YOKOHAMA MATHEMATICAL
JOURNAL VOL. 36, 1988

ON ANOTHER PROOF OF UMEZAWA'S THEOREM

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(Received March 31, 1987; revised July 20, 1988)

Let $f(z)$ be regular in $|z|\leqq 1$ and $f'(z)\neq 0$ on $|z|=1$.

Then, it is well known that if $f(z)$ satisfies the following condition

$$\int_{|z|=1}\left|1+\mathrm{Re}\frac{zf''(z)}{f'(z)}\right|d\theta<2\pi(p+1),$$

then $f(z)$ is at most p-valent in $|z|\leqq 1$.

In this paper, we want to give another proof of this theorem by applying of Ozaki's theorem.

1. Preliminary.

Let $f(z)$ be regular in $|z|\leqq 1$ and let us put

$$S=\{f(z)\,|\,|z|\leqq 1\},$$

$$S'=\{f(z)\,|\,|z|=1\}$$

and

$$\max_{\alpha\in S-S'}\frac{1}{2\pi}\int_{|z|=1}\left(\mathrm{Re}\frac{zf'(z)}{f(z)-\alpha}\right)d\theta=p \qquad (z=e^{i\theta}),$$

where $\alpha\in S-S'$ means that α is an element of S but it is not an element of S'.

From the elementary analytic function theory, we have that $f(z)$ is p-valent in $|z|<1$.

Let $A(p)$ be the class of function of the form

$$f(z)=\sum_{n=p}^{\infty}a_nz^n \quad (a_p\neq 0 \text{ and } p \text{ is a positive integer})$$

which are regular in $|z|\leqq 1$.

Theorem A. *Let* $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ *be regular in* $|z|\leqq 1$ *and* $f'(z)\neq 0$ *on* $|z|=1$. *If there holds the relation*

$$\int_0^{2\pi}\left|1+\mathrm{Re}\frac{zf''(z)}{f'(z)}\right|d\theta<4\pi \qquad (z=e^{i\theta}),$$

then $f(z)$ *is convex in one direction and hence* $f(z)$ *is univalent in* $|z|\leqq 1$.

We owe this theorem to [2, 4, 5, 6, 7, 8] and this theorem was generalized as follows [6, 8]:

Theorem B. *Let* $f(z)$ *be regular in* $|z|\leqq 1$ *and* $f'(z)\neq 0$ *on* $|z|=1$. *Suppose that*

$$\int_{|z|=1}|d\arg df(z)|=\int_0^{2\pi}\left|1+\mathrm{Re}\frac{zf''(z)}{f'(z)}\right|d\theta$$
$$<2\pi(p+1) \qquad (z=e^{i\theta}),$$

then $f(z)$ *is at most p-valent in* $|z|\leqq 1$.

In this paper we need the following lemma.

Lemma A. *Let* $f(z)$ *be regular in* $|z|\leqq 1$ *and* $f^{(k)}(z)\neq 0$ *for* $k=0, 1, 2, \cdots, p$ *on* $|z|=1$.
Then we have

$$\int_{|z|=1}|d\arg d^j f(z)|\leqq\int_{|z|=1}|d\arg d^{j+1}f(z)|$$

for $j=0, 1, 2, \cdots, p-1$ *or by a modification of the above inequalities*

$$\int_0^{2\pi}\left|j+\mathrm{Re}\frac{zf^{(j+1)}(z)}{f^{(j)}(z)}\right|d\theta$$
$$\leqq\int_0^{2\pi}\left|j+1+\mathrm{Re}\frac{zf^{(j+2)}(z)}{f^{(j+1)}(z)}\right|d\theta$$

for $j=0, 1, 2, \cdots, p-1$, *where* $z=e^{i\theta}$ *and* $0\leqq\theta\leqq 2\pi$.

We owe this lemma to Ozaki [2, 3].

2. Main theorem.

Theorem 1. *Let* $f(z)$ *be regular in* $|z|\leqq 1$, $f'(z)\neq 0$ *on* $|z|=1$ *and suppose that*

$$\int_{|z|=1}\left|1+\mathrm{Re}\frac{zf''(z)}{f'(z)}\right|d\theta<2\pi(p+1) \tag{1}$$

where p is a positive integer and $z=e^{i\theta}$, $0\leqq\theta\leqq 2\pi$.

Then $f(z)$ is at most p-valent in $|z|<1$.

Proof. Let α be an arbitrary element of $S-S'$ and put

$$g(z)=f(z)-\alpha.$$

Then from the definition of S and S', we have

$$g(z)\neq 0 \quad \text{on} \quad |z|=1,$$

$$g'(z)=f'(z) \quad \text{and} \quad g''(z)=f''(z).$$

From the assumption of Theorem 1, we have

$$g'(z)=f'(z)\neq 0 \quad \text{on} \quad |z|=1.$$

Applying Lemma A, we have

$$\begin{aligned}
&\operatorname*{Max}_{\alpha\in S-S'}\int_{|z|=1}\left(\mathrm{Re}\frac{zg'(z)}{g(z)}\right)d\theta \\
&=\operatorname*{Max}_{\alpha\in S-S'}\int_{|z|=1}\left(\mathrm{Re}\frac{zf'(z)}{f(z)-\alpha}\right)d\theta \\
&\leqq\operatorname*{Max}_{\alpha\in S-S'}\int_{|z|=1}\left|\mathrm{Re}\frac{zf'(z)}{f(z)-\alpha}\right|d\theta \\
&=\operatorname*{Max}_{\alpha\in S-S'}\int_{|z|=1}|d\arg g(z)| \\
&\leqq\operatorname*{Max}_{\alpha\in S-S'}\int_{|z|=1}|d\arg dg(z)| \\
&=\operatorname*{Max}_{\alpha\in S-S'}\int_{|z|=1}\left|1+\mathrm{Re}\frac{zg''(z)}{g'(z)}\right|d\theta \\
&=\operatorname*{Max}_{\alpha\in S-S'}\int_{|z|=1}\left|1+\mathrm{Re}\frac{zf''(z)}{f'(z)}\right|d\theta \\
&=\int_{|z|=1}\left|1+\mathrm{Re}\frac{zf''(z)}{f'(z)}\right|d\theta \\
&<2\pi(p+1)
\end{aligned}$$

where $z=e^{i\theta}$, $0\leqq\theta\leqq 2\pi$.

From the elementary analytic function theory,

$$\frac{1}{2\pi}\int_{|z|=1}\left(\mathrm{Re}\frac{\pi f'(z)}{f(z)-\alpha}\right)d\theta$$

must be a positive integer.

Therefore we have

$$\max_{\alpha\in S-S'} \frac{1}{2\pi}\int_{|z|=1}\left(\mathrm{Re}\frac{zf'(z)}{f(z)-\alpha}\right)d\theta \leqq p.$$

This shows that $f(z)$ is at most p-valent in $|z|<1$. This is another proof of Theorem B.

Corollary 1. *Let $f(z)\in A(p)$ and suppose that*

(2) $$p+\mathrm{Re}\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}>0 \quad \text{in} \quad |z|\leqq 1.$$

Then $f(z)$ is p-valent in $|z|<1$.

Proof. By applying the author's result [1, Lemma 9], by employing the same method as the proof of Theorem 1 and from the assumption (2), we have

$$f^{(k)}(z)\neq 0 \qquad \text{in} \quad 0<|z|\leqq 1$$

for $k=0, 1, 2, \cdots, p-1, p$.

Applying the same method as the proof of Theorem 1, by employing Lemma A and the assumption (2), we have

$$\begin{aligned}
&\max_{\alpha\in S-S'}\int_{|z|=1}\left(\mathrm{Re}\frac{zf'(z)}{f(z)-\alpha}\right)d\theta\\
&\leqq\int_{|z|=1}\left|1+\mathrm{Re}\frac{zf''(z)}{f'(z)}\right|d\theta\\
&=\int_0^{2\pi}\left|1+\mathrm{Re}\frac{zf''(z)}{f'(z)}\right|d\theta\\
&\leqq\int_0^{2\pi}\left|2+\mathrm{Re}\frac{zf''(z)}{f'(z)}\right|d\theta\\
&\leqq\cdots\cdots\cdots\cdots\cdots\cdots\\
&=\int_0^{2\pi}\left|p+\mathrm{Re}\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right|d\theta\\
&=\int_0^{2\pi}\left(p+\mathrm{Re}\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right)d\theta
\end{aligned}$$

Since $f^{(p)}(z)$ has no zero in $|z|\leqq 1$, we have

$$\int_0^{2\pi}\mathrm{Re}\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}d\theta=0$$

for $z=e^{i\theta}$, $0\leqq\theta\leqq 2\pi$.

Therefore we have

(3)
$$\underset{\alpha\in S-S'}{\mathrm{Max}}\int_{|z|=1}\left(\mathrm{Re}\frac{zf'(z)}{f(z)-\alpha}\right)d\theta$$
$$\leqq\int_0^{2\pi}\left(p+\mathrm{Re}\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right)d\theta$$
$$=2p\pi<2\pi(p+1).$$

On the other hand, we easily have

(4)
$$\int_{|z|=1}\left(\mathrm{Re}\frac{zf'(z)}{f(z)}\right)d\theta=2p\pi.$$

From (3) and (4), we have

$$\underset{\alpha\in S-S'}{\mathrm{Max}}\int_{|z|=1}\left(\mathrm{Re}\frac{zf'(z)}{f(z)-\alpha}\right)d\theta=2p\pi$$

and therefore $f(z)$ is p-valent in $|z|<1$.

Applying the same method as the proof of Corollary 1, we easily obtain the following corollaries:

Corollary 2. *Let $f(z)\in A(p)$ and suppose that*

$$\int_0^{2\pi}\left|p+\mathrm{Re}\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right|d\theta<2\pi(p+1)$$

where $z=e^{i\theta}$, $0\leqq\theta\leqq2\pi$.

Then $f(z)$ is p-valent in $|z|<1$.

Corollary 3. *Let $f(z)\in A(p)$ and suppose that*

$$\int_0^{2\pi}\left|1+\mathrm{Re}\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right|d\theta<4\pi$$

where $z=e^{i\theta}$, $0\leqq\theta\leqq2\pi$. Then $f(z)$ is p-valent in $|z|<1$.

The author is indebted to the referee for helpful comments and suggestions.

Acknowledgement. I would like to thank Prof. W. M. Causey for much help during the preparation of this paper at the University of Mississippi.

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