\in\tilde{V}(\mathcal{F})$ *is a* t.K.f. *of* $\mathcal{F}$, *then it holds that*

$$\Delta_D s=D_{\sigma(\tau)}s+\rho_D(s) \quad \textit{and} \quad \mathrm{div}_D s=0.$$

Let x_0 be a fixed point of M and $\rho(x)$ the distance from x_0 to $x\in M$. We set

$$B(2k)=\{x\in M\,|\,\rho(x)\leqq 2k\} \tag{1.13}$$

for and $k>0$. A function μ on R satisfies the following properties:

$$\begin{aligned} &0\leqq\mu(y)\leqq 1 && \text{on } R\\ &\mu(y)=1 && \text{for } y\leqq 1\\ &\mu(y)=0 && \text{for } y\geqq 2. \end{aligned} \tag{1.14}$$

Then we define a family $\{w_k\}$ of Lipschitz continuous functions on M:

$$w_k(x)=\mu(\rho(x)/k) \qquad k=1, 2, \cdots \tag{1.15}$$

for any $x\in M$. The family $\{w_k\}$ satisfies the following properties:

$$\begin{aligned} &0\leqq w_k(x)\leqq 1 && \text{for any } x\in M\\ &\mathrm{supp}\, w_k\subset B(2k)\\ &w_k(x)=1 && \text{for any } x\in B(k)\\ &\lim_{k\to\infty} w_k=1\\ &|dw_k|\leqq Ck^{-1} && \text{almost everywhere on } M \end{aligned} \tag{1.16}$$

where C is a positive constant independent of k ([1], [2], [6], [18], [19], [20]).

2. Let $\{X_i, X_a\}$ be the adapted frame to $\mathcal{F}$ and $\{e^i, e^a\}$ the dual frame to $\{X_i, X_a\}$. Let $\{t_a\}$ be the frame on Q such that $\pi(X_a)=t_a$, and let $\{\tilde{t}_a\}$ be the dual frame to $\{t_a\}$, that is, $\tilde{t}^a(u)=g_Q(t_a, u)$ for all $u\in\Gamma(Q)$. Since $D_X t_a=0$

for any $X\in\Gamma(E)$, we have that $t_a\in\tilde{V}(\mathcal{F})$. Moreover, we notice that $\sigma(t_a)=X_a$ and $D^*_X\tilde{t}^a=0$ for any $X\in\Gamma(E)$.

Let $\tilde{\Gamma}(Q^*)=\{\eta\in\Gamma(Q^*)\mid D^*_X\eta=0$ for any $X\in\Gamma(E)\}$. By the same way, we may define the spaces $\tilde{\Gamma}(\wedge^r Q^*)$.

We define a map $\Xi:\Gamma(Q^*)\to\wedge^{0,1}(M)$ by $\Xi(\eta)=\eta_a e^a$ for any $\eta=\eta_a\tilde{t}^a\in\Gamma(Q^*)$. It is trivial that $\Xi(\tilde{\Gamma}(Q^*))=\Delta^1(M)$ and Ξ preserves the local scalar products $\langle\,,\rangle$. The map Ξ may be extended to a map: $\Gamma(\wedge^r Q^*)\to\wedge^r(M)$ (say, the same letter Ξ). We notice that $\Xi(\tilde{\Gamma}(\wedge^r Q^*))=\Delta^r(M)$ and $\Xi(f)=f$ for any $f\in\Gamma(\wedge^0 Q^*)=\wedge^{0,0}(M)=C^\infty(M)$. Then we have

$$\Xi^{-1}\phi(u_1,\cdots,u_r)=\phi(\sigma(u_1),\cdots,\sigma(u_r)) \tag{2.1}$$

for any $\phi\in\Delta^r(M)$ and $u_1,\cdots,u_r\in\Gamma(Q)$. We define operators $\tilde{d}''$, $\tilde{*}''$ and $\tilde{\delta}''$ by

$$\begin{aligned}&\tilde{d}''=\Xi^{-1}\circ d''\circ\Xi \qquad \tilde{*}''=\Xi^{-1}\circ *''\circ\Xi\\ &\tilde{\delta}''=\Xi^{-1}\circ\delta''_b\circ\Xi\,.\end{aligned} \tag{2.2}$$

Then we have, for $\eta\in\tilde{\Gamma}(Q^*)$,

$$\tilde{d}''\eta(t,u)=(D^*_{\sigma(t)}\eta)(u)-(D^*_{\sigma(u)})\eta(t) \tag{2.3}$$

$$\tilde{\delta}''\eta=-g^{ab}(D^*_{X_a}\eta)(\pi(X_b)) \tag{2.4}$$

$$\tilde{\delta}''\eta=\delta''_b(\Xi(\eta))\,. \tag{2.5}$$

The operator $\tilde{\delta}''$ is the adjoint operator of $\tilde{d}''$ acting on $\tilde{\Gamma}(\wedge^r Q^*)$ with respect to $\langle\langle\,,\rangle\rangle$.

Proposition 2.1 *If $\mathcal{F}$ is minimal, then*

$$d(*(\Xi(\eta)))=-*\tilde{\delta}''\eta$$

for any $\eta\in\tilde{\Gamma}(Q^*)$.

Proof. We have

$$\begin{aligned}d(*(\Xi(\eta)))&=d''(*(\Xi(\eta)))\\ &=-*\delta''(\Xi(\eta))\\ &=-*\delta''_b(\Xi(\eta)) &&\text{(by Corollary 1.6)}\\ &=-*\tilde{\delta}''\eta &&\text{(by (2.5)).}\end{aligned}$$

We remark that $\tilde{d}''f=X_a(f)\tilde{t}^a$ and $\Xi(\tilde{d}''f)=d''f$ for any $f\in C^\infty(M)$.

Proposition 2.2 ([1], [2], [18]). *For any $\eta\in\tilde{\Gamma}(Q^*)$, it holds that*

$$\|\tilde{d}''w_k\otimes\eta\|^2_{B(2k)}\leqq C^2k^{-2}\|\eta\|^2_{B(2k)}$$

where $\|\cdot\|^2_{B(2k)}=\langle\langle\cdot,\ \cdot\rangle\rangle_{B(2k)}=\int_{B(2k)}\langle\cdot,\ \cdot\rangle*1.$

In fact, we have

$$\|dw_k\otimes\Xi(\eta)\|^2_{B(2k)}\leqq C^2k^{-2}\|\Xi(\eta)\|^2_{B(2k)} \qquad ([1],\ [2])$$

$$\|d''w_k\otimes\Xi(\eta)\|^2_{B(2k)}\leqq\|dw_k\otimes\Xi(\eta)\|^2_{B(2k)}$$

$$\|\tilde{d}''w_k\otimes\eta\|^2_{B(2k)}=\|d''w_k\otimes\Xi(\eta)\|^2_{B(2k)}$$

$$\|\Xi(\eta)\|^2_{B(2k)}=\|\eta\|^2_{B(2k)}.$$

For $\eta\in\tilde{\Gamma}(Q^*)$, we have $w_k^2\eta\in\Gamma_0(Q^*)$ and

$$\tilde{d}''(w_k^2\eta)=w_k^2\tilde{d}''\eta+2w_k\tilde{d}''w_k\wedge\eta \qquad \text{almost everywhere on } M \tag{2.6}$$

$$\tilde{\delta}''(w_k^2\eta)=w_k^2\tilde{\delta}''\eta-\tilde{*}''(2w_k\tilde{d}''w_k\wedge\tilde{*}''\eta) \qquad \text{almost everywhere on } M. \tag{2.7}$$

Hereafter, we omit the term of "almost everywhere on M" for simplicity.

We remark that, for any $s\in L^2(Q)\cap\tilde{V}(\mathcal{F})$ and $\xi\in L^2(Q^*)\cap\tilde{\Gamma}(Q^*)$, $w_ks\to s$ and $w_k\xi\to\xi$ as $k\to\infty$ in the strong sense.

For any $s\in\tilde{V}(\mathcal{F})$, we define an element $B_s\in\Gamma(\otimes^2Q^*)$ by

$$B_s(t,\ u)=(\Theta(s)g_Q)(t,\ u)-\frac{2}{q}(\mathrm{div}_Ds)\cdot g_Q(t,\ u) \tag{2.8}$$

for any $t,\ u\in\Gamma(Q)$. Then we have

Proposition 2.3 ([11], [20]). *It holds that*

$$B_s(t,\ u)=B_s(u,\ t), \qquad g^{ab}B_s(t_a,\ t_b)=0,$$

$$g^{ab}B_s(D_{X_a}s,\ t_b)=\langle B_s,\ B_s\rangle,$$

$$g^{ab}(D^*_{X_a}B_s)(t_b,\ s)=\langle-\Delta_Ds+\rho_D(s)+\left(1-\frac{2}{q}\right)\mathrm{grad}_D\,\mathrm{div}_Ds,\ s\rangle.$$

where $\langle B_s,\ B_s\rangle=g^{ab}g^{cd}B_s(t_a,\ t_c)\cdot B_s(t_b,\ t_d)$.

Proposition 2.4 ([11], [17]). *If* $B_s=0$, *then* s *is a* t.c.f. *of* $\mathcal{F}$.

The above propositions are proved by the direct calculation.

We set

$$\eta(t)=B_s(t,\ s) \tag{2.9}$$

for any $t\in\Gamma(Q)$. Then we have $\eta\in\tilde{\Gamma}(Q^*)$ and

$$\tilde{\delta}''(w_k^2\eta)=w_k^2\tilde{\delta}''\eta-\tilde{*}''(2w_k\tilde{d}''w_k\wedge\tilde{*}''\eta) \tag{2.10}$$
$$=-w_k^2g^{ab}(D^*_{X_a}B_s)(t_b,\ s)-w_k^2g^{ab}B_s(D_{X_a}s,\ t_b)$$
$$-\tilde{*}''(2w_k\tilde{d}''w_k\wedge\tilde{*}''\eta).$$

Let $\tilde{s}$ be the dual of $s\in\tilde{V}(\mathcal{F})$, that is, $\tilde{s}(t)=g_Q(s,\ t)$ for any $t\in\Gamma(Q)$. Then $\tilde{s}\in\tilde{\Gamma}(Q^*)$. We set

$$B_D(s)=-\Delta_Ds+\rho_D(s)+\left(1-\frac{2}{q}\right)\mathrm{grad}_D\,\mathrm{div}_Ds. \tag{2.11}$$

If $\mathcal{F}$ is minimal, then, by Stokes' theorem and Proposition 2.1, we have

$$0=\int_M d(*(\Xi(w_k^2\eta)))=-\int_M *\tilde{\delta}''(w_k^2\eta). \tag{2.12}$$

Then we have

Proposition 2.5 ([20]). *Suppose that $\mathcal{F}$ is minimal. Let $s\in\tilde{V}(\mathcal{F})$ and $\tilde{s}$ the dual of s. Then*

$$\langle\langle w_kB_D(s),\ w_ks\rangle\rangle_{B(2k)}+\langle\langle w_kB_s,\ w_kB_s\rangle\rangle_{B(2k)}$$
$$+4\langle\langle\tilde{d}''w_k\otimes\tilde{s},\ w_kB_s\rangle\rangle_{B(2k)}=0.$$

Proof of Theorem A. Let $s\in\tilde{V}(\mathcal{F})$ be an L^2-transverse field of $\mathcal{F}$ and satisfy

$$\Delta_Ds=\rho_D(s)+\left(1-\frac{2}{q}\right)\mathrm{grad}_D\,\mathrm{div}_Ds.$$

Thus we have $B_D(s)=0$. By Propositions 2.2 and 2.5, we have

$$\|w_kB_s\|^2_{B(2k)}$$
$$=-4\langle\langle\tilde{d}''w_k\otimes\tilde{s},\ w_kB_s\rangle\rangle_{B(2k)}$$
$$\leqq 4\|\tilde{d}''w_k\otimes\tilde{s}\|_{B(2k)}\|w_kB_s\|_{B(2k)}$$
$$\leqq 2\{4\|\tilde{d}''w_k\otimes\tilde{s}\|^2_{B(2k)}+\frac{1}{4}\|w_kB_s\|^2_{B(2k)}\}$$
$$\leqq 8C^2\cdot k^{-2}\|s\|^2_{B(2k)}+\frac{1}{2}\|w_kB_s\|^2_{B(2k)}.$$

Thus we have

$$\frac{1}{2}\|w_k B_s\|^2_{B(2k)} \leqq 8C^2 \cdot k^{-2} \|s\|^2_{B(2k)}.$$

When $k \to \infty$, we have $\|B_s\|^2 = 0$. Theerefore, we have $B_s = 0$. By Proposition 2.4, s is a t.c.f. of $\mathcal{F}$. The converse is Proposition 1.17.

If we set $\mathrm{div}_D s = 0$, then Theorem B is proved.

3. Let $\Omega^r(M, Q)$ (resp. $\Omega^r_0(M, Q)$) be the space of all Q-valued r-forms (resp. Q-valued r-forms with compact support) on M ([3], [4], [21]). On $\Omega^r_0(M, Q)$, we may introduce a global scalar product $\langle\langle\, ,\, \rangle\rangle$ ([4], [21]). Let d_D be the exterior differential operator: $\Omega^r(M, Q) \to \Omega^{r+1}(M, Q)$, and an operator $d_D^*: \Omega^r(M, Q) \to \Omega^{r-1}(M, Q)$ is also defined ([3], [4]). We note that d_D^* is the adjoint operator of d_D with respect to $\langle\langle\, ,\, \rangle\rangle$ ([21]). It is trivial that

(i) an element of $\Gamma(Q)$ is regarded as an element of $\Omega^0(M, Q)$, that is, there exists an identification: $\Gamma(Q) \to \Omega^0(M, Q)$,

(ii) the bundle map $\pi: TM \to Q$ is an element of $\Gamma^1(M, Q)$,

(iii) the identification: $(\Gamma_0(Q), \langle\langle\, ,\, \rangle\rangle) \to (\Omega^0_0(M, Q), \langle\langle\, ,\, \rangle\rangle)$ is isometric,

(iv) $d_D s(X) = D_X s$ for any $s \in \Gamma(Q)$ and $X \in \Gamma(TM)$.

Proposition 3.1 ([4], [21]). *For $s \in \tilde{V}(\mathcal{F})$, it holds that*

$$\Delta_D s = d_D^* d_D s.$$

For any $s \in \tilde{V}(\mathcal{F})$, we have $w_k^2 s \in \Gamma_0(Q)$ and

$$d_D(w_k^2 s) = w_k^2 d_D s + 2 w_k d w_k \otimes s \tag{3.1}$$

([21]). We have

$$|d'' w_k| \leqq |d w_k| \tag{3.2}$$

so that, by (1.16),

Proposition 3.2 ([1], [2], [21], [22]). *For any $s \in \tilde{V}(\mathcal{F})$, it holds that*

$$\|d'' w_k \otimes s\|^2_{B(2k)} \leqq C^2 k^{-2} \|s\|^2_{B(2k)}.$$

Proposition 3.3 ([23]). *For any $s \in \tilde{V}(\mathcal{F})$, it holds that*

$$\begin{aligned}&\langle\langle \Delta_D s, w_k^2 s \rangle\rangle_{B(2k)}\\ &= \langle\langle w_k D s, w_k D s \rangle\rangle_{B(2k)} + 2\langle\langle w_k D s, d'' w_k \otimes s \rangle\rangle_{B(2k)}\end{aligned}$$

In fact, we have

$$\begin{aligned}&\langle\langle \Delta_D s, w_k^2 s \rangle\rangle_{B(2k)}\\ &= \langle\langle D s, D(w_k^2 s) \rangle\rangle_{B(2k)} \qquad \text{(see [23])}\end{aligned}$$

$$=\int_{B(2k)} g^{ab} g_Q(D_{X_a} s,\ D_{X_b}(w_k^2 s))dM$$

and

$$D_{X_b}(w_k^2 s)=2w_k \cdot X_b(w_k)s+w_k^2 D_{X_b} s$$
$$=2w_k \cdot d'' w_k(X_b)s+w_k^2 D_{X_b} s.$$

Now, by the Schwarz inequality and Proposition 3.2, we have

(3.3) $$|2\langle\langle w_k Ds,\ d'' w_k \otimes s\rangle\rangle_{B(2k)}|$$
$$\leqq \frac{1}{2}\|w_k Ds\|^2_{B(2k)}+2C^2 k^{-2}\|s\|^2_{B(2k)}$$

for any $s\in \tilde{V}(\mathcal{F})$.

Since, for any $s\in \tilde{V}(\mathcal{F})$,

(3.4) $$\operatorname{div}(w_k^2 \sigma(s))=\operatorname{div}_D(w_k^2 s)-g_Q(w_k^2 s,\ \tau),$$

we have

Proposition 3.4 *Suppose that $\mathcal{F}$ is minimal. Then*

$$\int_{B(2k)} \operatorname{div}_D(w_k^2 s)dM=0$$

for any $s\in \tilde{V}(\mathcal{F})$.

Moreover, for any $s\in \tilde{V}(\mathcal{F})$, we have

(3.5) $$\operatorname{div}_D((w_k^2 \operatorname{div}_D s)s)$$
$$=2g_Q((w_k \operatorname{div}_D s)s,\ \operatorname{grad}_D w_k)+g_Q(w_k^2 s,\ \operatorname{grad}_D \operatorname{div}_D s)+(w_k \operatorname{div}_D s)^2$$

([22], [23]).

Proof of Theorem C. Let $s\in \tilde{V}(\mathcal{F})$ be an L^2-t. c. f. of $\mathcal{F}$, that is, s satisfies

$$\Delta_D s=\rho_D(s)+\left(1-\frac{2}{q}\right)\operatorname{grad}_D \operatorname{div}_D s.$$

Then we have

(3.6) $$\langle\langle \Delta_D s,\ w_k^2 s\rangle\rangle_{B(2k)}$$
$$=\langle\langle \rho_D(s),\ w_k^2 s\rangle\rangle_{B(2k)}+\left(1-\frac{2}{q}\right)\langle\langle \operatorname{grad}_D \operatorname{div}_D s,\ w_k^2 s\rangle\rangle_{B(2k)}.$$

Since $\mathcal{F}$ is minimal, Proposition 3.4 and (3.5) imply

$$(3.7)\qquad \begin{aligned}&\langle\langle \mathrm{grad}_D \mathrm{div}_D s,\ w_k^2 s\rangle\rangle_{B(2k)}\\ &=-\int_{B(2k)} 2g_Q((w_k \mathrm{div}_D s)s,\ \mathrm{grad}_D w_k)dM\\ &\quad-\int_{B(2k)} (w_k \mathrm{div}_D s)^2 dM.\end{aligned}$$

By the Schwarz inequality for the local scalar product $\langle\ ,\ \rangle$, we have

$$\begin{aligned}&|2g_Q((w_k \mathrm{div}_D s)s,\ \mathrm{grad}_D w_k)|\\ &=|2\langle (w_k \mathrm{div}_D s)s,\ \mathrm{grad}_D w_k\rangle|\\ &\leqq 2|(w_k \mathrm{div}_D s)s|\cdot|\mathrm{grad}_D w_k|\\ &\leqq 2Ck^{-1}|(w_k \mathrm{div}_D s)s| \qquad \text{(by (1.16) and (3.2))}\\ &=2|w_k \mathrm{div}_D s|\cdot|Ck^{-1}\cdot s|\\ &\leqq \frac{1}{2}(w_k \mathrm{div}_D s)^2+2|Ck^{-1}\cdot s|^2\\ &=\frac{1}{2}(w_k \mathrm{div}_D s)^2+2C^2k^{-2}\langle s,\ s\rangle.\end{aligned}$$

Thus we have

$$(3.8)\qquad \begin{aligned}&\int_{B(2k)} 2g_Q((w_k \mathrm{div}_D s)s,\ \mathrm{grad}_D w_k)dM\\ &\leqq \frac{1}{2}\int_{B(2k)} (w_k \mathrm{div}_D s)^2 dM+2C^2k^{-2}\|s\|^2_{B(2k)}\end{aligned}$$

By Proposition 3.3 and 3.4, (3.3), (3.6), (3.7) and (3.8), we have

$$\begin{aligned}&\langle\langle w_k Ds,\ w_k Ds\rangle\rangle_{B(2k)}\\ &=\langle\langle \rho_D(s),\ w_k^2 s\rangle\rangle_{B(2k)}-2\langle\langle w_k Ds,\ d'' w_k\otimes s\rangle\rangle_{B(2k)}\\ &\quad-\left(1-\frac{2}{q}\right)\int_{B(2k)} 2g_Q((w_k \mathrm{div}_D s)s,\ \mathrm{grad}_D w_k)dM\\ &\quad-\left(1-\frac{2}{q}\right)\int_{B(2k)} (w_k \mathrm{div}_D s)^2 dM\\ &\leqq \langle\langle \rho_D(s),\ w_k^2 s\rangle\rangle_{B(2k)}+|2\langle\langle w_k Ds,\ d'' w_k\otimes s\rangle\rangle_{B(2k)}|\\ &\quad+\left(1-\frac{2}{q}\right)\left|\int_{B(2k)} 2g_Q((w_k \mathrm{div}_D s)s,\ \mathrm{grad}_D w_k)dM\right|\\ &\quad-\left(1-\frac{2}{q}\right)\int_{B(2k)} (w_k \mathrm{div}_D s)^2 dM\\ &\leqq \langle\langle \rho_D(s),\ w_k^2 s\rangle\rangle_{B(2k)}+\frac{1}{2}\|w_k Ds\|^2_{B(2k)}+2C^2k^{-2}\|s\|^2_{B(2k)}\end{aligned}$$

$$+\frac{1}{2}\left(1-\frac{2}{q}\right)\int_{B(2k)}(w_k\,\mathrm{div}_D s)^2 dM+2\left(1-\frac{2}{q}\right)C^2k^{-2}\|s\|^2_{B(2k)}$$

$$-\left(1-\frac{2}{q}\right)\int_{B(2k)}(w_k\,\mathrm{div}_D s)^2 dM$$

$$=\langle\langle\rho_D(s),\ w_k^2 s\rangle\rangle_{B(2k)}+\frac{1}{2}\|w_k Ds\|^2_{B(2k)}$$

$$+2\left(2-\frac{2}{q}\right)C^2k^{-2}\|s\|^2_{B(2k)}$$

$$-\frac{1}{2}\left(1-\frac{2}{q}\right)\int_{B(2k)}(w_k\,\mathrm{div}_D s)^2 dM.$$

Thus we have

$$\frac{1}{2}\|w_k Ds\|^2_{B(2k)}\leqq\langle\langle\rho_D(s),\ w_k^2 s\rangle\rangle_{B(2k)}+2\left(2-\frac{2}{q}\right)C^2k^{-2}\|s\|^2_{B(2k)}$$
$$-\frac{1}{2}\left(1-\frac{2}{q}\right)\int_{B(2k)}(w_k\,\mathrm{div}_D s)^2 dM. \tag{3.9}$$

Since $2\left(2-\frac{2}{q}\right)C^2k^{-2}\|s\|^2_{B(2k)}\to 0$ as $k\to\infty$ and ρ_D is non-positive everywhere on M, we have

$$\limsup_{k\to\infty}\left\{\langle\langle\rho_D(s),\ w_k^2 s\rangle\rangle_{B(2k)}+2\left(2-\frac{2}{q}\right)C^2k^{-2}\|s\|^2_{B(2k)}\right.$$
$$\left.-\frac{1}{2}\left(1-\frac{2}{q}\right)\int_{B(2k)}(w_k\,\mathrm{div}_D s)^2 dM\right\}\leqq 0.$$

Thus, as $k\to\infty$, we have that $0\leqq\frac{1}{2}\|Ds\|^2\leqq 0$. Therefore, we have $Ds=0$, that is, s is D-parallel.

If ρ_D is non-positive everywhere and negative for at least one point of M, then (3.9) implies, as $k\to\infty$, that $\langle\langle\rho_D(s),\ s\rangle\rangle=0$. Therefore, we have $s=0$.

If we set $\mathrm{div}_D s=0$, then Theorem D is proved.

Remark. Recently the authors were informed that part of our results were also proved by S. Nishikawa and Ph. Tondeur [*Transversal infinitesimal automorphisms of harmonic foliations on complete manifolds*, Preprint].

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Toshihiko AOKI
Department of Liberal Arts
Kanazawa Institute of Technology
Nonoichi-machi, 921 Japan

Shinsuke YOROZU
Department of Mathematics
College of Liberal Arts
Kanazawa University
Kanazawa, 920 Japan