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ON THE INVARIANCE PRINCIPLE AND FUNCTIONAL LAW OF ITERATED LOGARITHM FOR NONERGODIC PROCESSES

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Abstract: Recently, nonergodic versions of several limit theorems for strictly stationary processes were given ([6], [7], [8]). The proofs were based on the fact that important properties of the stationary process which hold with respect to an invariant measure μ, are preserved with respect to its ergodic components. Using the same technique we shall prove two nonergodic versions of Heyde's functional CLT and log log law here.

1. Basic notions and the main results.

Let $(\Omega, \mathcal{A}, T, \mu)$ be a dynamical system where $(\Omega, \mathcal{A}, \mu)$ is a probability space with σ-algebra $\mathcal{A}$ of subsets of Ω, μ is a probability measure on the measure space $(\Omega, \mathcal{A})$ and the mapping $T:\Omega\rightarrow\Omega$ is bijective and bimeasurable, $\mu T^{-1}=\mu$. The collection $\mathcal{J}$ of all sets $A\in\mathcal{A}$ such that $A=TA$ is a σ-algebra. If $\mu(A)=0$ or $\mu(A)=1$ for each $A\in\mathcal{J}$, we say that μ is ergodic. A σ-algebra $\mathcal{M}\subset\mathcal{A}$ for which $\mathcal{M}\subset T^{-1}\mathcal{M}$ is called invariant (sometimes, an invariant σ-algebra is defined by the property $T^{-1}\mathcal{M}\subset\mathcal{M}$). Let us denote $\mathcal{M}_i=T^{-i}\mathcal{M}$, $i\in\boldsymbol{Z}$; $\mathcal{M}_{-\infty}=\bigcap_{i\in\boldsymbol{Z}}\mathcal{M}_i$ and $\mathcal{M}_\infty$ is the coarsest σ-algebra containing all $\mathcal{M}_i$, $i\in\boldsymbol{Z}$.

For a measurable function f, $(f\circ T^i)$ is a strictly stationary process. Let us suppose that f is square integrable and let us denote $X_i=f\circ T^i$, $S_n=\sum_{i=1}^n X_i$, $\sigma_n^2=E(S_n^2|\mathcal{J})$. Let $\varphi(t)=(2t\log\log t)^{1/2}$, $e<t<\infty$. For $\sigma_n^2>0$ a.s. we define

$$\theta_n(t)=\sigma_n^{-1}(S_k+(nt-k)X_{k+1}),\quad k\leqq nt\leqq k+1,\ k=0, 1, \cdots, n-1;$$

$$\eta_n(t)=(\varphi(\sigma_n^2))^{-1}(S_k+(nt-k)X_{k+1})\ \text{ if } \sigma_n^2>e,$$
$$=0\quad\text{otherwise},\ k\leqq nt\leqq k+1,\ k=0, 1, \cdots, n-1.$$

We put $g=\sup\{n:\sigma_n^2\leqq e\}$. Notice that for μ ergodic, the functions σ_n and g are constant almost surely; in Theorem 1 we shall consider them as numbers.

The functions θ_n and η_n belong to $C[0, 1]$, the space of all continuous functions on $[0, 1]$. Let K be the set of all absolutely continuous functions

$x \in C[0, 1]$ such that $x(0)=0$ and $\int_0^1 [\dot{x}(t)]^2 dt \leqq 1$ and $\dot{x}$ denotes the derivative of x determined almost everywhere with respect to the Lebesgue measure. By W we denote the standard Brownian motion on $[0, 1]$ and $\xrightarrow{d}$ denote the convergence of $C[0, 1]$-valued r.v. in distribution as defined in [1].

For $p \in \boldsymbol{Z}$, let $x_p = E(X_{-p} | \mathcal{M}_0) - E(X_{-p} | \mathcal{M}_{-p})$.

Theorem 1. (C.C. Heyde). *Let the measure μ be ergodic and*

(1) $E(X_0 | \mathcal{M}_\infty) = X_0, \quad E(X_0 | \mathcal{M}_{-\infty}) = 0,$

(2) $\sum_{m=1}^{\infty} \{\limsup_{n\to\infty} E(\sum_{p=m}^{n} x_p)^2 + \limsup_{n\to\infty} E(\sum_{p=m}^{n} x_{-p})^2\} < \infty.$

Then there exists a limit $\sigma = \lim_{n\to\infty} \sigma_n / n^{1/2}$ and $0 \leqq \sigma < \infty$. If $\sigma > 0$ then $\theta_n \xrightarrow{d} W$. Also, $g < \infty$, $\{\eta_n : n > g\}$ is relatively compact, and the set of its limit points coincides with K.

The proof is given in [2], [3].

Here we shall prove nonergodic version of Theorem 1. In that case, σ_n and σ should be considered as $\mathcal{I}$-measurable functions.

Theorem 2. *Let the assumptions* (1) *and* (2) *of Theorem* 1 *hold. Then there exists a limit $\sigma = \lim_{n\to\infty} \sigma_n / n^{1/2}$ and $0 \leqq \sigma < \infty$. If $\sigma > 0$ a.s. then $\theta_n \xrightarrow{d} W$, the family $\{\eta_n ; n \in N\}$ is relatively compact a.s., and the set of its limit points coincides with K.*

Theorem 3. *Let the assumption* (1) *of Theorem* 1 *hold and let*

(3) $\sum_{m=1}^{\infty} \{\limsup_{n\to\infty} E((\sum_{p=m}^{n} x_p)^2 | \mathcal{I}) + \limsup_{n\to\infty} E((\sum_{p=m}^{n} x_{-p})^2 | \mathcal{I}) < \infty.$

Then the conclusion of Theorem 2 *holds.*

2. Some auxiliary results and the proofs of Theorems 2, 3.

Let the invariant σ-algebra $\mathcal{M}$ be given and let the square integrable function f satisfy the assumptions of Theorem 2 or 3. We shall show that without loss of generality we can suppose that the dynamical system $(\Omega, \mathcal{A}, T, \mu)$ has some useful properties, namely that there exists an ergodic decomposition of the measure μ. The proofs of results presented here can be found in [5], [6], [7], [8].

We say that a σ-algebra $\mathcal{E} \subset \mathcal{A}$ is separable if there exists an at most countable collection of sets generating $\mathcal{E}$ (see [4]). It can be easily shown that there exists a separable σ-algebra $\mathcal{M}^* \subset \mathcal{M}$ such that $E(X_i | T^j \mathcal{M}^*) = E(X_i | T^j \mathcal{M})$ a.s. for all $i, j \in \boldsymbol{Z}$. Hence, there exists a separable σ-algebra $\mathcal{E} \subset \mathcal{A}$ such that

$T^{-1}\mathcal{E}=\mathcal{E}$, $\mathcal{M}_i^*\subset\mathcal{E}$ and X_i, $E(X_i|T^j\mathcal{M})$ are $\mathcal{E}$-measurable, $i, j\in \boldsymbol{Z}$. We can thus suppose that $\mathcal{A}$ and $\mathcal{M}$ are separable. Then, there exists a function g such that $\mathcal{A}$ is generated by $\{g^{-1}(A): A\subset \boldsymbol{R}$ is an interval$\}$ (we can take $g=\sum_{j=1}^{\infty}\frac{1}{3^j}\chi_{A_j}$ where $\{A_1, A_2, \cdots\}$ generates $\mathcal{A}$).

Let us define $\psi: \Omega\to\boldsymbol{R}^{\boldsymbol{Z}}$ by $(\psi\omega)_i=g(T^i\omega)$, $i\in\boldsymbol{Z}$. Let S be a shift in $\boldsymbol{R}^{\boldsymbol{Z}}$, i.e. $(Sz)_i=z_{i+1}$. We have $\psi\circ T=S\circ\psi$ and $\psi^{-1}(\mathcal{B}^{\boldsymbol{Z}})=\mathcal{A}$, $\mathcal{B}^{\boldsymbol{Z}}$ being the Borel σ-algebra in $\boldsymbol{R}^{\boldsymbol{Z}}$. The probability measure $\nu=\mu\psi^{-1}$ is S-invariant, i.e. $\nu=\nu S^{-1}$. So, $(\boldsymbol{R}^{\boldsymbol{Z}}, \mathcal{B}^{\boldsymbol{Z}}, S, \nu)$ is a dynamical system. For an $\mathcal{A}$-measurable function h on Ω there exists a $\mathcal{B}^{\boldsymbol{Z}}$-measurable function $\bar{h}$ on $\boldsymbol{R}^{\boldsymbol{Z}}$, $h=\bar{h}\circ\psi$. If h is integrable and $\bar{\mathcal{E}}\subset\mathcal{B}^{\boldsymbol{Z}}$ is a σ-algebra, it is $E_\mu(h|\psi^{-1}\bar{\mathcal{E}})=E_\nu(\bar{h}|\bar{\mathcal{E}})\circ\psi$ a.s. (μ). For the σ-algebra $\bar{\mathcal{J}}=\{A\in\mathcal{B}^{\boldsymbol{Z}}: A=SA\}$ we have $\mathcal{J}=\psi^{-1}(\bar{\mathcal{J}})$. So, we can assume that $(\Omega, \mathcal{A}, T, \mu)=(\boldsymbol{R}^{\boldsymbol{Z}}, \mathcal{B}^{\boldsymbol{Z}}, S, \nu)$. Under this assumption there exists a family $(m_\omega; \omega\in\Omega)$ of regular conditional probabilities induced by $\mathcal{J}$ with respect to μ; moreover, $m_\omega T^{-1}=m_\omega$ and the measures m_ω are ergodic for almost all (μ) $\omega\in\Omega$ (see [5]).

Let $\mathcal{E}\subset\mathcal{A}$ be a σ-algebra and let ν be a probability measure on the measure space $(\Omega, \mathcal{A})$. $L^2(\mathcal{E}, \nu)$ denotes the Hilbert space of all $\mathcal{A}$-measurable functions g such that $\int g^2 d\nu<\infty$ and there exists a $\mathcal{E}$-measurable function h, $g=h$ a.s. (ν). Functions from $L^2(\mathcal{E}, \nu)$ which are equal almost surely (ν) are considered as equal.

The projection operator onto $L^2(\mathcal{M}_i, \mu)\ominus L^2(\mathcal{M}_{i-1}, \mu)$ is denoted by P_i, $i\in\boldsymbol{Z}$. We have $P_ih=E(h|\mathcal{M}_i)-E(h|\mathcal{M}_{i-1})$, $h\in L^2(\mathcal{A}, \mu)$; so, $x_p=P_0X_{-p}$. By P_i^ω we denote the projection operator onto $L^2(\mathcal{M}_i, m_\omega)\ominus L^2(\mathcal{M}_{i-1}, m_\omega)$.

Lemma 1. *Let $h\in L^2(\mathcal{M}_\infty, \mu)$. Then for almost all (μ) $\omega\in\Omega$ we have $h\in L^2(\mathcal{M}_\infty, m_\omega)$ and $E(h|\mathcal{M}_i)=E_{m_\omega}(h|\mathcal{M}_i)$* a.s. (m_ω), *$E(h^2|\mathcal{M}_i)=E_{m_\omega}(h^2|\mathcal{M}_i)$ a.s. (m_ω), $P_i^\omega h=P_ih$ a.s. (m_ω).*

The proof of Lemma 1 is given in [5].

Corollary. *Let $h\in L^2(\mathcal{A}, \mu)$ and $h, h\circ T, h\circ T^2, \cdots$ be a martingale difference sequence in $L^2(\mathcal{A}, \mu)$. Then for almost all (μ) $\omega\in\Omega$, $h, h\circ T, h\circ T^2, \cdots$ is a martingale difference sequence in $L^2(\mathcal{A}, m_\omega)$.*

Proof. The σ-algebra generated by $h\circ T^{-n}$, $n\geqq 0$, is separable and invariant. We can thus assume that $h=P_0h$. Following Lemma 1 we have $h=P_0^\omega h$ a.s. (m_ω), hence $(h\circ T^i)$ is a martingale difference sequence in $L^2(\mathcal{A}, m_\omega)$ for a.e. (μ) $\omega\in\Omega$.

Let h_n, $n=1, 2, \cdots$ be $C[0, 1]$-valued random variables. We can view them as functions $h_n(t, \omega)$ of t and ω. The variables h_n converge to W in distribution iff their finite-dimensional distributions converge to those of W and the sequence (h_n) is tight. The first condition can be expressed by the use of characteristic functions:

(i) For each finite k, each collection $0 \leqq t_1 < \cdots < t_k \leqq 1$, and $(s_1, \cdots, s_k) \in \boldsymbol{R}^k$ it is

$$\int \exp(i \sum_{j=1}^{k} s_j \cdot h_n(t_j, \cdot)) d\mu \xrightarrow[n\to\infty]{} \exp\Big(-\frac{1}{2}(t_1 \cdot (\sum_{j=1}^{k} s_j)^2 + (t_2 - t_1) \cdot (\sum_{j=2}^{k} s_j)^2 + \cdots)\Big).$$

The second condition can be expressed by (ii), (iii), compare [2], as follows:

(ii) $\sup_n \mu(|h_n(0, \cdot)| > \lambda) \xrightarrow[\lambda\to\infty]{} 0.$

(iii) For each $\varepsilon > 0$, $\sup_n \mu(\sup_{|s-t|<\delta} |h_n(s, \cdot) - h_n(t, \cdot)| > \varepsilon) \xrightarrow[\delta\downarrow 0]{} 0.$

The convergence of h_n is related to the measure μ; in the case described above we say that h_n converge in distribution with respect to μ.

Lemma 2. *Let for almost all* (μ) $\omega \in \Omega$, h_n *converge to* W *in distribution with respect to* m_ω. *Then* h_n *converge to* W *in distribution with respect to* μ.

Proof. Condition (i) immediately follows from the fact that

$$\int \exp(i \cdot \sum_{j=1}^{k} s_j \cdot h_n(t_j, \cdot)) d\mu = \int\int \exp(i \cdot \sum_{j=1}^{k} s_j \cdot h_n(t_j, \cdot)) dm_\omega d\mu(\omega).$$

Let $\varepsilon > 0$ be fixed. For $\tau > 0$ we put

$$A(\lambda, \tau) = \mu\{\omega : \sup_n m_\omega(|h_n(0, \cdot)| > \lambda) < \tau\}$$

and

$$B(\delta, \tau) = \mu\{\omega : \sup_n m_\omega(\sup_{|s-t|<\delta} |h_n(s, \cdot) - h_n(t, \cdot)| > \varepsilon) < \tau\}.$$

For each n it holds $\mu(|h_n(0, \cdot)| > \lambda) \leqq \tau + 1 - A(\lambda, \tau)$ and

$$\mu(\sup_{|s-t|<\delta} |h_n(s, \cdot) - h_n(t, \cdot)| > \varepsilon) \leqq \tau + 1 - B(\delta, \tau).$$

We have $A(\lambda, \tau) \xrightarrow[\lambda\to\infty]{} 1$ and $B(\delta, \tau) \xrightarrow[\delta\downarrow 0]{} 1$ hence the conditions (ii), (iii) hold for μ, too.

Proof of Theorem 2. Following [2], pp. 141-142, there exist functions $g, h \in L^2(\mathcal{A}, \mu)$ such that $f = g + h - h \circ T$ and $(g \circ T^i)$ is a martingale difference sequence. In the ergodic case, the theorem can be derived from the existence of this decomposition (see [3], [2]). According to the Corollary to Lemma 1, the decomposition exists in a.e. (μ) probability space $(\Omega, \mathcal{A}, m_\omega)$. So, the functional CLT and log log law hold in a.e. (μ) probability space $(\Omega, \mathcal{A}, m_\omega)$. We get the invariance principle in $(\Omega, \mathcal{A}, \mu)$ by Lemma 2. The proof of the functional log log law is even easier: Let A be the set of all elements of Ω

for which $\{\eta_n : n \in N\}$ is relatively compact and has K as the set of limit points. We have $m_\omega(A)=1$ for a.e. (μ) $\omega \in \Omega$, so $\mu(A)=1$ which finishes the proof of Theorem 2.

Proof of Theorem 3. According to Lemma 1, the assumptions of Theorem 1 are fulfilled in almost all (μ) probability spaces $(\Omega, \mathcal{A}, m_\omega)$. Similarly as in the proof of Theorem 2 we get the desired result from its validity in spaces $(\Omega, \mathcal{A}, m_\omega)$.

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