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# CONGRUENCE THEOREMS FOR PROPER SEMI-RIEMANNIAN HYPERSURFACES IN A REAL SPACE FORM

By

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## § 0. Introduction.

For Riemannian hypersurfaces in a real space form, several authors investigated problems related to congruity under some assumptions. P. J. Ryan ([2], [3]) established a local congruence theorem in a real space form for Riemannian hypersurfaces whose shape operators have at most two mutually distinct constant eigenvalues. It is a natural question to consider this problem for semi-Riemannian hypersurfaces in a semi-Riemannian real space form. The main purpose of this paper is obtain a congruence theorem for proper semi-Riemannian hypersurfaces analogous to that of Ryan.

Throughout this paper, all manifolds are smooth and connected and geometrical objects are assumed to be smooth unless mentioned otherwise.

## § 1. Preliminaries.

In this section, we prepare the basic facts about submanifolds in a semi-Riemannian manifold and especially in a space of constant curvature. We call a non-degenerate and symmetric tensor field of type (0, 2) on a manifold $M$ a semi-Riemannian metric of $M$ and a manifold $M$ with a semi-Riemannian metric a semi-Riemannian manifold. Let $T_pM$ be the tangent space of $M$ at $p\in M$. A semi-Riemannian manifold $M^n$ isometrically imbedded into a semi-Riemannian manifold $N^m$ by an imbedding $f$ is called a semi-Riemannian submanifold of $N$. In the sequel, we shall identify $f(M)$ with $M$. Especially if $n=m-1$, then $M$ is called a semi-Riemannian hypersurface of $N$. Let $T_p^{\perp}M$ be the normal space of $M$ in $N$ at $p\in M$ and $\langle , \rangle$ be semi-Riemannian metrics of $N$ and $M$. The Levi-Civita connections on $N$ and $M$ are denoted by $\tilde{\nabla}$ and $\nabla$, respectively.

For the tangent vector fields $X$ and $Y$ of $M$, we have the Gauss formula:

$$\tilde{\nabla}_X Y=\nabla_X Y+h(X, Y)\,, \tag{1.1}$$

where $\nabla_X Y$ and $h(X, Y)$ are the tangential and the normal components of $\tilde{\nabla}_X Y$ respectively. It is easy to show that $h$ is symmetric. The tensor field $h$ is said to be the second fundamental form of the semi-Riemannian submanifold $M$.

If $X$ is a tangent vector field of $M$ and $E$ is a normal vector field of $M$, then we have the Weingarten formula:

$$\tilde{\nabla}_X E = -A_E X + D_X E, \tag{1.2}$$

where $-A_E X$ and $D_X E$ are the tangential and the normal components of $\tilde{\nabla}_X E$ respectively. It is easy to verify that $D$ is a connection of the normal bundle of $M$. We called $A$ the second fundamental tensor of the semi-Riemannian submanifold $M$. It follows that

$$\langle h(X, Y)E\rangle = \langle A_E X, Y\rangle \tag{1.3}$$

for any tangent vectors $X$ and $Y$ of $M$ and any normal vector $E$ of $M$.

Let $\tilde{R}$ and $R$ be the curvature tensors of $N$ and $M$, respectively. The equations of Gauss and Codazzi are given by

$$\langle \tilde{R}(X, Y)Z, W)=\langle R(X, Y)Z, W\rangle + \langle h(X, Z), h(Y, W)\rangle - \langle h(X, W), h(Y, Z)\rangle, \tag{1.4}$$

$$(\tilde{R}(X, Y)Z)^{\perp} = (\nabla'_X h)(Y, Z) - (\nabla'_Y h)(X, Z) \tag{1.5}$$

for any tangent vectors $X, Y, Z$ and $W$ of $M$, where $(\tilde{R}(X, Y)Z)^{\perp}$ is the normal component of $\tilde{R}(X, Y)Z$ and $(\nabla'_X h)(Y, Z) = D_X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$.

Assume that $N$ is of constant curvature $c$. Then the equation (1.5) can be rewritten as

$$(\nabla_X(A_E))Y = (\nabla_Y(A_E))X \tag{1.6}$$

for a parallel normal vector field $E$ of $M$. Especially, if $M$ is a semi-Riemannian hypersurface of $N$, the equation (1.4) can be rewritten as

$$R(X, Y) = cX\wedge Y + \varepsilon A_E X \wedge A_E Y, \tag{1.7}$$

where $X\wedge Y$ is defined by

$$(X\wedge Y)Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y$$

for any tangent vector $Z$ of $M$, $E$ is a unit normal vector of $M$ and $\varepsilon = \langle E, E\rangle$.

For a semi-Riemannian hypersurface $M$ of $N$, we shall call $\varepsilon = \langle E, E\rangle$ the sign of the semi-Riemannian hypersurface $M$, where $E$ is a unit normal vector of $M$.

## § 2. A congruence theorem.

Let $M_{(1)}$ and $M_{(2)}$ be semi-Riemannian submanifolds of a semi-Riemannian manifold $N$. $M_{(1)}$ and $M_{(2)}$ are said to be congruent if there exists an isometry $\tilde{\phi}$ of $N$ such that $\tilde{\phi}(M_{(1)})=M_{(2)}$. Let $R_\nu^n$ be an $n$-dimensional real vector space together with an inner product of signature $(\nu, n-\nu)$ given by

$$\langle x, x\rangle=-\sum_{i=1}^{\nu} x_i^2+\sum_{j=\nu+1}^{n} x_j^2,$$

where $x=(x_1, \cdots, x_n)$ is the natural coordinate of $R_\nu^n$. $R_\nu^n$ is called an $n$-dimensional semi-Euclidean space. We take any point $O$ of $M_{(1)}$ and any curve $\alpha$ in $M_{(1)}$ starting at $O$. We set $\bar{O}=\phi O$ and $\bar{\alpha}=\phi\alpha$. Let $P_\alpha$ (resp. $P_{\bar{\alpha}}$) be the parallel translation from $O$ (resp. $\bar{O}$) along the curve $\alpha$ (resp. $\bar{\alpha}$) with respect to the normal connection of $M_{(1)}$ (resp. $M_{(2)}$). Let $h_1$ (resp. $h_2$) be the second fundamental form of $M_{(1)}$ (resp. $M_{(2)}$). O'Neill proved the following theorem in [1].

**Theorem A.** *Let $M_{(1)}$ and $M_{(2)}$ be semi-Riemannian submanifolds of $R_\nu^n$ such that there exists an isometry $\phi: M_{(1)}\to M_{(2)}$. Then there is an isometry $\tilde{\phi}$ of $R_\nu^n$ such that $\tilde{\phi}|M_{(1)}=\phi$ if and only if, at a point $O$ of $M$, there exists a linear isometry $F_O: T_O^\perp M_{(1)}\to T_{\bar{O}}^\perp M_{(2)}$ with the following property* (*): *If $\alpha$ is any curve in $M_{(1)}$ starting at $O$, then, for each $s$ the linear isometry*

$$F_{\alpha(s)}=P_{\bar{\alpha}(s)}\circ F_O\circ P_{\alpha(s)}{}^{-1}: T_{\alpha(s)}^\perp(M_{(1)})\to T_{\bar{\alpha}(s)}^\perp(M_{(2)})$$

*preserves second fundamental form, that is,*

$$F_{\alpha(s)}h_1(v, w)=h_2(\phi_*v, \phi_*w)$$

*for any $v$ and $w\in T_{\alpha(s)}M_{(1)}$.*

We define semi-Riemannian manifolds $S_\nu^n(c)$ and $H_\nu^n(c)$ as follows:

$$S_\nu^n(c)=\{(x_1, \cdots, x_{n+1})\in R_\nu^{n+1}\,|\,-\sum_{i=1}^{\nu} x_i^2+\sum_{i=\nu+1}^{n+1} x_i^2=1/c\},\qquad (c>0),$$

$$H_\nu^n(c)=\{(x_1, \cdots, x_{n+1})\in R_{\nu+1}^{n+1}\,|\,-\sum_{i=1}^{\nu+1} x_i^2+\sum_{i=\nu+2}^{i=\nu+2} x_i^2=1/c\},\qquad (c<0).$$

These spaces are complete and of constant curvature $c$. $S_\nu^n(c)$ and $H_\nu^n(c)$ are called a semi-sphere and a semi-hyperbolic space, respectively. $S_\nu^n(c)$ is diffeomorphic to $R^\nu\times S^{n-\nu}$ and $H_\nu^n(c)$ is diffeomorphic to $S^\nu\times R^{n-\nu}$. We call these spaces and the semi-Euclidean space real space forms.

**Remark.** Among these spaces, $S_{n-1}^n(c)$ and $H_1^n(c)$ are not simply connected.

As a generalization of Theorem A, we prove the following theorem.

**Theorem 2.1.** *Let $M_{(1)}$ and $M_{(2)}$ be semi-Riemannian submanifolds of a real space form $N$ such that there exists an isometry $\phi: M_{(1)} \to M_{(2)}$. Then there is an isometry $\tilde{\phi}$ of $N$ such that $\tilde{\phi}|M_{(1)}=\phi$ if and only if, at a point $O$ of $M_{(1)}$, there exists a linear isometry $F_O: T_O^{\perp}M_{(1)} \to T_{\bar{O}}^{\perp}M_{(2)}$ with the property (*) in Theorem A, where $T_O^{\perp}M_{(1)}$ (resp. $T_{\bar{O}}^{\perp}M_{(2)}$) is the normal space of $M_{(1)}$ (resp. $M_{(2)}$) in $N$ at $O$ (resp. $\bar{O}$).*

**Proof.** The necessity is obvious and hence we shall prove the sufficiency.
Assume that there exists a linear isometry $F_O: T_{O(1)}^{\perp}M \to T_{\bar{O}}^{\perp}M_{(2)}$ with the property (*). To apply Theorem A, we imbed the real space form $N$ into a semi-Euclidean space $R_\nu^{m+1}$ as a hypersurface in the natural way. Let $\tilde{T}_p^{\perp}M_{(1)}$ (resp. $\tilde{T}_{q(2)}^{\perp}M$) be the normal space of $M_{(1)}$ (resp. $M_{(2)}$) in $R_\nu^{m+1}$ at $p \in M_{(1)}$ (resp. $q \in M_{(2)}$).

First we define a linear isometry $\tilde{F}_O: \tilde{T}_O^{\perp}M_{(1)} \to \tilde{T}_{\bar{O}}^{\perp}M_{(2)}$ as follows:

$$\tilde{F}_O(Y_1+b\bar{E}_O)=F_O(Y_1)+b\bar{E}_{\bar{O}} \tag{2.1}$$

for any $Y_1 \in T_O^{\perp}M_{(1)}$ and any real number $b$, where $\bar{E}$ is a unit normal vector field of $N$ in $R_\nu^{m+1}$.

We will prove that $\tilde{F}_O$ satisfies the property (*). Easily we can show

$$\tilde{P}_\alpha|_{T_O^{\perp}M_{(1)}}=P_\alpha \quad \text{and} \quad \tilde{P}_{\bar{\alpha}}|_{T_{\bar{O}}^{\perp}M_{(2)}}=P_{\bar{\alpha}}\,, \tag{2.2}$$

where $\tilde{P}_\alpha$ (resp. $\tilde{P}_{\bar{\alpha}}$) is the parallel translation with respect to the normal connection of $M_{(1)}$ (resp. $M_{(2)}$) in $R_\nu^{m+1}$.

Let $\bar{h}$ and $\tilde{h}_a$ be the second fundamental forms of $N$ and $M_{(a)}$ $(a=1, 2)$ in $R_\nu^{m+1}$, respectively. We set $\tilde{F}_{\alpha(s)}=\tilde{P}_{\bar{\alpha}(s)}\circ\tilde{F}_O\circ\tilde{P}_{\alpha(s)}{}^{-1}$. If follows from (2.1) and (2.2) that

$$\tilde{F}_{\alpha(s)}h_1(v, w)=F_{\alpha(s)}h_1(v, w) \tag{2.3}$$

for any $v$ and $w \in T_{\alpha(s)}M_{(1)}$. Since $F_{\alpha(s)}h_1(v, w)=h_2(\phi_*v, \phi_*w)$, we have

$$\tilde{F}_{\alpha(s)}h_1(v, w)=h_2(\phi_*v, \phi_*w)\,. \tag{2.4}$$

On the other hand, since $N$ is totally umbilic in $R_\nu^{m+1}$, we have

$$\tilde{F}_{\alpha(s)}\bar{h}(v, w)=\bar{h}(\phi_*v, \phi_*w)\,. \tag{2.5}$$

Therefore, it is clear from (2.4) and (2.5) that

$$\tilde{F}_{\alpha(s)}\tilde{h}_1(v, w)=\tilde{h}_2(\phi_*v, \phi_*w)\,, \tag{2.6}$$

which implies that $\tilde{F}_O$ satisfies the property (*).

By virtue of Theorem A, it follows that there exists a linear isometry $\tilde{\phi}$ of $R_\nu^{m+1}$ such that

$$\tilde{\phi}\,|\,M_{(1)}=\phi\,.$$

Moreover, $\tilde{\phi}\,|\,N$ is clearly an isometry of $N$. This completes the proof. Q.E.D.

**Corollary 2.2.** *Let $M_{(1)}$ and $M_{(2)}$ be totally geodesic semi-Riemannian submanifolds of a real space form $N$ such that there exists an isometry $\phi: M_{(1)}\to M_{(2)}$. Then $M_{(1)}$ and $M_{(2)}$ are congruent.*

**Corollary 2.3.** *Let $M_{(1)}$ and $M_{(2)}$ be semi-Riemannian hypersurfaces of a real space form $N$ such that there exists an isometry $\phi: M_{(1)}\to M_{(2)}$. Assume that $M_{(1)}$ and $M_{(2)}$ have no geodesic points. Then there is an isometry $\tilde{\phi}$ of $N$ such that $\tilde{\phi}\,|\,M_{(1)}=\phi$ if and only if, at each point $p$ of $M_{(1)}$, $\phi_* A^1_{E_1}=\pm A^2_{E_2}\phi_*$ holds for any unit normal vector $E_1$ (resp. $E_2$) of $M_{(1)}$ (resp. $M_{(2)}$) at $p$ (resp. $\phi(p)$), where $A^1$ (resp. $A^2$) is the second fundomental tensor of $M_{(1)}$ (resp. $M_{(2)}$).*

## § 3. Decomposition of proper semi-Riemannian hypersurfaces.

Let $M$ be a semi-Riemannian manifold with a semi-Riemannian metric $\langle\,,\rangle$, $TM$ the tangent bundle of $M$ and $T$ a distribution on $M$, that is, a subbundle of $TM$. If $\nabla_X Y\in T$ for any $X\in TM$ and any $Y\in\Gamma(T)$, then $T$ is said to be parallel, where $\Gamma(T)$ is the module of all cross sections of the subbundle $T$ of $TM$. If $\langle\,,\rangle\,|\,T_p$ $(p\in M)$ is non-degenerate, then $T$ is called a non-degenerate distribution. The following three results are stated in [5].

**Theorem B.** *Let $T$ be a non-degenerate parallel distribution on a semi-Riemannian manifold $M$. Let $M'$ be the maximal integral manifold of $T$ through a point of $M$. Then $M'$ is a totally geodesic submanifold of $M$. If $M$ is complete, then so is $M'$.*

**Theorem C.** *Let $T$ be a non-degenerate parallel distribution on a semi-Riemannian manifold $M$ and $T^\perp$ the orthogonal complement of $T$. Then $T^\perp$ is a non-degenerate parallel distribution. Let $M'$ (resp. $M''$) be the maximal integral manifold of $T$ (resp. $T^\perp$) through $p\in M$. Then there exists an isometry $\phi$ of a neighborhood of $p$ in $M$ into the product semi-Riemannian manifold $M'\times M''$.*

**Theorem D.** *Let $T$ be a non-degenerate parallel distribution on a complete and simply connected semi-Riemannian manifold $M$ and $T^\perp$ the orthogonal complement of $T$. Let $M'$ (resp. $M''$) be the maximal integral manifold of $T$ (resp. $T^\perp$) through $p\in M$. Then there exists an isometry $\phi$ of $M$ onto the product semi-Riemannian manifold $M'\times M''$. Moreover $M'$ and $M''$ are complete and simply* *connected.*

Now we prepare some lemmas on a tensor field of type (1, 1). At first we can prove the following lemma by using local frame fields.

**Lemma 3.1.** *Let $B_1$ and $B_2$ be real vector bundles on a manifold $M$. Let $\Phi$ be a continuous bundle homomorphism of $B_1$ into $B_2$. Set $k(p)=\dim(\mathrm{Ker}\,\Phi_p)$ and $r(p)=\dim(\mathrm{Im}\,\Phi_p)$ $(p\in M)$. Then $r$ is lower semi-continuous and $k$ is upper semi-continuous. Moreover, if $\Phi$ is smooth and $k$ or $r$ is constant on $M$, then $\mathrm{Ker}\,\Phi$ is a subbundle of $B_1$ and $\mathrm{Im}\,\Phi$ is a subbundle of $B_2$.*

Let $A$ be a tensor field of type (1, 1) on $M$. If $A$ can be expressed by a real diagonal matrix with respect to a linear frame at each point of $M$, then $A$ is said to be diagonalizable.

**Lemma 3.2.** *Let $A$ be a diagonalizable tensor field of type (1.1) on $M$. Assume that the number of mutually distinct eigenvalues of $A$ is exactly $s$ at each point of $M$. Then the eigenvalues $\lambda_1, \cdots, \lambda_s$ $(\lambda_1>\cdots>\lambda_s)$ are smooth on $M$ and $T_{\lambda_i}=\mathrm{Ker}\,(A-\lambda_i I)$ is a smooth subbundle of $TM$ $(1\leq i\leq s)$.*

**Proof.** Since $\lambda_1, \cdots, \lambda_s$ are continuous function on $M$ (see [2], for example), $(A-\lambda_1 I)$ and $(A-\lambda_2 I)\cdots(A-\lambda_s I)$ are continuous bundle homomorphisms of $TM$. The minimal polynomial of $A$ is $(t-\lambda_1)\cdots(t-\lambda_s)$. We have $T_{\lambda_1}=\mathrm{Im}((A-\lambda_2 I)\cdots(A-\lambda_s I))$. From Lemma 3.1, $\dim(T_{\lambda_1})_p$ is constant on $M$. Similarly $\lambda_2, \cdots, \lambda_s$ have constant multiplicities on $M$. Using the Cauchy's integral representation, we can show that $\lambda_1, \cdots, \lambda_s$ are smooth on $M$. Hence $(A-\lambda_i I)$ is smooth $(1\leq i\leq s)$. From Lemma 3.1, $T_{\lambda_i}$ is a smooth subbundle $(1\leq i\leq s)$.

Q. E. D.

**Lemma 3.3.** *Let $A$, $\lambda_i$ and $T_{\lambda_i}$ $(1\leq i\leq s)$ be as the above lemma. Assume that $(\nabla_X A)Y=(\nabla_Y A)X$ holds for any $X$ and $Y\in TM$. Then distributions $T_{\lambda_1}, \cdots, T_{\lambda_s}$ are involutive. Moreover if $\dim T_{\lambda_i}>1$, then $X\lambda_i=0$ holds for any $X\in T_{\lambda_i}$.*

**Proof.** Let $X$ and $Y$ be elements of $\Gamma(T_{\lambda_1})$, where $\Gamma(T_{\lambda_1})$ is the module of all the cross sections of the subbundle $T_{\lambda_1}$ of $TM$. From $(\nabla_X A)Y=(\nabla_Y A)X$ we have

$$(A-\lambda_1 I)[X, Y]=(X\lambda_1)Y-(Y\lambda_1)X. \tag{3.1}$$

By multiplying $(A-\lambda_2 I)\cdots(A-\lambda_s I)$ to both sides of (3.1), the left hand side is equal to zero since $(t-\lambda_1)\cdots(t-\lambda_s)$ is the minimal polynomial of $A$ and the right hand side is equal to

$$(\lambda_1-\lambda_2)\cdots(\lambda_1-\lambda_s)\{(X\lambda_1)Y-(Y\lambda_1)X\}.$$

Therefore both sides of (3.1) is equal to zero, that is,

$$[X, Y]\in T_{\lambda_1} \quad \text{and} \quad (X\lambda_1)Y-(Y\lambda_1)Y=0\,. \tag{3.2}$$

which means that $T_{\lambda_1}$ is involutive. Similarly $T_{\lambda_2}, \cdots, T_{\lambda_s}$ are involutive. Especially if $\dim T_{\lambda_1}>1$, then we have $X\lambda_1=0$ from (3.2) by choosing $X$ and $Y$ to be linearly independent. This completes the proof. Q. E. D.

If $A$ can be expressed by a real diagonal matrix with respect to an orthonormal frame at each point of the semi-Riemannian manifold $M$, then $A$ is said to be proper.

**Lemma 3.4.** *Let $A$, $\lambda_i$ and $T_{\lambda_i}$ $(1\leq i\leq s)$ be as in Lemma 3.3. Assume that $A$ is proper and $s=2$. Then $T_{\lambda_1}$ and $T_{\lambda_2}$ are non-degenerate and $TM=T_{\lambda_1}\oplus T_{\lambda_2}$ (orthogonal direct sum). Moreover, assume that $\lambda_1$ and $\lambda_2$ are constant over $M$. Then $T_{\lambda_1}$ and $T_{\lambda_2}$ are parallel.*

**Proof.** By Lemma 3.3, distributions $T_{\lambda_1}$ and $T_{\lambda_2}$ are smooth and involutive. For $X\in\Gamma(T_{\lambda_1})$ and $Y\in\Gamma(T_{\lambda_2})$, we have, from (1.6),

$$(A-\lambda_1 I)\nabla_Y X=(A-\lambda_2 I)\nabla_X Y\,. \tag{3.3}$$

Since $(t-\lambda_1)(t-\lambda_2)$ is the minimal polynomial of $A$, the left hand side of (3.3) is in $T_{\lambda_2}$ and the right hand side is in $T_{\lambda_1}$. Therefore both sides of (3.3) are equal to zero, that is,

$$\nabla_Y X\in\Gamma(T_{\lambda_1}) \quad \text{and} \quad \nabla_X Y\in\Gamma(T_{\lambda_2})\,. \tag{3.4}$$

If $Z\in T_{\lambda_1}$, then we have

$$\langle\nabla_Z X, Y\rangle+\langle X, \nabla_Z Y\rangle=\nabla_Z\langle X, Y\rangle=0\,. \tag{3.5}$$

On the other hand, it follows from (3.4) that

$$\nabla_Z Y\in T_{\lambda_2}, \text{ that is, } \langle X, \nabla_Z Y\rangle=0\,.$$

which together with (3.5) yields

$$\nabla_Z X\in(T_{\lambda_2})^{\perp}=T_{\lambda_1}\,. \tag{3.6}$$

From (3.4) and (3.6), we can see that the distribution $T_{\lambda_1}$ is parallel. Similarly $T_{\lambda_2}$ is parallel. Q. E. D.

Let $A$ be the second fundamental tensor of a semi-Riemannian hypersurface $M$ of a semi-Riemannian manifold $N$. $M$ is said to be proper if $A_E$ is proper for a unit normal vector $E$ at each point of $M$.

**Theorem 3.5.** *Let $M^n$ be a proper semi-Riemannian hypersurface of a real* *space form* $N^{n+1}(\bar{c})$ *whose sign is* $\varepsilon$. *Assume that there exist constants* $\lambda$ *and* $\mu$ $(\lambda\neq\mu)$ *and the set of all eigenvalues of* $A_E$ *is* $\{\lambda, \mu\}$ *for a unit normal vector* $E$ *at each point of* $M$. *Then the maximal integral manifolds of* $T_\lambda$ (*resp.* $T_\mu$) *are locally isometric to the real space form* $M(c_1)$ (*resp.* $M(c_2)$) *of constant curvature* $c_1=\bar{c}+\varepsilon\lambda^2$ (*resp.* $c_2=\bar{c}+\varepsilon\mu^2$) *and* $\bar{c}+\varepsilon\lambda\mu=0$ *holds.* $M$ *is locally isometric to* $M(c_1)\times M(c_2)$.

**Proof.** Fix a point $p$ of $M$. Let $E$ be a unit normal vector field of $M$ defined on a neighborhood $U$ of $p$. Let us restrict ourselves to the neighborhood $U$. By Lemma 3.4, $T_\lambda$ and $T_\mu$ are non-degenerate and parallel. Let $M'$ (resp. $M''$) be the maximal integral manifold of $T_\lambda$ (resp. $T_\mu$) through $p$. Since $M'$ and $M''$ are totally geodesic in $M$ from Theorem B, the curvature tensors of $M'$ and $M''$ are just the restriction of the curvature tensor $R$ of $M$. Hence we denote them by the same letter $R$ without confusion. For any $X$ and $Y\in T_\lambda$, $R(X, Y)=(\bar{c}+\varepsilon\lambda^2)(X\wedge Y)$ holds by virtue of (1.7). Thus $M'$ is of constant curvature $c_1$. Similarly $M''$ is of constant curvature $c_2$. Therefore it follows from Theorem C that there exists an isometry $\phi$ of a neighborhood $V$ of $p$ in $U$ into the product of real space forms $M(c_1)$ and $M(c_2)$. Take any unit vectors $X\in T_\lambda$ and $Y\in T_\mu$. Since the distribution $T_\mu$ is parallel, we have $R(X, Y)Y\in T_\mu$, which together with (1.7) implies $\bar{c}+\varepsilon\lambda\mu=0$. Q.E.D.

Therefore we obtain the following theorem from Theorem D.

**Theorem 3.6.** *Let* $M^n$ *be a complete and simply connected proper semi-Riemannian hypersurface of a real space form* $N^{n+1}(\bar{c})$ *whose sign is* $\varepsilon$. *Assume that there exist constants* $\lambda$ *and* $\mu$ $(\lambda\neq\mu)$ *and the set of all eigenvalues of* $A_E$ *is* $\{\lambda, \mu\}$ *for a unit normal vector* $E$ *at each point of* $M$. *Then* $M$ *is isometric to the product of* $M(c_1)$ *and* $M(c_2)$ *which are complete, simply connected and of constant curvature* $c_1=\bar{c}+\varepsilon\lambda^2$ *and* $c_2=\bar{c}+\varepsilon\mu^2$, *and* $\bar{c}+\varepsilon\lambda\mu=0$ *holds.*

## §4. Models of proper semi-Riemannian hypersurfaces.

In this section we introduce some complete proper semi-Riemannian hypersurfaces $M$ of real space forms $N$ which will serve as models in our discussion. Let $A$ be the second fundamental tensor of $M$. Let $E$ be a unit normal vector of $M$. We put $\varepsilon=\langle E, E\rangle$.

Let us consider the following hypersurfaces $M$ of real space forms $N$ (and take a connected component of $M$ if necessary).

(R) The hypersurfaces of $N=R_\nu^{n+1}$.

(R-1) $$R_\nu^n=\{x=(x_1, \cdots, x_{n+1})\in N \mid x_{n+1}=0\},$$

$$A_E=0\,,\qquad \varepsilon=1\,.$$

(R-2) $$S_\nu^n(c)=\{x\in N\mid -\sum_{i=1}^{\nu}x_i^2+\sum_{i=\nu+1}^{n+1}x_i^2=1/c\}\qquad (c>0)\,,$$

$$A_E=\pm\sqrt{c}I,\qquad \varepsilon=1.$$

(R-3) $$R_{\nu-1}^n=\{x\in N\mid x_1=0\},$$

$$A_E=0\,,\qquad \varepsilon=-1\,.$$

(R-4) $$H_{\nu-1}^n(c)=\{x\in N\mid -\sum_{i=1}^{\nu}x_i^2+\sum_{i=\nu+1}^{n+1}x_i^2=1/c\}\qquad (c<0),$$

$$A_E=\pm\sqrt{-c}I\,,\qquad \varepsilon=-1\,.$$

(R-5) $$R_\tau^r\times S_{\nu-\tau}^{n-r}(c)=\Big\{\{x\in N\mid -\sum_{i=\tau+1}^{\nu}x_i^2+\sum_{i=\nu+r-\tau+1}^{n+1}x_i^2=\frac{1}{c}\Big\}\quad (c>0)\,,$$

$$A_E=\pm(0_r\oplus\sqrt{c}I_{n-r})\,,\qquad \varepsilon=1\,.$$

(R-6) $$R_\tau^r\times H_{\nu-\tau-1}^{n-r}(c)=\Big\{x\in N\mid -\sum_{i=\tau+1}^{\nu}x_i^2+\sum_{i=\nu+r-\tau+1}^{n+1}x_i^2=\frac{1}{c}\Big\}\quad (c<0)\,,$$

$$A_E=\pm(0_r\oplus\sqrt{-c}I_{n-r})\,,\qquad \varepsilon=-1\,.$$

(S) The hypersurfaces of $N=S_\nu^{n+1}(\bar{c})$ ($\subset R_\nu^{n+2}$).

(S-1) $$S_\nu^n(c)=\{x\in N\mid x_{n+2}=\sqrt{1/\bar{c}-1/c}\}\qquad (\bar{c}\le c)\,,$$

$$A_E=\pm\sqrt{c-\bar{c}}I\,,\qquad \varepsilon=1\,.$$

(S-2) $$R_{\nu-1}^n=\{x\in N\mid x_1=x_{n+2}+t_0\}\qquad (t_0>0)\,,$$

$$A_E=\pm\sqrt{\bar{c}}I\,,\qquad \varepsilon=-1\,,$$

(S-3) $$S_{\nu-1}^n(c)=\{x\in N\mid x_1=\sqrt{1/c-1/\bar{c}}\}\qquad (0<c\le\bar{c})\,,$$

$$A_E=\pm\sqrt{\bar{c}-c}I\,,\qquad \varepsilon=-1\,.$$

(S-4) $$H_{\nu-1}^n(c)=\{x\in N\mid x_{n+2}=\sqrt{1/\bar{c}-1/c})\qquad (c<0)\,,$$

$$A_E=\pm\sqrt{\bar{c}-c}I\,,\qquad \varepsilon=-1\,.$$

(S-5) $$S_\tau^r(c_1)\times S_{\nu-\tau}^{n-r}(c_2)=\Big\{x\in N\mid -\sum_{i=1}^{\tau}x_i^2+\sum_{i=\nu+1}^{\nu+r-\tau+1}x_i^2=\frac{1}{c_1}\,,$$

$$-\sum_{i=\tau+1}^{\nu}x_i^2+\sum_{i=\nu+r-\tau+2}^{n+2}x_i^2=\frac{1}{c_2}\Big\}$$

$$(1/c_1+1/c_2=1/\bar{c},\ c_1>0,\ c_2>0)\,,$$

$$A_E=\pm(\sqrt{c_1-\bar{c}}I_r\oplus(-\sqrt{c_2-\bar{c}})I_{n-r})\,,\qquad \varepsilon=1\,.$$

(S-6) $$S^r_\tau(c_1)\times H^{n-r}_{\nu-\tau-1}(c_2)=\Big\{x\in N\,|\,-\sum_{i=1}^{\tau}x_i^2+\sum_{i=\nu+1}^{\nu+r-\tau+1}x_i^2=\frac{1}{c_1},\ -\sum_{i=\tau+1}^{\nu}x_i^2+\sum_{i=\nu+r-\tau+2}^{n+2}x_i^2==\frac{1}{c_2}\Big\}$$

$$(1/c_1+1/c_2=1/\bar{c},\ c_1>0,\ c_2<0),$$

$$A_E=\pm(\sqrt{\bar{c}-c_1}I_r\oplus\sqrt{\bar{c}-c_2}I_{n-r}),\qquad \varepsilon=-1.$$

(H) The hypersurfaces of $N=H^{n+1}_\nu(\bar{c})$ $(\subset R^{n+2}_{\nu+1})$.

(H-1) $$R^n_\nu=\{x\in N\,|\,x_1=x_{n+2}+t_0\}\qquad (t_0>0),$$

$$A_E=\pm\sqrt{-\bar{c}}I,\qquad \varepsilon=1.$$

(H-2) $$S^n_\nu(c)=\{x\in N\,|\,x_1=\sqrt{1/c-1/\bar{c}}\}\qquad (c>0),$$

$$A_E=\pm\sqrt{c-\bar{c}}I,\qquad \varepsilon=1.$$

(H-3) $$H^n_\nu(c)=\{x\in N\,|\,x_{n+2}=\sqrt{1/\bar{c}-1/c}\}\qquad (\bar{c}\leq c<0),$$

$$A_E=\pm\sqrt{c-\bar{c}}I,\qquad \varepsilon=1.$$

(H-4) $$H^n_{\nu-1}(c)=\{x\in N\,|\,x_1=\sqrt{1/c-1/\bar{c}}\}\qquad (c\leq\bar{c}),$$

$$A_E=\pm\sqrt{\bar{c}-c}I,\qquad \varepsilon=-1.$$

(H-5) $$S^r_\tau(c_1)\times H^{n-r}_{\nu-\tau}(c_2)=\Big\{x\in N\,|\,-\sum_{i=1}^{\tau}x_i^2+\sum_{i=\nu+2}^{\nu+r-\tau+2}x_i^2=\frac{1}{c_1},\ -\sum_{i=\tau+1}^{\nu+1}x_i^2+\sum_{i=\nu+r-\tau+3}^{n+2}x_i^2=\frac{1}{c_2}\Big\}$$

$$(1/c_1+1/c_2=1/\bar{c},\ c_1>0,\ c_2<0),$$

$$A_E=\pm(\sqrt{c_1-\bar{c}}I_r\oplus\sqrt{c_2-\bar{c}}I_{n-r}),\qquad \varepsilon=1.$$

(H-6) $$H^r_{\tau-1}(c_1)\times H^{n-r}_{\nu-\tau}(c_2)=\Big\{x\in N\,|\,-\sum_{i=1}^{\tau}x_i^2+\sum_{i=\nu+2}^{\nu+r-\tau+2}x_i^2=\frac{1}{c_1},\ -\sum_{i=\tau+1}^{\nu+1}x_i^2+\sum_{i=\nu+r-\tau+3}^{n+2}x_i^2=\frac{1}{c_2}\Big\}$$

$$(1/c_1+1/c_2=1/\bar{c},\ c_1<0,\ c_2<0),$$

$$A_E=\pm(\sqrt{\bar{c}-c_1}I_r\oplus(-\sqrt{\bar{c}-c_2})I_{n-r}),\qquad \varepsilon=-1.$$

**Remark.** We note that the each model satisfies either of the following (I) or (II):

(I) The hypersurface $M$ is totally umbilic in $N$, that is, $A_E=\lambda I$ holds, and

$M$ is of constant curvature $c$ $(=\bar{c}+\varepsilon\lambda^2)$.

(II) $A_E$ has exactly two mutually distinct eigenvalues $\lambda$ and $\mu$, $M$ is isometric to the product of real space forms of constant curvature $c_1(=\bar{c}+\varepsilon\lambda^2)$ and $c_2(=\bar{c}+\varepsilon\mu^2)$, and furthermore $\bar{c}+\varepsilon\lambda\mu=0$ holds.

Since the calculations of the second fundamental tensors of the models are much the same, we shall work on the case (S-2), for example. Namely, we consider the following situation:

$$R^n_{\nu-1}=M=\{p\in R^{n+2}_\nu \mid p_1=p_{n+2}+t_0\}\cap S^{n+1}_\nu(\bar{c}) \qquad (\subset R^{n+2}_\nu)\,,$$

where $p=(p_1, \cdots, p_{n+2})$. Let $T^\perp_p M$ (resp. $\tilde{T}^\perp_p M$) be the normal space of $M$ in $S^{n+1}_\nu(\bar{c})$ (resp. $R^{n+2}_\nu$) at $p\in M$. In the sequel, we shall identify $T_x R^{n+2}_\nu$ $(x\in R^{n+2}_\nu)$ with $R^{n+2}_\nu$ in the natural way. By straightforward calculations, we have

$$\tilde{T}^\perp_p M=\left\{x\in R^{n+2}_\nu \mid x_2=\frac{(x_1-x_{n-2})p_2}{t_0}, \cdots, x_{n+1}=\frac{(x_1-x_{n+2})p_{n+1}}{t_0}\right\},$$

$$T_p S^{n+1}_\nu(\bar{c})=\left\{x\in R^{n+2}_\nu \mid -\sum_{i=1}^{\nu} x_i p_i+\sum_{i=\nu+1}^{n+2} x_i p_i=0\right\},$$

where $x=(x_1, \cdots, x_{n+2})$ and $p=(p_1, \cdots, p_{n+2})$. Therefore it follows that

$$\begin{aligned} T^\perp_p M&=\tilde{T}^\perp_p M\cap T_p S^{n+1}_\nu(\bar{c}) \\ &=\left\{x\in R^{n+2}_\nu \mid x_1=\left(\frac{t_0{}^2\bar{c}}{1+t_0 c p_{n+2}}+1\right)x_{n+2},\ x_2=\frac{p_2 t_0\bar{c}x_{n+2}}{1+t_0 c p_{n+2}}, \right. \\ &\qquad \left. \cdots, x_{n+1}=\frac{p_{n+1}t_0\bar{c}x_{n+2}}{1+t_0 c p_{n+2}}\right\}. \end{aligned}$$

Consequently, we can construct a unit normal vector field $E$ of $M$ in $S^{n+1}_\nu(\bar{c})$ as follows:

$$E(p)=\pm(p_1\sqrt{\bar{c}}+1/(t_0\sqrt{\bar{c}}),\ p_2\sqrt{\bar{c}},\ \cdots,\ p_{n+1}\sqrt{\bar{c}},\ p_{n+2}\sqrt{\bar{c}}+1/(t_0\sqrt{\bar{c}})) \tag{4.1}$$

for each point $p$ of $M$.

Let $\tilde{\nabla}$ be the Levi-Civita connection on $R^{n+2}_\nu$. If $X\in TM$, then, from (1.1) and (1.2), we have

$$A_E X=-(\tilde{\nabla}_X E)^T\,, \tag{4.2}$$

where $(\tilde{\nabla}_X E)^T$ is the component tangential to $M$ of $\tilde{\nabla}_X E$. Let $\tilde{E}$ be a vector field on $R^{n+2}_\nu$ defined by

$$\tilde{E}(x)=\pm(x_1\sqrt{\bar{c}}+1/(t_0\sqrt{\bar{c}}),\ x_2\sqrt{\bar{c}},\ \cdots,\ x_{n+1}\sqrt{\bar{c}},\ x_{n+2}\sqrt{\bar{c}}+1/(t_0\sqrt{\bar{c}}))\,, \tag{4.3}$$

for each point $x$ of $R^{n+2}_\nu$. We note that the vector field $\tilde{E}$ is the extension of $E$. Let $X^k$, $\tilde{E}^k$ and $(\tilde{\nabla}_X E)^k$ $(1\leq k\leq n+2)$ be the components of $X$, $\tilde{E}$ and $\tilde{\nabla}_X E$, respectively. Then we have

(4.4) $$(\tilde{\nabla}_X E)^k = \sum_{i=1}^{n+2} X^i(\partial \tilde{E}^k/\partial x_i).$$

On the other hand, it follows from (4.3) that

(4.5) $$\partial \tilde{E}^k/\partial x_i = \pm\sqrt{\bar{c}}\,\delta_i^k,$$

where $\delta_i^j$ is the Kronecker's delta. Therefore, from (4.2), (4.4) and (4.5), we obtain $A_E = \pm\sqrt{\bar{c}}I$.

## § 5. Main theorem.

Now we prove the following main theorem.

**Theorem 5.1.** *Let $M^n$ be a proper semi-Riemannian hypersurface of a real space form $N^{n+1}(\bar{c})$. Assume that there exist constants $\lambda$ and $\mu$ $(\lambda \neq \mu)$ and the set of all eigenvalues of $A_E$ is $\{\lambda, \mu\}$ or $\{\lambda\}$ for a unit normal vector $E$ at each point of $M$. Then $M$ is locally congruent to one of the models in § 4.*

**Proof.** Since the number of mutually distinct eigenvalues of $A_E$ is at most two, the following two cases can be considered:

(I) $M$ is totally umbilic in $N^{n+1}(\bar{c})$.
(II) $A_E$ has exactly two mutually distinct eignvalues $\lambda$ and $\mu$ $(\lambda \neq \mu)$.

Case (I). Assume that $A_E = \lambda I$. From (1.7), $M$ is of constant curvature $c$ given by $\bar{c} + \varepsilon\lambda^2$, where $\varepsilon = \langle E, E\rangle$. Then, the following tweleve cases can be considered:

| | |
|---|---|
| (1) $\bar{c}=0$, $c=0$ and $\varepsilon=1$, | (2) $\bar{c}=0$, $c>0$ and $\varepsilon=1$, |
| (3) $\bar{c}=0$, $c=0$ and $\varepsilon=-1$, | (4) $\bar{c}=0$, $c<0$ and $\varepsilon=-1$, |
| (5) $0<\bar{c}\leq c$ and $\varepsilon=1$, | (6) $\bar{c}>0$, $c=0$ and $\varepsilon=-1$, |
| (7) $\bar{c}\geq c>0$ and $\varepsilon=-1$, | (8) $\bar{c}>0$, $c<0$ and $\varepsilon=-1$, |
| (9) $\bar{c}<0$, $c=0$ and $\varepsilon=1$, | (10) $\bar{c}<0$, $c>0$ and $\varepsilon=1$, |
| (11) $\bar{c}\leq c<0$ and $\varepsilon=1$, | (12) $0>\bar{c}\geq c$ and $\varepsilon=-1$. |

According to (1), (2), ⋯, (11) or (12), we can compare $M$ with (R-1), (R-2), (R-3), (R-4), (S-1), (S-2), (S-3), (S-4), (H-1), (H-2), (H-3) or (H-4), respectively.

Let $\bar{M}$ be the model corresponding to $M$. Let $\phi$ be a local isometry of $M$ into $\bar{M}$ and $\bar{A}$ the second fundamental tensor of $\bar{M}$. From (1.7), we have

$\lambda=\pm\sqrt{\varepsilon(c-\bar{c})}$. Hence, at each point $p$ of $M$, $\bar{A}_{\bar{E}}\phi_*=\pm\phi_*A_E$ holds for any unit normal vector $E$ (resp. $\bar{E}$) of $M$ (resp. $\bar{M}$). Therefore from Corollary 2.2 and Corollary 2.3, we can see that $M$ is locally congruent to $\bar{M}$.

Case (II). Assume that $A_E=\lambda I_r\oplus\mu I_{n-r}$ $(\lambda\neq\mu)$. By Theorem 3.5, $M$ is locally isometric to the product of real space form $M(c_1)$ of constant curvature $c_1(=\bar{c}+\varepsilon\lambda^2)$ and real space form $M(c_2)$ of constant curvature $c_2(=\bar{c}+\varepsilon\mu^2)$ and $\bar{c}+\varepsilon\lambda\mu=0$ holds. Moreover we can take a local isometry $\phi$ of $M$ into $M(c_1)\times M(c_2)$ such that $\phi_*T_\lambda=TM(c_1)$ around each point of $M$, where $\varepsilon=\langle E, E\rangle$ and $T_\lambda=\mathrm{Ker}\,(A_E-\lambda I)$. Then the following six cases can be considered:

(1)′ $\bar{c}=0$, $c_1=0$, $c_2>0$ and $\varepsilon=1$,

(2)′ $\bar{c}=0$, $c_1=0$, $c_2<0$ and $\varepsilon=-1$,

(3)′ $\bar{c}>0$, $c_1>0$, $c_2>0$ and $\varepsilon=1$,

(4)′ $\bar{c}>0$, $c_1>0$, $c_2<0$ and $\varepsilon=-1$,

(5)′ $\bar{c}<0$, $c_1>0$, $c_2<0$ and $\varepsilon=-1$,

(6)′ $\bar{c}<0$, $c_1<0$, $c_2<0$ and $\varepsilon=-1$,

According to (1)′, (2)′, ⋯ (5)′ or (6)′, we are able to compare $M$ with (R-5), (R-6), (S-5), (S-6), (H-5) or (H-6) respectively.

Let $\bar{M}$ be the model corresponding to $M$ and $\bar{A}$ the second fundamental tensor of $\bar{M}$. Let $\phi$ be a local isometry of $M$ into $\bar{M}$ such that $\phi_*T_\lambda=TM(c_1)$. From (1.7) we have $\lambda=\pm\sqrt{\varepsilon(c_1-\bar{c})}$ and $\mu=\pm\sqrt{\varepsilon(c_2-\bar{c})}$ $(\lambda\mu=-\varepsilon\bar{c})$. Hence, it follows that $\bar{A}_{\bar{E}}\phi_*=\pm\phi_*A_E$ holds for any unit normal vector $E$ of $M$ at $p$ and any unit normal vector $\bar{E}$ of $\bar{M}$ at $\phi(p)$. From Corollary 2.3, $M$ is locally congruent to $\bar{M}$. Q. E. D.

As a global version of this theorem, we have the following.

**Theorem 5.2.** *Let $M^n$ be a complete proper semi-Riemannian hypersurface of a real space form $N^{n+1}(\bar{c})$. Assume that there exist constants $\lambda$ and $\mu$ $(\lambda\neq\mu)$ and the set of all eigenvalues of $A_E$ is $\{\lambda, \mu\}$ or $\{\lambda\}$ for a unit normal vector $E$ at each point of $M$. Furthermore assume that $M$ is locally isometric to neither type of $S^n_{n-1}$, $H^n_1$, $S^k_{k-1}\times M'$ nor $H^k_1\times M'$ $(1\leq k\leq n-1)$, where $M'$ is a semi-Riemannian manifold. Then $M$ is congruent to one of the models in* § 4.

**Proof.** By the preceding Theorem 5.1, $M$ is locally congruent to one of the models in § 4. Let $\bar{M}$ be the model locally congruent to $M$. Let $\pi: \hat{M}\to M$ be the universal semi-Riemannian covering of $M$ and $f$ be the imbedding of $M$ into $N$. Then by for, $\hat{M}$ is immersed into $N$. By the way, since $\pi$ is a local isometry, $M$ and $\hat{M}$ (i. e., $\hat{M}$ and $\bar{M}$) are locally congruent. Let $\hat{A}$ (resp. $\bar{A}$) be the second fundamental tensor and $\hat{E}$ (resp. $\bar{E}$) be a unit normal vector field of $\hat{M}$ (resp. $\bar{M}$). Since $M$ is complete, so is $\hat{M}$.

At first, we consider the case where $\bar{M}$ is a real space form. Since $\bar{M}$ is neither $S^n_{n-1}$ nor $H^n_1$, there exists an isometry $\phi$ of $\hat{M}$ onto $\bar{M}$. It follows from $\hat{A}_{\hat{E}}=\pm\sqrt{\varepsilon(c-\bar{c})}I$ and $\bar{A}_{\bar{E}}=\pm\sqrt{\varepsilon(c-\bar{c})}I$ that $\bar{A}_{\bar{E}}\phi_*=\pm\phi_*\hat{A}_{\hat{E}}$. Then, from Corollary 2.2 and Corollary 2.3, $\hat{M}$ and $\bar{M}$ are congruent. Thus it follows from this fact that $\pi$ is injective and $\pi$ is an isometry of $\hat{M}$ onto $M$. Therefore, $M$ and $\bar{M}$ are congruent.

Next, we consider the case where $\bar{M}$ is a product of the space forms. It follows from Theorem 3.6 that there exists an isometry $\phi$ of $\hat{M}$ onto $M(c_1)\times M(c_2)$ such that $\phi_*T_\lambda=TM(c_1)$. Moreover, $M(c_1)$ and $M(c_2)$ are real space forms by the assumption. It follows from $\bar{A}_{\bar{E}}=\pm(\sqrt{\varepsilon(c_1-\bar{c})}I_r\oplus\pm\sqrt{\varepsilon(c_2-\bar{c})}I_{n-r})$ and $\hat{A}_{\hat{E}}=\pm(\sqrt{\varepsilon(c_1-\bar{c})}I_r\oplus\pm\sqrt{\varepsilon(c_2-\bar{c})}I_{n-r})$ that $\bar{A}_{\bar{E}}\phi_*=\pm\phi_*\hat{A}_{\hat{E}}$. Hence, from Corollary 2.3, $\hat{M}$ and $\bar{M}$ are congruent. Thus, $\pi$ is injective and $\pi$ is an isometry of $\hat{M}$ onto $M$. Therefore $M$ and $\bar{M}$ are congruent. Q.E.D.

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