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# THE J-SUM OF BANACH ALGEBRAS AND SOME APPLICATIONS

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## Introduction.

James space $J$[11, 12] is the first example of a non-reflexive Banach space isomorphic with its bidual $J^{**}$. Andrew and Green [1] have studied $J$ as a Banach algebra. James [13] and Lindenstrauss [17] proved that every separable Banach space $Z$ can be realized as the quotient space of the bidual $E^{**}$ by $E$ for a suitable Banach space $E$. Bellenot [2] defined and studied the $J$-sum of Banach spaces and obtained similar results which have the advantage that if $Z$ is a concrete example then so is $E$. The purpose of this paper is to define and study $J(X_n)$, the $J$-sum of a sequence $(X_n)$ of Banach algebras and apply them to obtain Banach algebra versions of such results. We use the theory of decompositions in Banach spaces [19, 20] and also orthogonal bases developed by Husain and others [6, 7, 8, 9].

## 1. Notation, Terminology and Basic Theory of Schauder Decompositions of Banach Algebras.

For the basic theory of Banach algebras we refer to [3] and for that of bases and decompositions in Banach spaces to [18, 19, 20] rather than going to original sources. As in [19, 20] for a subset $S$ of a Banach space $X$, $[S]$ denotes the closed linear span of $S$ in $X$ and we say that $X$ is topologically spanned by $S$ if $[S]=X$. All our spaces will be over the field $R$ of real numbers.

Let $(E, \|\circ\|)$ be a Banach algbra, $E^*$ its dual space and $E^{**}$ its bidual i.e. the second dual space of $E$. $E^{**}$ can be made into a Banach algebra under the Arens product and the canonical map $a \rightarrow \hat{a}$ is an isometric monomorphism of $E$ into $E^{**}$. Let L($E$) and B(E) be the algebras of linear mappings and bounded linear operators on $E$ to itself.

For a locolly convex algebra E, let $\Phi_E$ be the space of (non zero, continuous) multiplicative linear functionals (in short, mlf's) on $E$ with the weak* topology given by basic neighbourhoods of the type

$$V(\phi\,;\,S\,;\,\varepsilon)=\{\psi\in\Phi_E:\ |\psi(y)-\phi(y)|<\varepsilon \text{ for all } y\in S\}$$

where $S$ is a finite subset of E, $\varepsilon>0$ and $\phi\in\Phi_E$.

We now come to the basic theory of orthogonal Schauder decompositions in Banach algebras most of which follows in a natural way from the theory of Schauder decompositions in Banach spaces as presented in [19, 20]. T. Husaín and others [10] are developing a detailed theory of orthogonal decompositions in topological algebras but we shall confine our attention to the following independent development of ours which is to be used in the next section.

**(1.1)** Let $\langle G_n\rangle$ be a sequence of non-zero closed subalgebras of $E$ such that for $n, m\in N$, the set of natural numbers, we have $G_n\cdot G_m=\{0\}$ for $n\neq m$ i.e. $G_n$'s are mutually orthogonal. Let $D(G_n)=\prod_{n\in N} G_n$ be the algebra of sequences $\langle y_n\rangle$ with $y_n\in G_n$ for each $n\in N$ under pointwise operations. It is commutative if and only if each $G_n$ is so. For $y=\langle y_n\rangle\in D(G_n)$ and $j\in N$ let $p_j\langle y_n\rangle=y_j$ and let $y^{(j)}$ be the sequence given by $(y^{(j)})_n=y_n$ for $n\leqq j$ and $(y^{(j)})_n=0$ for $n>j$. Then $y^{(j)}\to y$ in the topology $\tau_p$ of pointwise convergence on $D(G_n)$.

Let $D_0(G_n)=\{\langle y_n\rangle\in D(G_n):\ y_n=0$ for all but a finite number of $n$'s$\}$,

$$C_0(G_n)=\{\langle y_n\rangle\in D(G_n):\ y_n \longrightarrow 0 \text{ in } E\}$$

and $D_\infty(G_n)=\{\langle y_n\rangle\in D(G_n):\ \langle y_n\rangle$ is bounded in $E\}$. Then $D_\infty(G_n)$ endowed with the norm $\|\langle y_n\rangle\|_\infty=\sup_n\|y_n\|$ is a Banach algebra and $C_0(G_n)$ is a closed ideal in it. Also $D_0(G_n)$ is a dense ideal in $(C_0(G_n),\ \|\cdot\|_\infty)$ as well as in $(D(G_n),\ \tau_p)$.

For $y=\langle y_n\rangle\in D(G_n)$ let $\|y\|=\sup_{n\in N}\left\|\sum_{j=1}^{n} y_j\right\|$.

By Proposition III. 15.2 [20] the space $D_1(G_n)=\{\langle y_n\rangle\in D(G_n):\ \sum y_n$ converges in $E\}$ endowed with the norm $\|\cdot\|$ is a Banach space and also by Discussion on p. 500 [20], the space $D_2(G_n)=\{y=\langle y_n\rangle\in D(G_n):\ \|y\|<\infty\}$ endowed with the norm $\|\cdot\|$ is a Banach space.

Let $y=\langle y_n\rangle\in D(G_n)$. Then

(i) $\|y\|_\infty\leqq 2\|y\|$,

(ii) $y\in D_1(G_n)$ if and only if $\langle y^{(j)}\rangle$ is a Cauchy sequence in $(D_1(G_n),\ \|\cdot\|)$ if and only if $\|y^{(j)}-y\|\to 0$ and

(iii) $y\in D_2(G_n)$ if and only if $\langle y^{(j)}\rangle$ is a bounded sequence in $(D_1(G_n,\ \|\cdot\|)$.

Consequently $D_0(G_n)$ is a dense subspace of $(D_1(G_n),\ \|\cdot\|)$ and $D_1(G_n)\subset D_2(G_n)\subset D_\infty(G_n)$. Further if for each $n$, $S_n$ is a dense subset of $G_n$ containing 0 then $D_0(S_n)=\{\langle y_n\rangle\in D_0(G_n):\ y_n\in S_n$ for each $n\}$ is dense in $D_1(G_n)$.

It is an easy consequence of mutual orthogonality of $G_n$'s that $D_1(G_n)$ and $D_2(G_n)$ are algebras, $D_1(G_n)$ is an ideal in $D_2(G_n)$ and $\|\cdot\|$ is submultiplicative Therefore, $(D_1(G_n),\ \|\cdot\|)$ and $(D_2(G_n),\ \|\cdot\|)$ are Banach algebras and $D_1(G_n)$ is a closed ideal in $(D_2(G_n), \|\cdot\|)$.

So we can talk of right (left) multipliers on $D(G_n)$ to itself and $D_j(G_n)$ to itself and $D_j(G_n)$ to $D_k(G_n)$ $(j, k=1, 2, \infty; (j, k)\neq(\infty, 1), (\infty, 2))$.

As noted in the proof (3) of Theorem III. 15.1 [20], the sequence $\langle F_n\rangle$ of subspaces of $D_1(G_n)$ defined by $F_n=\prod_{j\in N} H_j$ where $H_j=\{0\}$ for $j\neq n$ and $H_n=G_n$ is a Schauder decomposition of $D_1(G_n)$ which is also monotone ([20], Definition III. 15.13). The fact that $D_0(G_n)$ is dense in $(D_1(G_n), \|\cdot\|)$ follows from this observation as well.

If $\langle G_n\rangle$ is a Schauder decomposition of $E$ with $\langle \nu_n\rangle$ as the associated sequence of co-ordinate projections then $\langle G_n\rangle$ can be thought of as an algebraic Schauder decomposition of $E$ in the sense that $\nu_n$'s are homomorphisms. Further, using Proposition III. 15.3 [20] $D_1(G_n)$ is isomorphic to $E$ by the mapping $\langle y_n\rangle\rightarrow\sum_{j=1}^{\infty} y_j$ and $\|\cdot\|$ and $|||\cdot|||$ are equivalent in $E$, where $|||x|||=\|\langle \nu_n x\rangle\|$. Clearly $\|\cdot\|=|||\cdot|||$ on $G_n$ for each $n$. If $\|\cdot\|=|||\cdot|||$, then $\nu=\sup_n\left\|\sum_{j=1}^{n}\nu_j\right\|=1$. Thus $\|\nu_1\|\leqq 1$ and $\|\nu_n\|\leqq 2$ for each $n$. Since $\nu_1$ is a projection, $\|\nu_1\|\geqq 1$, and, therefore, $\|\nu_1\|=1$. Consequently $\langle G_n\rangle$ is monotone. From now onwards we assume that $(G_n)$ is monotone.

For each $n$, let $\nu_n^*$ be the adjoint of $\nu_n$ defined on $E^*$ to $E^*$ and $\nu_n^{**}$ the adjoint of $\nu_n^*$ defined on $E^{**}$ to $E^{**}$. Further for $n, m\in N$ we have $\nu_n\nu_m=\delta_{nm}\nu_n$, $\nu_m^*\nu_n^*=\delta_{nm}\nu_n^*$ and $\nu_n^{**}\nu_m^{**}=\delta_{nm}\nu_n^{**}$. By Theorem 6.1 of [4] each $\nu_n^{**}$ is a homomorphism. The following proposition gives a little more.

**(1.2) Proposition.** *Let $m, n\in N$. Then*

(a) *for $\phi, \psi\in E^{**}$, $(\nu_n^{**}\phi)(\nu_m^{**}\psi)=\delta_{nm}\nu_n^{**}(\phi\psi)$,*

(b) $(\nu_n^{**}E^{**})(\nu_m^{**}E^{**})\subset\delta_{nm}\nu_n^{**}E^{**}$.

**Proof.** It is enough to prove (a).

Let $f\in E^*$. Since for $x, y\in E$

$$(\nu_n x)(\nu_m y)=\delta_{nm}(\nu_n x)(\nu_n y)=\delta_{nm}\nu_n(xy),$$

we have that

$$\nu_m^*(f(\nu_n x))=\delta_{nm}(\nu_n^* f)x \qquad \text{for each } x \text{ in } E.$$

So

$$\begin{aligned}(\nu_n^*((\nu_m^{**}\psi)f))(x)&=\psi(\nu_m^*(f(\nu_n x)))\\ &=\psi(\delta_{nm}(\nu_n^* f)x)=\delta_{nm}(\psi(\nu_n^* f))(x) \qquad \text{for each } x \text{ in } E.\end{aligned}$$

Therefore,

$$\begin{aligned}(\nu_n^{**}\phi)((\nu_m^{**}\psi)f)&=\delta_{nm}\phi(\psi(\nu_n^* f))\\ &=\delta_{nm}(\nu_n^{**}(\phi\psi))(f).\end{aligned}$$

Hence

$$(\nu_n^{**}\phi)(\nu_m^{**}\psi)=\delta_{nm}\nu_n^{**}(\phi\psi).$$

By Proposition III. 15.6(c) [20], the map $\tau$ on $E^{**}$ defined by $\tau(\phi)=\langle\nu_n^{**}\phi\rangle$ is a norm-decreasing map of $E^{**}$ onto the Banach space

$$D_3(G_n)=D_2(\nu_n^{**}E^{**})=\left\{\langle\phi_n\rangle : \phi_n\in\nu_n^{**}E^{**} \text{ for each } n,\ \sup_n\left\|\sum_{j=1}^{n}\phi_j\right\|<\infty\right\}$$

equipped with the norm $\|\langle\phi_n\rangle\|=\sup_n\left\|\sum_{j=1}^{n}\phi_j\right\|$. By Theorem III. 15.13 [20], $\tau$ is an isomorphism (or equivalently $\tau$ is one-one) if and only if $\langle G_n\rangle$ is shrinking and in that case monotonicity of $\langle G_n\rangle$ and Proposition III. 15.6(b) [20] give that $\tau$ is an isometry: Moreover, by Proposition III. 15.7(b) [20], for each $n$, $\nu_n^{**}(E^{**})$ can be identified with $G_n^{**}$ via the linear map $\nu_n^{**}(\phi)\to\Sigma_{n,\phi}$ given by $\Sigma_{n,\phi}(f\,|\,G_n)=(\nu_n^{**}\phi)(f)=\phi(\nu_n^{*}f)$. It can be easily shown that $\Sigma_{n,\phi}(f\,|\,G_n)=((\nu_n^{**}\phi)f)\,|\,G_n$ ([3], p. 50) for $\phi$ in $E^{**}$ and $f$ in $E^{*}$. So using Proposition (1.2) above we have that $\Sigma_{n,\phi}\Sigma_{n,\psi}=\Sigma_{n,\phi\psi}$. Therefore, $\nu_n^{**}(E^{**})$ and $G_n^{**}$ can be identified as Banach algebras as well via the same map. Thus we have the following

**(1.3) Theorem.** *If $\langle G_n\rangle$ is a monotone shrinking decomposition of $E$ then $E^{**}$ can be identified with $D_3(G_n)$ as a Banach algebra. If, in addition, each $G_n$ is reflexive then $E^{**}$ can be identified with $D_2(G_n)$.*

We note some simple properties of $D_1(G_n)$ and $D_2(G_n)$ which have to be often used in sequel. We shall write $D_0$, $D$, $D_1$, $D_2$, $D_\infty$ and $C_0$ in place of $D_0(G_n)$, $(D(G_n), \tau_p)$, $(D_1(G_n), \|\cdot\|)$, $(D_2(G_n), \|\cdot\|)$, $(D_\infty(G_n), \|\cdot\|_\infty)$ and $(C_0(G_n), \|\cdot\|_\infty)$ respectively.

**(1.4) Remarks.** (i): $\langle y_n\rangle\in D$ is an idempotent (a right identity, a left identity, the identity) in $D$ if and only if for each $n$, $y_n$ is an idempotent (a right identity, a left identity, the identity) in $G_n$.

(ii) $\langle y_n\rangle$ is an idempotent in $D_1$ if and only if $\langle y_n\rangle\in D_0$ and for each $n$, $y_n$ is an idempotent in $G_n$. In particular, $D_1$ has no right or left identity.

(iii) If $\langle y_n\rangle\in D$ is such that for each $n$, $y_n$ is (a right, a left, the) identity in $G_n$ then $\langle y^{(j)}\rangle$ is a (right, left, two-sided) approximate identity for $D_1$ which is further bounded if and only if $\langle y_n\rangle$ is in $D_2$.

(iv) $D_2$ has a right identity implies that $D_1$ has a bounded sequential right approximate identity. In view of Theorem (1.3) above this may be thought of as a little improvement in this particular case of the well known result ([3], p. 146 and [5]) which says that if the bidual of a Banach algebra $A$ has a right identity then $A$ has a bounded right approximate identity. It is noteworthy that the bidual of a commutative Banach algebra need note be commutative ([5]) but Theorem 1.3 above gives that $E^{**}$ is commutative if $E$ is commutative and each $G_n$ is reflexive.

**(1.5) Proposition.** *The spaces $\Phi_{C_0}$, $\Phi_D$ and $\Phi_{D_1}$ of multiplicative linear functionals on $C_0$, $D$ and $D_1$ with their respective weak* topologies respectively can all be identified with the disjoint topological sum of $\Phi_{G_n}$'s via*

$$\bigcup_{n\in N}\{f\circ p_n : f\in\Phi_{G_n}\}.$$

**Proof.** Since $D_0(G_n)$ is the linear span of $F$'s and $F_jF_k\subset\delta_{jk}F_j$, the set of multiplicative linear functionals on $D_0(G_n)$ can be identified with $\{f\circ p_j : f$ is a multiplicative linear functional on $G_j$, $j\in N\}$.

Let $E$ denote $C_0$, $D$ or $D_1$.

Since for each $n$, $p_n$ is continuous on $E$ onto $G_n$ and $D_0(G_n)$ is dense in $E$ we have the desired identification set theoretically.

We shall now show that the expression on the right is a disjoint topological sum:

Let $\phi=f\circ p_j$ for some $f\in\Phi_{G_j}$, some $j\in N$. There exists $x_j\in G_j$ such that $f(x_j)\neq 0$. Let $0<\varepsilon\leqq\frac{|f(x_j)|}{2}$. Let $x=\langle\delta_{nj}x_j\rangle_n$. Then $x\in D_0(G_n)$ and for $n\neq j$, for all $g\in\Phi_{G_n}$, $g\circ p_n(x)=0$ so that $|(g\circ p_n)(x)-(f\circ p_j)(x)|=|f(x_j)|\geqq 2\varepsilon$.

This shows that each $\Phi_{G_j}\circ p_j$ is open in $\Phi_E$. Since the sets $\Phi_{G_j}\circ p_j$ are mutually disjoint and their union is $\Phi_E$ we have that each $\Phi_{G_j}\circ p_j$ is also closed in $\Phi_E$. The fact that the topology induced on $\Phi_{G_j}\circ p_j$ by that of $\Phi_E$ is same as that induced by that of $\Phi_{G_j}$ follows from the observation that for a nonempty finite subset $S$ of $E$, $\varepsilon>0$ and $\phi=f\circ p_j$ with $f\in\Phi_{G_j}$

$$V(\phi, S, \varepsilon)\cap\Phi_{G_j}\circ p_j=V(f, p_j\circ S, \varepsilon)\circ p_j.$$

**(1.6)** Given a $\langle T_n\rangle\in\prod_{n\in N}L(G_n)$, we can define a linear mapping $T=\beta(\langle T_n\rangle)$ on $D$ to itself by $T(\langle x_n\rangle)=\langle T_nx_n\rangle$, $T$ leaves each $F_n$ and therefore, $D_0$ invariant; further, it is a right (left) multiplier ([14], [16]) if and only if each $T_n$ is so. In case $T_n\in B(G_n)$ for each $n$, $T$ is the only linear mapping which is continuous on $D=(D(G_n), \tau_p)$ and coincides with $T_n$ on $G_n$ (identified with $F_n$). In case $T|D_0(G_n)$ is continuous on $(D_0(G_n), \|\cdot\|)$ we have that $TD_2\subset D_2$ and $T|D_2$ is continuous on $D_2$ with $\|T|D_2\|=\|T|D_0(G_n)\|$. Since $D_0(G_n)$ is dense in $D_1$, this further gives that $TD_1\subset D_1$ and $T|D_1$ is the only continuous extension of $T|D_0(G_n)$ to $D_1$.

On the other hand if $T$ is a linear mapping on $D$ to itself leaving each $F_n$ invariant, the restriction of $T$ to $F_n$ induces a $T_n\in L(G_n)$, and if $T$ is continuous on $D$ to itself then $T_n\in B(G_n)$ for each $n$; we shall write $\langle T_n\rangle=\theta(T)$. In case each $G_n$ has no right (left) divisors of zero, a right (left) multiplier $T$ on $D$ to itself or on $D_j$ to $D_k$, $j$, $k=1, 2, \infty$; $(j, k)\neq(\infty, 1), (\infty, 2)$ leaves each $F_n$

invariant. Further, in this case a right (left) multiplier $T$ on $D_2$ to itself is zero iff $T|D_1$ is zero.

**(I.7) Proposition.**

(i) *If $T$ is a bounded linear operator on $D_j$ to $D_k$ $(j, k=1, 2, \infty)$ leaving each $F_n$ invariant then for ecah $n$, $T_n$ is bounded and $\|T_n\|\leqq\|T\|$. Further $TD_1\subset D_1$ if $j=k=2$.*

(ii) *Let $\langle T_n\rangle\in\prod_{n\in N}B(G_n)$ and $T=\beta(\langle T_n\rangle)$,*

(a) *If $\langle\|T_n\|\rangle$ is bounded then $T$ defines a bounded linear operator on $D_\infty$ to itself and $\|T\|=\sup_n\|T_n\|$.*

(b) *If $\langle\|T_n\|\rangle\in l_1$ then $T$ defines a bounded linear operator on $D_2$ to itself, $TD_1\subset D_1$ and $\|T\|\leqq 2\sum_{n=1}^{\infty}\|T_n\|$.*

(c) *Let $1\leqq p, q, r\leqq\infty$ be such that $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$, suppose that $\langle G_n\rangle$ is $p$-Besselian with constant $c$ and $r$-Hilbertian with constant $C$* ([20], *Theorem* III. 15.16). *If $\langle\|T_n\|\rangle\in l_q$ then $T$ defines a bounded linear operator on $D_2$ to itself, $TD_1\subset D_1$ and $\|T\|\leqq\frac{C}{c}\|\langle T_n\rangle\|_q$.*

**Proof.** (i) and (ii)(a) follow from the observation that for $y_{n_0}\in G_{n_0}$, $h_n=0$, $n\neq n_0$, $h_{n_0}=y_{n_0}$

$$\|y_{n_0}\|=\|\langle h_n\rangle\|=\|\langle h_n\rangle\|_\infty.$$

For (ii)(c) we first note that $\dfrac{1}{\left(\frac{p}{r}\right)}+\dfrac{1}{\left(\frac{q}{r}\right)}=1$ if $r\neq\infty$ and $p=q=\infty$ if $r=\infty$ so the result follows after simple computations using Theorem III. 15.16 [20]. (ii)(b) follows from (ii)(c) since every monotone decomposition is $\infty$-Besselian with constant $c=\frac{1}{2}$ and 1-Hilbertian with constant $C=1$.

We shall refine these results in some special cases in the next section.

**(1.8) Remarks.** If each $G_n$ is one-dimensional and $G_n^2=G_n$ or, equivalently, if each $G_n$ is spanned by a single $x_n$ with $x_n^2=x_n\neq 0$ then we have an orthogonal base $\langle x_n\rangle$ for $E$ in the sense of ([6], [7], [8] and [9]) (and not of [19]).

Many properties and examples of such bases have been given in these papers. To mention a few $E$ is commutative and semi-simple, and the maximal ideal space of $E$ consists of coordinate functionals and can thus be identified with $N$. Also obviously since $E$ has no right or left divisors of zero i. e. it is without order and therefore as in ([16], Theorem 1.1.1 or [14]) we have that every multiplier on $E$ is bounded. Further we note that for each $n$, $L(G_n)=B(G_n)$ can be identified with $R$ via $T_n\leftrightarrow t_n$, $T_n(y_n)=t_ny_n$. $T=\beta(\langle t_n\rangle)$ gives a linear mapping of $E$ to itself if and only if the corresponding $\langle t_n \rangle$ is a multiplier sequence of $E$ ([19], Definition 5.1) and in that case by Proposition I. 5.4 [19], $T$ is bounded and $\|\langle t_n \rangle\|_\infty \leqq \|T\|$. Let us write $\langle t_n \rangle = \gamma_E(T)$. Apart from drawing obvious analogues of Proposition 1.7 above we can deduce from Theorem II. 16.5 of [19] that if $\langle x_n \rangle$ is unconditional then the Banach algebra $l_\infty$ of bounded sequences is isomorphic to the algebra of bounded multipliers on $E$ via $\gamma_E$.

In general this map $\gamma_E$ is not an isomorphism e. g. in the case of Banach algebra $J$, whose multiplier algebra ([2], Theorem 3.1) is $J^{**}$ which is separable and is, therefore, not isomorphic to $1_\infty$. We note that the natural basis of $J$ is not unconditional. On the other hand $\gamma_E$ becomes an isometry in case $\langle x_n \rangle$ is hyperorthogonal in the sense of ([19], Definition II. 20.2).

The following variant of Singer's example ([19], Example II. 20.1) shows that $\gamma_E$ need not be an isometry if the basis is not hyperorthogonal, even if the basis is unconditional.

(**1.9**) **Example.** Let $E$ be as in ([19], Example II. 20.1) the two-dimensional Banach space of all pairs of scalars $x=(\xi_1, \xi_2)$ endowed with norm $\|x\| = \max(|\xi_1|, |\xi_2|, |\xi_1+\xi_2|)$, $(x=(\xi_1, \xi_2)\in E)$. Then $E$ is an algebra in which $\|\cdot\|$ is not submultiplicative, but $\|\cdot\|'=2\|\cdot\|$ is submultiplicative. As argued in [19], the natural basis of $E$ is not hyperorthogonal. Let $t_1=1$ and $t_2=-1$. Then $\|(t_1, t_2)\|_\infty=1$. On the other hand for $x=(1, -1)$, $\|x\|'=2$ and $\|Tx\|'=\|(1, 1)\|'=4$ so that $\|T\|\geqq 2$.

(**1.10**) Now we suppose that for each $n$, $G_n$ has an orthogonal basis $\{e_j^n : j\in M_n\}$, $M_n=N$ or $M_n=\{j : 1\leqq j\leqq k_n\}$ for some $k_n\in N$. Let $\Lambda=\{(n, j): j\in M_n, n\in N\}$. For $(n, j), (n', j')\in\Lambda$, $e_j^n\cdot e_{j'}^{n'}=\delta_{nn'}\delta_{jj'}e_j^n$. In view of Proposition 1.5 and (1.8) above, the set of multiplicative linear functionals on $D_1$ can be identified with $\Lambda$ via

$$\phi_{m,k}\Big(\langle \sum_{j\in M_n} \alpha_j^n e_j^n \rangle\Big)=\alpha_k^m .$$

As in (1.8) above each multiplier on $G_n$ to itself is bounded and the set of multipliers on $G_n$ to itself can be identified via $\gamma_{G_n}$ with the set $\Gamma_n$ of multiplier sequences of $G_n$ (defined on $M_n$ to $R$). Let $\Gamma=\prod_{n\in N}\Gamma_n$. Then $\Gamma\subset \prod_{n\in N}\Big(\prod_{j\in M_n} R\Big)=R^\Lambda$.

Using Proposition (1.7) and (1.8) above we have:

**Proposition.** (i) *The set of multipliers on $D$ to itself can be identified with $\Gamma$ via $T\to\gamma(T)=\langle\gamma_{G_n}(T_n)\rangle$, where $\langle T_n\rangle=\Theta(T)$.*

(ii) *For a multiplier $T$ on $D_j$ to $D_k$, $j$, $k=1, 2, \infty$, $(j, k)\neq(\infty, 1), (\infty, 2)$,*

*$\gamma(T)$ considered as a function on $\Lambda$ to $R$ is bounded and $\|\gamma(T)\|_\infty \leqq \|T\|$; further $\gamma$ is injective.*

(iii) *If $\alpha$ is a bounded function on $\Lambda$ to $R$ and for each $n$, the basis $\langle e_j^n\rangle$ of $G_n$ is hyperorthogonal then $\alpha$ gives rise to a multiplier $T$ on $D_\infty$ to itself and $\|\alpha\|_\infty = \|T\|$.*

(1.11) **Proposition.** *Let $1 \leqq p, q, r \leqq \infty$ be such that $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. Suppose that the decomposition $\langle G_n\rangle$ of $E$ is $p$-Besselian with constant $c$ and $r$-Hilbertian with constant $C$* ([20], Theorem III. 15.16). *Let $\alpha$ be a function on $\Lambda$ to $R$, $\alpha = \langle\langle \alpha_j^n\rangle_{j\in M_n}\rangle_{n\in N}$.*

(a) *If for each $n$, the basis $\langle e_j^n\rangle$ of $G_n$ is hyperorthogonal and $\|\langle \|\langle \alpha_j^n\rangle_j\|_\infty\rangle_n\|_q < \infty$, then $\alpha$ gives rise to a bounded multiplier $T$ on $D_2$ to itself and $\|T\| \leqq \frac{C}{c}\|\langle \|\langle \alpha_j^n\rangle_j\|_\infty\rangle\|_q$.*

(b) *Let $\langle p_n\rangle$, $\langle q_n\rangle$, $\langle r_n\rangle$, $\langle C_n\rangle$, $\langle c_n\rangle$ be sequences of positive numbers such that for each $n$, $\frac{1}{p_n}+\frac{1}{q_n}=\frac{1}{r_n}$, and $1 \leqq p_n, q_n, r_n \leqq \infty$. If for each $n$, the basis $\langle e_j^n\rangle$ of $G_n$ is $p_n$-Besselian with constant $c_n$ and $r_n$-Hilbertian with constant $C_n$* ([20], Theorem III. 15.16) *and $\left\|\langle \frac{C_n}{c_n}\|\langle \alpha_j^n\rangle_j\|_{q_n}\rangle_n\right\|_q < \infty$ then $\alpha$ gives rise to a multiplier on $D_2$ to itself and $\|T\| \leqq \frac{C}{c}\left\|\langle \frac{C_n}{c_n}\|\langle \alpha_j^n\rangle_j\|_{q_n}\rangle_n\right\|_q$.*

In the next section we shall be able to refine these results in a special example.

## 2. The $J$-sum of Banach Algebras.

(**2.1**) Let $\langle X_n\rangle$ be an increasing sequence of closed subalgebras of a Banach algebra $Z$ with $X_1 \neq \{0\}$. We introduce an equivalent norm $\|\cdot\|$ on $Z$, if the need be, so that $\|xy\| \leqq \frac{1}{2}\|x\|\cdot\|y\|$ for $x$, $y$ in $Z$. We now construct the $J$-sum of $X_n$'s on the lines of [2].

Let $\mathcal{P}$ be the set of finite strictly increasing sequences $P=\{p_1, p_2, \cdots, p_k\}$ of non-negative integers. For $x=\langle x_n\rangle$, $x_n \in X_n$ and $P \in \mathcal{P}$ let

$$\|x\|_P^2 = \sum_{i=1}^{k-1} \|x_{p_i} - x_{p_{i+1}}\|^2 + \|x_{p_k}\|^2,$$

where for notational convenience we set $x_0=0$.

Define
$$2\|x\|_J^2 = \sup_{P\in\mathcal{P}} \|x\|_P^2.$$

Then $\sup_n \|x_n\| \leqq \|x\|_J$.

So using Schwarz-inequality and the stringent submultiplicativity condition on $\|\cdot\|$ above we have that for $x=\langle x_n\rangle$, $y=\langle y_n\rangle$, $xy=\langle x_n y_n\rangle$, $\|xy\|_J \leqq \|x\|_J \|y\|_J$. Further, $\|\cdot\|_J$ is a norm on the space $\Phi(X_n)=\{\langle x_n\rangle : x_n$ is zero for all but finitely many $n$'s$\}$ and its completion is called the $J$-sum of $X_n$'s and is denoted by $J(X_n)$. Thus $J(X_n)$ is a Banach algebra. Further $\langle X_n\rangle$ is a monotone shrinking decomposition for $J(X_n)$ if we identify $X_n$ with $G_n=\prod_{j\in N} X_j^n$, where $X_n^n=X_n$ and $X_j^n=\{0\}$ for $j\neq n$.

Moreover, $J(X_n)^{LIM}=\{\langle x_n\rangle : \|\langle x_n\rangle\|_J<\infty\}$ and $(J(X_n), \|\cdot\|_J)$ and $(J(X_n)^{LIM}, \|\cdot\|_J)$ can be looked upon as $D_1(G_n)$ and $D_2(G_n)$ respectively. Thus applying Theorem 1.3 to $J(X_n)$ we have

**Theorem.** *If $X_n$ is reflexive for each $n$ then $J(X_n)^{LIM}$ is isometrically algebraically isomorphic with $J(X_n)^{**}$.*

In view of ([2], Remark 7, p. 98) we have the following algebraic analogue of Theorem 1.1 III of [2].

(2.2) **Theorem.** *Let $Z$ and $\langle X_n\rangle$ be as above such that $[\bigcup_n X_n]=Z$ and each $X_n$ is reflexive. Then $Z$ is isometrically algebraically isomorphic with $J(X_n)^{**}/J(X_n)$.*

**Proof.** We shall use proof of ([2], Theorem 1.1). Let $\Omega(X_n)$ be the algebra under pointwise operations of eventually constant sequences $\langle x_n\rangle$ with seminorm $\|x\|_\Omega=\lim_{n\to\infty}\|x_n\|$. Then $\|\cdot\|_\Omega$ is submultiplicative and, therefore, $\ker\|\cdot\|_\Omega$ is an ideal in $\Omega(X_n)$. Thus $\|\cdot\|_\Omega$ induces a submultiplicative norm on $\Omega/\ker\|\cdot\|_\Omega$.

So $\bar{\Omega}(X_n)$, the completion of the space $(\Omega/\ker\|\cdot\|_\Omega, \|\cdot\|_\Omega)$ is a Banach algebra. We first note that the map $\theta$ in ([2], Theorem 1.1) is in fact the unique continuous extension $q_1$ of the quotient map $q:(\Omega, \|\cdot\|_J)$ to $(\Omega/\ker\|\cdot\|_\Omega, \|\cdot\|_\Omega)$. Since $q$ is multiplicative, so is $q_1$. Consequently its kernel $J(X_n)$ is a closed ideal in $J(X_n)^{LIM}$. Also $J(X_n)^{LIM}/J(X_n)$ is isometrically isomorphic to $\bar{\Omega}(X_n)$ from ([2], Theorem 1.1). But since $[\bigcup_n X_n]=Z$, $\bar{\Omega}(X_n)$ is isometrically algebraically isomorphic with $Z$ and the result follows from the theorem in (2.1) above.

(2.3) **Remarks.** (i) In fact as is clear from Proposition (2.8) to be proved later that $\langle x_n\rangle\to\lim x_n$ for $\langle x_n\rangle\in J(X_n)^{LIM}$ is the desired quotient map in the above theorem.

(ii) It follows immediately from the above theorem that a reflexive Banach algebra $Z$ is isometrically algebraically isomorphic with $X^{**}/X$ with $X=J(X_n)$, $X_n=Z$ for every $n$.

Bellenot further deduces from ([2], Theorem 1.1 III) in ([2), Corollary 1.3) that if $Z$ is a separable Banach space, then there is an $X$ with a shrinking bimonotone $FDD$ so that $X^{**}/X$ is isometric to $Z$. This is possible because every finitely generated normed linear space is finite dimensional and, therefore, reflexive. But every finitely generated Banach algebra need not be reflexive as for instance, is the case with $C[0, 1]$. So we cannot use Theorem 2.2 above to have an analogue of the above quoted part of ([2], Cor. 1.3).

However, if we assume that $Z$ has an orthogonal basis $(z_n)$ then for each $n$, $X_n=[z_j: 1\leqq j\leqq n]$ is a subalgebra of $Z$ and thus there is a Banach algebra $X$ e.g. $J(X_n)$ with an orthogonal basis $\{e_j^n: 1\leqq j\leqq n, n\in N, (e_j^n)_j=z_j$ and $(e_j^n)_k=0, k\neq j\}$ such that $Z$ is isometrically algebraically isomorphic to $X^{**}/X$. This can be compared with ([13], Theorem 1) and ([17], Corollary 1). (See [15] for a survey of such results in Banach spaces.)

**(2.4)** We dualize the general $J$-sum of Banach spaces as follows. Let $\langle Y_n\rangle$ be a sequence of Banach spaces and let for each $n$, $\psi_n: Y_{n+1}\to Y_n$ be a norm-decreasing linear mapping. We denote $\psi_j\psi_{j+1}\cdots\psi_{n-1}$ by $\psi_j^n(j<n)$ and define the $J$-sum, $\tilde{J}(Y_n)$ of $Y_n$'s as follows. For $y=\langle y_n\rangle$, $y_n\in Y_n$, $P=\{p_1, \cdots, p_k\}\in\mathcal{P}$ let

$$\|y\tilde{\|}_P^2=\sum_{i=1}^{k-1}\|y_{p_i}-\psi_{p_i}^{p_{i+1}}y_{p_{i+1}}\|^2+\|y_{p_k}\|^2.$$

Let $2\|y\tilde{\|}^2=\sup_{P\in\mathcal{P}}\|y\tilde{\|}_P^2$. Then $\sqrt{2}\|y\tilde{\|}\geqq\sup_n\|y_n\|$. Further $\|\cdot\tilde{\|}$ is a norm on $\Phi(Y_n)$ and $\tilde{J}(Y_n)$ is defined to be the completion of $(\Phi(Y_n), \|\cdot\tilde{\|})$. Then $\langle Y_n\rangle$ can be thought of as a monotone decomposition of $\tilde{J}(Y_n)$ (the norm induced on $Y_n$ may not be equal to that of $Y_n$ but is equivalent to it) and

$$\tilde{J}(Y_n)^{LIM}=\{y=\langle y_n\rangle: \|y\tilde{\|}<\infty\}.$$

An example is provided by taking $Y_n=X_n^*$, $\psi_n=\phi_n^*$ where $\langle(X_n, \phi_n)\rangle$ is as in §1 of [2]. But we shall be concerned with the example: for $n\in N$, $Y_n=B(X_n, Z)$ and $\psi_n$ the restriction map where $Z$, $X_n$ are as in (2.1) above. $Y_n$ may also be thought of as the space of bounded linear mappings $T$ from $Z$ to itself with domain $D(T)$ equal to $X_n$. We follow the convention that for linear mappings $T$ and $S$ from $Z$ to itself, $T+S$ is the linear mapping with domain $D(T)\cap D(S)$ satisfying $(T+S)x=Tx+Sx$ for $x$ in $D(T)\cap D(S)$. Then for a sequence $\langle T_n\rangle$, $T_n\in Y_n$ we can also write $T_{n_1}+T_{n_2}$ instead of $T_{n_1}+\psi_{n_1}^{n_2}T_{n_2}$ for $n_1<n_2$.

We note that in the case $Z=R=X_n=Y_n$ for each $n$, $\tilde{J}(Y_n)=J(X_n)=J$, the James space $J$ and $J(X_n)^{LIM}=\tilde{J}(Y_n)^{LIM}$ is the bidual $J^{**}$.

(2.5) For $x=\langle x_n\rangle$, $x_n\in X_n$; $P=\{p_1, \cdots, p_k\}\in\mathcal{P}$, let

$$|||x|||_P=\frac{\|x\|_P}{\sqrt{k}}\text{; and}$$

let

$$\sqrt{2}\,|||x|||=\sup_{p\in\mathcal{P}}|||x|||_p.$$

Then $\sqrt{2}\,|||x|||\geqq\sup\|x_n\|$. Further $|||\cdot|||$ is a norm on $\Phi(X_n)$, we denote its completion by $J_T(X_n)$ and note that the inclusion map on $J(X_n)$ to $J_T(X_n)$ is (defined and is) norm-decreasing. Further, letting

$$J_T^{LIM}(X_n)=\{x=\langle x_n\rangle : |||x|||<\infty\},$$

inclusion map on $J(X_n)^{LIM}$ to $J_T^{LIM}(X_n)$ is norm-decreasing. In the setting of (2.4) above $|||\cdot|||^{\sim}$, $J_T^{\sim}$ and $\tilde{J}_T^{LIM}(Y_n)$ can be also defined in the obvious way. Let

$$\operatorname{Inv} B(J(X_n)^{LIM})=\{T\in B(J(X_n)^{LIM}) : TX_n\subset X_n \text{ for each } n\}$$

$$=\operatorname{Inv}_J B(J(X_n)^{LIM})=\{T\in B(J(X_n)^{LIM}) :$$

$$\text{for each } n,\ TX_n\subset X_n \text{ and } T(J(X_n))\subset J(X_n)\},$$

$$\operatorname{Inv}\tilde{J}(Y_n)^{LIM}=\{T=\langle T_n\rangle\in\tilde{J}(Y_n)^{LIM} : T_nX_n\subset X_n \text{ for each } n\},$$

and similarly for other such spaces.

(2.6) **Theorem.** (i) $\beta : \operatorname{Inv}\tilde{J}(Y_n)^{LIM}\to\operatorname{Inv}_J B(J(X_n)^{LIM})$ *is continuous.*
(ii) $\beta : \operatorname{Inv}\tilde{J}_T^{LIM}(Y_n)\to\operatorname{Inv}_J B(J_T^{LIM}(X_n))$ *is continuous.*
(iii) $\theta : \operatorname{Inv} B(J(X_n)^{LIM})\to\operatorname{Inv}\tilde{J}_T^{LIM}(Y_n)$ *is continuous.*
(iv) $\theta : \operatorname{Inv} B(J_T^{LIM}(X_n))\to\operatorname{Inv}\tilde{J}_T^{LIM}(Y_n)$ *is continuous.*

**Proof.** (i): Let $\langle T_n\rangle\in\operatorname{Inv}\tilde{J}(Y_n)^{LIM}$. Let $x=\langle x_n\rangle\in J(X_n)^{LIM}$. Then for each $n$, $y_n=T_nx_n\in X_n$. Let $y=\langle y_n\rangle$. For

$$p<m,\ y_p-y_m=T_px_p-T_mx_m=(T_p-T_m)x_p+T_m(x_p-x_m),$$

so,

$$\|y_p-y_m\|\leqq\|T_p-T_m\|\,\|x_p\|+\|T_m\|\cdot\|x_p-x_m\|$$

$$\leqq\|T_p-T_m\|\,\|x\|_J+\sqrt{2}\,\|\langle T_n\rangle\|^{\sim}\,\|x_p-x_m\|$$

and

$$\|y_m\|\leqq\|T_m\|\cdot\|x_m\|\leqq\sqrt{2}\,\|\langle T_n\rangle\|^{\sim}\,\|x_m\|.$$

Therefore, for any $P=\{p_1, \cdots, p_k\}\in\mathcal{P}$

$$\|y\|_P^2=\sum_{i=1}^{k-1}\|y_{p_i}-y_{p_{i+1}}\|^2+\|y_{p_k}\|^2$$

$$\leqq \sum_{i=1}^{k-1} (\|T_{p_i}-T_{p_{i+1}}\| \|x\|_J+\sqrt{2}\|\langle T_n\rangle\tilde{\|} \|x_{p_i}-x_{p_{i+1}}\|)^2$$

$$+\|\langle T_n\rangle\tilde{\|}^2\|x_{p_k}\|^2$$

$$\leqq\Big[\Big(\sum_{i=1}^{k-1}\|T_{p_i}-T_{p_{i+1}}\|^2\|x\|_J^2\Big)^{1/2}+\Big(\sum_{i=1}^{k-1}2\|\langle T_n\rangle\tilde{\|}^2\|x_{p_i}-x_{p_{i+1}}\|^2$$

$$+2\|\langle T_n\rangle\tilde{\|}^2\|x_{p_k}\|^2\Big)^{1/2}\Big]^2$$

$$\leqq(\|\langle T_n\rangle\tilde{\|}_P\|x\|_J+\sqrt{2}\|\langle T_n\rangle\tilde{\|} \|x\|_P)^2$$

$$\leqq(\sqrt{2}\|\langle T_n\rangle\tilde{\|} \|x\|_J+2\|\langle T_n\rangle\tilde{\|} \|x\|_J)^2.$$

$$=(2+\sqrt{2})^2\|\langle T_n\rangle\tilde{\|}^2 \|x\|_J^2.$$

Therefore, $\|y\|_J\leqq(\sqrt{2}+1)\|\langle T_n\rangle\tilde{\|} \|x\|_J$. Hence

$$\|\beta(\langle T_n\rangle)\|\leqq(\sqrt{2}+1)\|\langle T_n\rangle\tilde{\|} \text{ and } \beta(\langle T_n\rangle)J(X_n)\subset J(X_n).$$

(ii) Let

$$\langle T_n\rangle\in \operatorname{Inv}\tilde{J}_T^{LIM}(Y_n)$$

and

$$x=\langle x_n\rangle\in J_T^{LIM}(X_n).$$

Let

$$y_n=T_nx_n,\ n\in N, \text{ and } y=\langle y_n\rangle.$$

For any partition $P=\{p_1, \cdots, p_k\}\in\mathcal{P}$

$$\frac{\|y\|_P}{\sqrt{k}}\leqq\sqrt{2}\frac{\|\langle T_n\rangle\tilde{\|}_P}{\sqrt{k}}|||x|||+2|||\langle T_n\rangle\tilde{|||}\frac{\|x\|_P}{\sqrt{k}}.$$

So $|||y|||\leqq\sqrt{2}(|||\langle T_n\rangle\tilde{|||})\cdot|||x|||+2|||\langle T_n\rangle\tilde{|||}\cdot|||x|||$. Therefore,

$$|||\beta(\langle T_n\rangle)|||\leqq(\sqrt{2}+2)|||\langle T_n\rangle\tilde{|||}$$

and

$$\beta(\langle T_n\rangle)J_T(X_n)\subset J_T(X_n).$$

(iii) Let $T\in \operatorname{Inv} B(J(X_n)^{LIM})$ and $\langle T_n\rangle=\theta(T)$. Then by Proposition (1.7(i) $\|T_n\|\leqq\|T\|$ for each $n$. Let $\varepsilon>0$ and $P=\{p_1, \cdots, p_k\}\in\mathcal{P}$ be arbitrary. For $1\leqq i\leqq k$ there exists $x_{p_i}\in X_{p_i}$ such that $\|x_{p_i}\|\leqq1$; for $1\leqq i\leqq k-1$, $\|(T_{p_i}-T_{p_{i+1}})x_{p_i}\|^2\geqq\|T_{p_i}-T_{p_{i+1}}\|^2-\frac{\varepsilon}{k}$ and $\|T_{p_k}x_{p_k}\|^2\geqq\|T_{p_k}\|^2-\frac{\varepsilon}{k}$. Put $x_n=0$ for $n\neq p_i$, $1\leqq i\leqq k$. Let $x=\langle x_n\rangle$. Then $x\in J(X_n)^{LIM}$ and $\|x\|_J\leqq\sqrt{(2k)}$. Also

$$\begin{aligned}
&2\|T\|^2\cdot\|x\|_J^2\geqq 2\|Tx\|_J^2\geqq\|Tx\|_P^2\\
&=\sum_{i=1}^{k-1}\|T_{p_i}x_{p_i}-T_{p_{i+1}}x_{p_{i+1}}\|^2+\|T_{p_k}x_{p_k}\|^2\\
&=\sum_{i=1}^{k-1}\|(T_{p_i}-T_{p_{i+1}})x_{p_i}+T_{p_{i+1}}(x_{p_i}-x_{p_{i+1}})\|^2+\|T_{p_k}x_{p_k}\|^2\\
&\geqq\sum_{i=1}^{k-1}\Big(\frac{1}{2}\|(T_{p_i}-T_{p_{i+1}})x_{p_i}\|^2-\|T_{p_{i+1}}(x_{p_i}-x_{p_{i+1}})\|^2\Big)+\|T_{p_k}x_{p_k}\|^2\\
&\geqq\sum_{i=1}^{k-1}\frac{1}{2}\Big(\|T_{p_i}-T_{p_{i+1}}\|^2-\frac{\varepsilon}{k}\Big)-\sum_{i=1}^{k-1}\|T_{p_{i+1}}\|^2\|x_{p_i}-x_{p_{i+1}}\|^2+\|T_{p_k}\|^2-\frac{\varepsilon}{k}\\
&=\sum_{i=1}^{k-1}\frac{1}{2}\|T_{p_i}-T_{p_{i+1}}\|^2+\|T_{p_k}\|^2-\varepsilon-\sum_{i=1}^{k-1}\|T\|^2\|x_{p_i}-x_{p_{i+1}}\|^2\\
&\geqq\frac{1}{2}\|\langle T_n\rangle\tilde{\|}_P^2-\varepsilon-\|T\|^2\|x\|_P^2\geqq\frac{1}{2}\|\langle T_n\rangle\tilde{\|}_P^2-\varepsilon-2\|T\|^2\|x\|_J^2.
\end{aligned}$$

So

$$8k\|T\|^2\geqq 4\|T\|^2\|x\|_J^2\geqq\frac{1}{2}\|\langle T_n\rangle\tilde{\|}_P^2-\varepsilon.$$

Thus

$$\frac{\|\langle T_n\rangle\tilde{\|}_P^2}{k}\leqq\frac{2\varepsilon}{k}+16\|T\|^2.$$

Since $\varepsilon>0$ is arbitrary we have

$$\frac{\|\langle T_n\rangle\tilde{\|}_P}{\sqrt{k}}\leqq 4\|T\|.$$

Therefore, $|||\langle T_n\rangle\tilde{|||}\leqq 2\sqrt{2}\|T\|$ i.e. $|||\theta(T)\tilde{|||}\leqq 2\sqrt{2}\|T\|$.

(iv) Let $T\in\operatorname{Inv}B(J_T^{LIM}(X_n))$ and $\langle T_n\rangle=\theta(T)$. Then

$$\sup_n\|T_n\|\leqq|||T|||.$$

Let $\varepsilon>0$ and $P=\{p_1,\cdots,p_k\}\in P$ be arbitrary and let $x_{p_i}$, $1\leqq i\leqq k$, $x$ be as in the proof of (iii) above. Then $x\in J_T^{LIM}(X_n)$ and $|||x|||\leqq\sqrt{2}$. Also

$$2|||Tx|||^2\geqq\frac{\|Tx\|_P^2}{k}\geqq\frac{1}{2}\frac{\|\langle T_n\rangle\tilde{\|}_P^2}{k}-\frac{\varepsilon}{k}|||T|||^2\frac{\|x\|_P^2}{k}.$$

So

$$2|||T|||^2|||x|||^2\geqq\frac{1}{2}\frac{\|\langle T_n\rangle\tilde{\|}_P^2}{k}-\frac{\varepsilon}{k}-|||T|||^2\cdot 2|||x|||^2.$$

Thus

$$8|||T|||^2\geqq\frac{1}{2}\frac{\|\langle T_n\rangle\tilde{\|}_P^2}{k}-\frac{\varepsilon}{k}.$$

Since $\varepsilon \in 0$ is arbitrary, we have

$$8|||T|||^2 \geqq \frac{\|\langle T_n\rangle\tilde{\|}_P^2}{k}.$$

So

$$8|||T|||^2 \geqq |||\langle T_n\rangle\tilde{|||}^2.$$

So

$$|||\langle T_n\rangle\tilde{|||} \leqq 2\sqrt{2}\,|||T|||.$$

(2.7) We now give a few basic results for the spaces $J(X_n)$ and $J(X_n)^{LIM}$ which will be used in investigating spaces of multiplicative linear functionals and multipliers of algebras $J(X_n)$ and $J(X_n)^{LIM}$. We would like to note here that as far as we know even in the setting of $Z$ a Banach space (as in [2]) these results are new and can be thought of as generalisations of the motivating result of James [11] that $J^{**}$ is the linear span of $J$ and the constant sequence 1. Let $r \in N$. For a $t \in X_r$ let $t_{(r)} = \langle x_n\rangle$ such that $x_n = t$ for $n \geqq r$ and 0 otherwise. For a subset $S$ of $X_r$, let $S_{(r)}$ and $S^{(r)}$ denote the subspaces of $J(X_n)^{LIM}$ given by

$$S_{(r)} = \{t_{(r)} : t \in S\}$$

and

$$S^{(r)} = \{\langle \delta_{nr} t\rangle : t \in S\}.$$

Then the mapping $t \to t_{(r)}$ on $X_r$ onto $X_{r(r)}$ is an isometric algebraic isomorphism. For an $x = \langle x_n\rangle$, $x_n$ in $X_n$ and $r \in N$, let $x_{(r)} = x_{r(r)}$, and define $\bar{x}^{(r)}$ by $\bar{x}_n^{(r)} = x_n$ for $n \leqq r$, $\bar{x}_n^{(r)} = x_r$ for $n > r$. Then $\bar{x}^{(r)} = \sum_{k<r} \langle \delta_{nk} x_k\rangle + x_{(r)}$. As in the proof of ([2], Theorem 1.1 I] $\bar{x}^{(j)} \to x$ as $j \to \infty$ for $x \in J(X_n)^{LIM}$.

(2.8) **Proposition.** (a) $J(X_n)^{LIM} \subset \{\langle x_n\rangle : x_n \in X_n$ *for all* $n$, $\langle x_n\rangle$ *convergent in* $Z\}$.
(b) *The map* $\lambda$ *on* $J(X_n)^{LIM}$ *to* $Z$ *defined by* $\lambda(\langle x_n\rangle) = \lim x_n$ *is a homomorphism of norm one with range* $= [UX_n]$.

**Proof.** (a) Let $\langle x_n\rangle \in J(X_n)^{LIM}$. If it is not convergent then there exists an $\varepsilon > 0$ and a sequence $\langle n_j\rangle$ in $N$: $n_1 < n_2 < \cdots$ and $\|x_{n_{2j-1}} - x_{n_{2j}}\| \geqq \varepsilon$ for each $j$. Let $P_k = \{n_1, n_2, \cdots, n_{2k}\}$. Then $\|\langle x_n\rangle\|_{P_k}^2 \geqq k\varepsilon^2$ and therefore $\|\langle x_n\rangle\|_J = \infty$ in contradiction to the assumption that $\langle x_n\rangle \in J(X)^{LIM}$.
(b) It is clearly a homomorphism of norm one. Let $t \in [UX_n]$. Then there exists a sequence $\langle t_n\rangle$ with $t_n \in X_n$ for all $n$ such that $t_n \to t$.

We can inductively define a strictly increasing sequence $(m_j)$ in $N$ such that

$$\|t_n - t_m\| < \frac{1}{2^j} \text{ for all } n, m \geqq m_j.$$

Put

$$x_n=0, \quad n<m_1,$$

$$=t_{m_j} \quad \text{for } m_j \leqq n < m_{j+1} \quad (j=1, 2, \cdots)$$

Then $\langle x_n \rangle$ is in $J(X_n)^{LIM}$ and $\lim x_n = t$.

**(2.9) Propostion.** *If $X_n = X_r$ for $n \geqq r$, $X_{r-1} \subsetneqq X_r$ then $J(X_n)^{LIM}$ is the topological direct sum of $J(X_n)$ and $X_r$ in the sense that $J(X_n)^{LIM} = \{\langle y_n + z_n \rangle :$ $\langle y_n \rangle \in J(X_n)$ and $\langle z_n \rangle \in X_{r(r)}\}$.*

**Proof.** Let $\langle x_n \rangle \in J(X_n)^{LIM}$. Then by the above proposition there exists $t \in X_r$ such that $x_n \to t$. Let $z_n = t$ for $n \geqq r$ and $z_n = 0$ for $n < r$ and $y_n = x_n - z_n$ for all $n$. Then

$$\|\langle z_n \rangle\|_J = \|t\| \leqq \|\langle x_n \rangle\|_J$$

and

$$\|\langle y_n \rangle\|_J \leqq 2 \|\langle x_n \rangle\|_J .$$

We note, as in the proof of ([2], Theorem 1.1(I)) that $\sup\limits_{\substack{P \in \mathcal{P} \\ p_1 \leqq n}} \sum\limits_{i=1}^{k-1} \|x_{p(i)} - x_{p(i+1)}\|^2$ $\to 0$ as $n \to \infty$ because otherwise $\exists \varepsilon > 0$ and a sequence $(P^{(n)})$ in $\mathcal{P}$, $P^{(n)} = \{p_1^{(n)}, p_2^{(n)}, \cdots, p_{k_n}^{(n)}\}$ such that $p_1^{(n+1)} \geqq p_{k_n}^{(n)}$ and

$$\sum_{i=1}^{k_n - 1} \|x_{(p_i^{(n)})} - x_{(p_{i+1}^{(n)})}\|^2 \leqq \varepsilon \text{ for each } n .$$

Setting $P_{(n)} = \cup \{p^{(j)} : 1 \leqq j \leqq n\}$ we have

$$2\|\langle x_n \rangle\|_J^2 \geqq \sum_{j=1}^{n} \sum_{i=1}^{k_j - 1} \|x_{p_i^{(j)}} - x_{p_{i+1}^{(j)}}\|^2 \geqq n\varepsilon$$

for each $n$ which is not so.

Also for $n, m \geqq r$, $\|y_n - y_m\| = \|x_n - x_m\|$ and $\|y_n\| = \|x_n - t\|$ tends to zero as $n \to \infty$. So $y^{(n)} \to y$ in $J(X_n)^{LIM}$. Therefore, $y \in J(X_n)$.

**(2.10) Proposition.** *Let for each $n$, $S_n$ be a dense subset of $X_n$. Then the linear span of $\{S_{r(r)}, S_r^{(r)} : r \in N\}$ is dense in $J(X_n)^{LIM}$. In particular if each $X_n$ is separable, then $J(X_n)^{LIM}$ is separable.*

**Proof.** Let $x = \langle x_n \rangle \in J(X_n)^{LIM}$ and $\varepsilon \in 0$. Then for each $j$ in $N$ there exists $s_j \in S_j$ such that $\|x_j - s_j\| < \frac{\varepsilon}{2^j}$. Now for $j \in N$, $\|\bar{x}^{(j)} - (\sum\limits_{k<j} \langle \delta_{nk} s_k \rangle +$ $s_{j(j)})\|_J \leqq \sum\limits_{k \leqq j} \|x_k - s_k\| < \varepsilon$. Since $\bar{x}^{(j)} \to x$ as $j \to \infty$ we have the desired result.

**(2.11) Theorem.** (i) *$\Phi_{J(X_n)}$ can be identified with the disjoint topological sum $\bigcup\limits_{n \in N} \{n\} \times \Phi_{X_n}$ via $\bigcup\limits_{n \in N} \{\phi_n \circ p_n : \phi_n \in \Phi_{X_n}\}$.*

(ii) *Suppose that* $Z=[\cup X_n]$.

$\Phi_{J(X_n)^{LIM}}$ *can be identified with the disjoint union* $\bigcup_{n\in N}\{\phi_n\circ p_n:\phi_n\in\Phi_{X_n}\}$ $\cup\{\phi\circ\lambda:\phi\in\Phi_Z\}$ *in which* $\bigcup_{n\in N}\{\phi_n\circ p_n:\phi_n\in\Phi_{X_n}\}$ *is sequentially dense and for each* $n$, $\{\phi_n\circ p_n:\phi_n\in\Phi_{X_n}\}$ *is an open closed subset.*

**Proof.** (i) The proof follows directly from Proposition 1.5.

(ii) We first note that since for each $j\in N$, $p_j$ is continuous on $J(X_n)^{LIM}$ we have that for each $j\in N$ and $f\in\Phi_{X_j}$, $\phi=f\circ p_j$ is a multiplicative linear functional on $J(X_n)^{LIM}$. Let $\phi\in\Phi_{J(X_n)}$ such that $\phi(a)\neq 0$ for some $a\in J(X_n)$. Then by Proposition 1.5 $\phi|J(X_n)=f\circ p_j$ for some $j\in N$ and some $f\in\Phi_{X_j}$.

Now for $b\in J(X_n)^{LIM}$, $ab\in J(X_n)$, so

$$\phi(b)=\frac{\phi(ab)}{\phi(a)}=\frac{f\circ p_j(ab)}{f\circ p_j(a)}=\frac{f\circ p_j(a)f\circ p_j(b)}{f\circ p_j(a)}=f\circ p_j(b).$$

Thus $\phi=f\circ p_j$ on $J(X_n)^{LIM}$.

We now note that in view of Proposition (2.8) above if $\phi\in\Phi_Z$ then $\psi$ given by $\psi(\langle x_n\rangle)=\phi(\lim x_n)$ is an mlf on $J(X_n)^{LIM}$ which is zero on $J(X_n)$. Since $\cup X_n$ is dense in $Z$, $\phi(t)\neq 0$ for some $n\in N$ and some $t\in X_n$. Then $\psi(t_{\langle n\rangle})=\phi(t)\neq 0$.

On the other hand if $\psi$ is an mlf on $J(X_n)^{LIM}$ which is zero on $J(X_n)$, then we can define for each $n\in N$ a homomorphism $\phi_n: X_n\to R$ by $\phi_n(t)=\psi(t_{\langle n\rangle})$. Then $\phi_n=0$ or an mlf on $X_n$ and, therefore, for each $t$ in $X_n$ $\|\phi_n(t)\|\leqq\|t\|$. Let $n\geqq m$. Since $\psi=0$ on $J(X_n)$ we have for $t\in X_m$, $\phi_m(t)=\psi(t_{\langle m\rangle})=\phi_n(t)$ so that $\phi_n|X_m=\phi_m$. Let $a\in J(X_n)^{LIM}$ be such that $\psi(a)\neq 0$. Then $0\neq\psi(a)=\psi(\lim \bar{a}^{\langle n\rangle})=\lim\psi(\bar{a}^{\langle n\rangle})=\lim\phi_n(a_n)$ so $\phi_n\neq 0$ for all $n\geqq m$ for some $m\in N$. Thus we can define a non-zero homomorphism on $\cup X_n$ to $R$ by $\phi(t)=\phi_n(t)$ for $t\in X_n$ which extends uniquely to an mlf say $\phi$ on $Z$ because $\phi$ is continuous from [3, § 15, Proposition 3].

Now let $x\in J(X_n)^{LIM}$. Then by Proposition (2.8) $\lim x_n$ exists in $Z$ and $\psi(x)=\lim\psi(\bar{x}^{\langle n\rangle})=\lim\psi(x_{n\langle n\rangle})=\lim\phi_n(x_n)=\lim\phi(x_n)=\phi(\lim x_n)$. Thus as a set $\Phi_{J(X_n)^{LIM}}$ is the disjoint union $\bigcup_{n\in N}\{\phi_n\circ p_n:\phi_n\in\Phi_{X_n}\}\cup\{\phi\circ\lambda:\phi\in\Phi_Z\}$. Since $\forall\phi\in\Phi_Z$, $(\phi|X_n)\circ p_n\to\phi\circ\lambda$ in the weak* topology we conclude that $\bigcup_{n\in N}\{\phi_n\circ p_n:\phi_n\in\Phi_{X_n}\}$ is sequentially dense in $\Phi_{J(X_n)^{LIM}}$.

Now consider a $\psi_0=\phi_m\circ p_m$ and a finite subset $S$ of $J(X_n)^{LIM}$. There exists an $a_m\in X_m$ such that $\phi_m(a_m)\neq 0$. Put $a_n=0$, $n\neq m$, and let

$$S_1=SU\{\langle a_n\rangle\}.$$

Then for $0<\varepsilon<|\phi_m(a_m)|$,

$$V(\psi_0;S\cup\{\langle a_n\rangle\};\varepsilon)=\{\psi\in\Phi_{J(X_n)^{LIM}}:|\psi(x)-\psi_0(x)|<\varepsilon,\ x\in S_1\}$$

$$=\{\phi'_m\circ p_m:|\phi'_m(x_m)-\phi_m(x_m)|<\varepsilon,\ x\in S_1\}$$

$=V(\phi_m;\{x_m : x\in S_1\};\varepsilon)\circ p_m$.

So for each $n$, $\{\phi_n\circ p_n : \phi_n\in\Phi_{X_n}\}$ is open in $\Phi_{J(X_n)^{LIM}}$. Let us now consider $\psi_0=\phi\circ\lambda$ for some $\phi\in\Phi_Z$. Let $m\in N$. Since $\phi\neq 0$ on $Z=[\cup X_n]$, for some $j>m$ $\exists a_j\in X_j$: $\phi(a_j)\neq 0$. Put $\varepsilon=|\phi(a_j)/2|$. Put

$$\begin{aligned} a_n &= a_j \quad \text{for} \quad n\geqq j \\ &=0, \quad n<j. \end{aligned}$$

Then $\langle a_n\rangle\in J(X_n)^{LIM}$ and

$$V(\psi_0;\{\langle a_n\rangle\};\varepsilon)\cap\{\phi_m\circ p_m : \phi_m\in\Phi_{X_m}\}$$

is empty. Thus $\psi_0$ is not a limit point of $\{\phi_m\circ p_m : \phi_m\in\Phi_{X_m}\}$. Hence $\{\phi_m\circ p_m : \phi_m\in\Phi_{X_m}\}$ is closed in $\Phi_{J(X_n)^{LIM}}$. Let $\psi_0=\phi\circ\lambda$ for some $\phi\in\Phi_Z$. Then the family $\mathcal{F}=\{V(\psi_0;S;\varepsilon)\setminus\bigcup_{n\in F}\Phi_{X_n}\circ p_n : S\subset J(X_n)^{LIM}$, $S$ finite, $F\subset N$, $F$ finite, $\varepsilon>0\}$ forms a base of neighbourhoods at $\psi_0$. Now let $S$ be any finite subset of $J(X_n)^{LIM}$, $F$ any finite subset of $N$. For an arbitrary $\varepsilon>0$, there is a least $m_\varepsilon\in N$ (depending on $\varepsilon$ and $S$ both) such that $|\phi(b_j)-\phi(\lim b_n)|<\frac{\varepsilon}{2}\ \forall j\geqq m_\varepsilon$, $\forall b\in S$. Put $G_\varepsilon$ $(=G_{\varepsilon,F,S})=F\cup\{j : j<m_\varepsilon\}$. Now for $j\geqq m_\varepsilon$ and $\phi_j\in\Phi_{X_j}$ satisfying $|\phi_j(b_j)-\phi|X_j(b_j)|<\frac{\varepsilon}{2}$, we have

$$|\phi_j(b_j)-\phi(\lim b_n)|<\varepsilon.$$

On the other hand if for $j\geqq m_\varepsilon$, $\phi_j\in\Phi_{X_j}$ is such that $|\phi_j\circ p_j(b)-\phi\circ\lambda(b)|=|\phi_j(b_j)-\phi(\lim b_n)|<\varepsilon$ then $|\phi_j(b_j)-(\phi|X_j)(b_j)|<\frac{3\varepsilon}{2}$. Put for $\varepsilon>0$

$$W(\psi_0;S;G_\varepsilon;\varepsilon)=(V(\phi;\lim S;\varepsilon)\circ\lambda)\cup\Big(\bigcup_{j\in N\setminus G_\varepsilon}V(\phi|X_{j;\,p_j;\,S;\,\varepsilon})\circ p_j\Big).$$

Then using the fact that $G_{\varepsilon/2}\supset G_\varepsilon\supset G_{3\varepsilon/2}$ and above observations we have

$$\begin{aligned} &W(\psi_0;S;G_{\varepsilon/2};\varepsilon/2)\subset V(\psi_0;S;\varepsilon)\setminus\bigcup_{n\in G_{\varepsilon/2}}\{\phi_n\circ p_n : \phi_n\in\Phi_{X_n}\} \\ &\subset V(\psi_0;S;\varepsilon)\setminus\bigcup_{n\in G_\varepsilon}\{\phi_0\circ p_n : \phi_n\in\Phi_{X_n}\} \\ &\subset W(\psi_0;S;G_{3\varepsilon/2};3\varepsilon/2). \end{aligned}$$

So

$$W(\psi_0;S;G_{\varepsilon/2};\varepsilon/2)\supset V(\psi_0;S;\varepsilon/3)\setminus\bigcup_{n\in G_{\varepsilon/3}}\{\phi_n\circ p_n : \phi_n\in\Phi_{X_n}\}$$

Now $G_{\varepsilon/3}$ is a finite subset and $G_{\varepsilon/3}\supset G_\varepsilon\supset F$. But $V(\psi_0;S;\varepsilon/3)\setminus\bigcup_{n\in G_{\varepsilon/3}}\{\phi_n\circ p_n : \phi_n\in\Phi_{X_n}\}\in\mathcal{F}$, so $W(\psi_0;S;G_{\varepsilon/2};\varepsilon/2)$ is a neighbourhood of $\psi_0$ contained in $V(\psi_0;S;\varepsilon)\setminus\bigcup_{n\in F}\{\phi_n\circ p_n : \phi_n\in\Phi_{X_n}\}$. So $\{W(\psi_0;S;G_{\varepsilon,F,S};\varepsilon)$: $\varepsilon>0$, $F\subset N$, $F$ finite, $S\subset J(X_n)^{LIM}$, $S$ finite$\}$ is also a neighbourhood base for $\psi_0$.

(2.12) **Remarks.** (i) Suppose that $X_n$ is an ideal in $X_{n+1}$ for each $n$ and $[\cup X_n]=Z$. The technique used in the proof of Theorem 2.11(ii) to extend $f\circ p_j$ on $J(X_n)$ to $J(X_n)^{LIM}$ can also be used to show that if $\phi$ is an mlf on an ideal $A$ of a Banach algebra $B$ with $\phi(a)\neq 0$ for an $a\in A$, then $\phi(b)=\dfrac{\phi(ab)}{\phi(a)}$ gives the unique extension of $\phi$ to an mlf on $B$. Thus for a fixed $m$, given any $\phi_m\in\Phi_{X_m}$ with $\phi_m|X_{m-1}=0$, we can inductively define a unique sequence $\langle\phi_n\rangle_{n\geq m}$ with $\phi_n\in\Phi_{X_n}$, such that $\phi_n|X_{n-1}=\phi_{n-1}\forall n>m$ which in turn gives an mlf $\phi$ on the linear span of $X_n$ defined by $\phi(x)=\phi_n(x)$ if $x\in X_n$. Since $\cup X_n$ is dense in $Z$ this gives us a unique mlf $\phi$ on $Z$ such that $\phi|X_m=\phi_m$.

On the other hand if $\phi\in\Phi_Z$ then since $\cup U_n$ is dense in $Z$ there is a least $m$ such that $\phi|X_m\neq 0$ and therefore $\phi|X_m\in\Phi_{X_m}$. Thus $\phi|X_m$ can be used as $\phi_m$ above to give back $\phi$. Hence $\Phi_Z$ can be identified with $\bigcup_{n\in N}\Phi_{X_n}$ which as a set can further be identified with the disjoint union $\bigcup_{n\in N}\{n\}\times\{\phi_n\in\Phi_{X_n}:\phi_n=0 \text{ on } X_{n-1}\}$. If for some $n$, $X_n=X_{n-1}$ then the set $\{\phi_n\in\Phi_{X_n}:\phi_n=0 \text{ on } X_{n-1}\}$ is empty.

(ii) If for each $n$, $X_n$ has an orthogonal basis $\{e_j^n: j\in\Lambda_n\}$, $\Lambda_n\subset N$, $\Lambda_n\subset\Lambda_{n+1}$ and for $j\in\Lambda_n$, $e_j^n=e_j^{n+1}$, then $\Phi_{X_n}$ can be identified with $\Lambda_n$ and $\{\phi_n\in\Phi_{X_n}:\phi_n=0 \text{ on } X_{n-1}\}$ with $\Lambda_n\setminus\Lambda_{n-1}$ where $\Lambda_0=\phi$. So $\Phi_{J(X_n)}$ can be identified with $\bigcup_{n\in N}\{n\}\times\Lambda_n$. For $\phi_m=\phi_m\circ p_m\in\Phi_{J(X_n)}$ for some $m\in N$ with $\phi(e_j^m)\neq 0$, taking $\varepsilon=|\phi(e_j^m)|/2$ and $\langle a_n\rangle=\langle\delta_{nm}e_j^m\rangle$ we observe that the neighbourhood $V(\phi;\{\langle a_n\rangle\};\varepsilon)$ of $\phi$ is a singleton. This shows that $\Phi_{J(X_n)}$ in this case has the discrete topology. $\Phi_{J(X_n)^{LIM}}$ can be identified with $(\bigcup_{n\in N}\{n\}\times\Lambda_n)\cup(\bigcup_{n\in N}\Lambda_n)$, where as in (1.10) above for $k\in\Lambda_m$, $\phi_{m,k}(x)=\alpha_k^m$ and $\phi_k(x)=\lim_{m\to\infty}\alpha_k^m=\lim_{m\to\infty}\phi_{m,k}(x)$, $x=\langle x_n\rangle=\langle\sum_{j\in\Lambda_n}\alpha_j^n e_j^n\rangle$. In fact it is this result which generalizes Proposition 2.7 of [1] in the first place. In case each $\Lambda_n$ is finite, $\{e_k^m: k\in\Lambda_m, m\in N\}$ forms an orthogonal basis for $J(X_n)$ where $e_k^m$ is identified with $\langle\delta_{nm}e_k^m\rangle$ and $e_k^m$ is placed before $e_{k'}^{m'}$ if $m<m'$ or $m=m'$ together with $k<k'$. Further in this case if for some $r$ $X_n=X_r$ for all $n>r$ and $X_{r-1}\subsetneqq X_r$ $(X_0=\{0\})$ then in view of Proposition (2.9) $J(X_n)^{LIM}$ also has a basis given by $\{f_k^r: k\in\Lambda_r\}\cup\{e_k^m: K\in\Lambda_m, m\in N\}$ where for $k\in\Lambda_r$, $f_k^r=(e_k^r)_{(r)}$, the sequence with elements $e_k^r$ from $r^{th}$ place onwards and zero elsewhere; clearly, this is not an orthogonal basis.

We now come to multipliers of $J(X_n)$ and $J(X_n)^{LIM}$. As in Proposition in (1.10) above the multipliers on these spaces can be identified with certain spaces of functions $\alpha=\langle\langle\alpha_j^n\rangle_{j\in\Lambda_n}\rangle_{n\in N}$. Applying Theorem (2.6) above we have the following result which generalizes Theorem 3.1 of [1].

**(2.13) Theorem.** *Let $\alpha=\langle\langle\alpha_j^n\rangle_{j\in\Lambda_n}\rangle_{n\in N}$. If $\alpha$ gives a multiplier on $J(X_n)$ to $J(X_n)^{LIM}$ then for each $j\in\cup\{\Lambda_n: n\in N\}$, $\langle 0, \cdots, 0, \alpha_j^{n_j}, \alpha_j^{n_j+1}, \cdots\rangle$ is in $J^{**}$ where $n_j$ is the least $n$ such that $j\in\Lambda_n$. Further if $\cup\{\Lambda_n: n\in N\}$ is finite then this condition is sufficient for $\alpha$ to give a multiplier on $J(X_n)^{LIM}$ to itself.*

**Proof.** Let $j\in\cup\{\Lambda_n: n\in N\}$ and $n_j$ be the least $n$ such that $j\in\Lambda_n$. Let $x=\langle x_n\rangle$ be given by $x_n=0$, $n<n_j$, $x_n=e_j^n$, $n\geqq n_j$. Then $x\in J(X_n)^{LIM}$ and $\|x\|_J=\|e_j^{n_j}\|$. Now let $\alpha$ give a multiplier $T$ on $J(X_n)^{LIM}$ and write $\langle T_n\rangle=\theta(T)$. Then for $n_j\leqq n\leqq m$,

$$\|T_n x_n-T_m x_m\|^2=\|(\alpha_j^n-\alpha_j^m)e_j^{n_j}\|^2$$
$$=|\alpha_j^n-\alpha_j^m|^2\|e_j^{n_j}\|^2$$

and

$$\|T_n x_n\|^2=|\alpha_j^n|^2\|e_j^{n_j}\|^2.$$

Since $\langle T_n x_n\rangle=Tx\in J(X_n)^{LIM}$ we have the desired result. Now let $T_n e_j^n=\alpha_j^n e_j^n$ $(n\in N,\ j\in\Lambda_n)$ and $T=\beta(\langle T_n\rangle)$. If $\cup\{\Lambda_n: n\in N\}$ is finite say having $r$ elements then each base is unconditional and, therefore, under an equivalent norm, for $n<m$,

$$\|T_n-T_m\|^2=\sup_{j\in\Lambda_n}|\alpha_j^n-\alpha_j^m|^2$$
$$\leqq\sum_{j\in\Lambda_n}|\alpha_j^n-\alpha_j^m|^2.$$

Similarly $\|T_n\|^2\leqq\sum_{j\in\Lambda_n}|\alpha_j^n|^2$ for all $n\in N$. So

$$\|\langle T_n\rangle\tilde{\|}^2\leqq\sum_{1\leq j\leq r}\|\alpha_{(j)}\|_J^2,$$

where $\alpha_{(j)}=\langle 0, \cdots 0, \alpha_j^{n_j}, \alpha_j^{n_j+1}, \cdots\rangle$

Thus the result follows from Theorem 2.6(1) above. We would like to thank the referee for his critical remarks, useful comments and suggestions.

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