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MARTINGALE DECOMPOSITIONS OF STATIONARY PROCESSES

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I. Introduction and main results.

1. Two propositions from ergodic theory.

Let $(\Omega, \mathcal{A}, T, \mu)$ be a dynamical system where $\mathcal{A}$ is a σ-algebra of subsets of Ω, T is a surjective and measure preserving map of Ω onto itself and μ is a T-invariant probability measure on $(\Omega, \mathcal{A})$, i.e. $\mu(A)=\mu(T^{-1}A)$ for each A from $\mathcal{A}$. By a σ-algebra we shall always mean a sub-σ-algebra of $\mathcal{A}$. Any set or partition will be considered to be measurable as well. For an at most countable and measurable partition ξ of Ω and a σ-algebra $\mathcal{C}$ let $H_\mu(\xi|\mathcal{C})$ denote the conditional entropy of ξ, i.e. $H_\mu(\xi|\mathcal{C})=E\Sigma_{A\in\xi}\phi(\mu(A|\mathcal{C}))$ where $\mu(A|\mathcal{C})$ means the conditional probability and $\phi(t)=-t\cdot\log(t)$ for $0<t<1$, 0 otherwise. If there is no danger of confusion we shall omit the referrence to μ and for $\mathcal{C}$ trivial $H(\xi)=H(\xi|\mathcal{C})$ is called the entropy of ξ. A σ-algebra $T^{-1}\mathcal{M}\subset\mathcal{M}$ will be called *invariant*. If the σ-algebras $T^i\mathcal{M}$ exist we denote by $\mathcal{M}_\infty=\vee_{i\in Z}T^i\mathcal{M}$ the smallest σ-algebra containing all $T^i\mathcal{M}$. By $\mathcal{M}_{-\infty}$ we denote $\bigcap_{i=1}^{\infty}T^{-i}\mathcal{M}$.

There exists a σ-algebra $\mathcal{P}$ with the following properties

i) $\mathcal{P}$ is invariant and for each invariant σ-algebra $\mathcal{M}\subset\mathcal{P}$ it is $\mathcal{M}=T^{-1}\mathcal{M}$ mod μ,
ii) $\mathcal{P}$ is the maximal σ-algebra with property (i),

We shall call $\mathcal{P}$ the Pinsker σ-algebra (the definition is equivalent to the usual one as given in [9], see section III).

Theorem 1. *Let $\mathcal{M}$ be an invariant σ-algebra, Q an invariant sub-σ-algebra of the Pinsker σ-algebra $\mathcal{P}$ and ξ be an $\mathcal{M}$-measurable partition of Ω with finite entropy. Then for* $k=1, 2, \cdots$

$$H(\xi|T^{-k}\mathcal{M})=H(\xi|Q\vee T^{-k}\mathcal{M})$$

and

$$H(\xi|\mathcal{M}_{-\infty})=H(\xi|Q\vee\mathcal{M}_{-\infty})$$

where $Q\vee\mathcal{M}$ *denotes the smallest* σ*-algebra containing* Q *and* $\mathcal{M}$. *If* $\mathcal{M}_\infty$ *exists then we can take* ξ $\mathcal{M}_\infty$*-measurable.*

As a special case of Theorem 1 we shall obtain Theorems 10 and 11 on p. 67 in [9].

Theorem 2. *Let* $\mathcal{M}$, Q *be given as in Theorem* 1. *The* σ*-algebras* $\mathcal{M}$ *and* Q *are then conditionally independent with respect to* $\mathcal{M}_{-\infty}$, $T^{-k}\mathcal{M}$, $k=1, 2, \cdots$. *For any* $\mathcal{M}$*-measurable and integrable function* f *it holds*

$$E(f\,|\,T^{-k}\mathcal{M})=E(f\,|\,Q\vee T^{-k}\mathcal{M}), \qquad k=1, 2, \cdots,$$

$$E(f\,|\,\mathcal{M}_{-\infty})=E(f\,|\,Q\vee\mathcal{M}_{-\infty}).$$

If $\mathcal{M}_\infty$ *exists then the conditional independence holds for it (in the place of* $\mathcal{M}$) *and we can take* f $\mathcal{M}_\infty$*-measurable.*

The invariant σ-algebra $\mathcal{M}$ is often defined by the relation $\mathcal{M}\subset T^{-1}\mathcal{M}$ (see [4], [6]). It is not difficult to get versions of Theorems 1,2 for that case. Notice that if $\mathcal{M}\subset T^{-1}\mathcal{M}$ then the σ-algebras $T^i\mathcal{M}$ all exist and we can consider T as a bijective and bimeasurable transformation.

2. Decomposition of stationary processes.

Let $L^p(\mathcal{A}, \mu)$, $p=1, 2$ denote the spaces of integrable, resp. square integrable functions. If there is no danger of confusion we shall write $L^p(\mathcal{A})$. For an invariant σ-algebra $\mathcal{M}$ we define

$$P_i f=E(f\,|\,T^{-i}\mathcal{M})-E(f\,|\,T^{-i-1}\mathcal{M})$$

whenever f is integrable and the σ-algebras exist (are sub-σ-algebras of $\mathcal{A}$). The operators P_i will be called *difference projection operators* generated by $\mathcal{M}$; notice that in $L^2(\mathcal{A})$, P_i is the projection operator onto $L^2(T^{-i}\mathcal{M})\ominus L^2(T^{-i-1}\mathcal{M})$. As $E(f\,|\,\mathcal{M})\circ T=E(f\circ T\,|\,T^{-1}\mathcal{M})$,

$$(P_i f)\circ T=P_{i+1}(f\circ T).$$

$(P_k f)\circ T^i$ is thus a reversed martingale difference sequence for any integrable f where $P_k f$ and $(P_k f)\circ T^i$ are defined. The σ-algebra $\sigma\{f\circ T^i\colon i\geqq 0\}$ generated by functions $f\circ T^i$, $i\geqq 0$ is invariant so each reversed martingale difference sequence can be expressed in this way. Let us suppose that T is invertible and bimeasurable.

Theorem 3. *Let* $\mathcal{M}$ *be an invariant* σ*-algebra,* P_i *the difference projection operators generated by* $\mathcal{M}$, Q *an invariant sub-*σ*-algebra of the Pinsker* σ*-algebra* $\mathcal{P}$ *and* $\bar{P}_i$ *the difference projection operators generated by the invariant* σ*-algebra*

$\mathcal{M}\vee Q$. *Then for each function* $f\in L^1(\mathcal{M})$ *it holds*

$$E(f\,|\,\mathcal{M}_{-\infty})=E(f\,|\,Q\wedge\mathcal{M}_{-\infty})$$

and

$$\bar{P}_i f=P_i f$$

whenever P_i *are defined. If the* σ*-algebra* $\mathcal{M}_\infty$ *exists we can take* $f\in L^1(\mathcal{M}_\infty)$ *and* $i\in Z$.

Example. The assumption that f is $\mathcal{M}_\infty$-measurable is important. Let $(X, \mathcal{F}, \lambda)$ be a probability space where $X=\{-1, 1\}$, $\mathcal{F}$ is the collection of all subsets of X and $\lambda(-1)=1/2=\lambda(1)$. $(\Omega_1, \mathcal{A}_1, T_1, \mu_1)$ is the dynamical system where $\Omega_1=X^Z$, $\mathcal{A}_1=\mathcal{F}^Z$, $\mu_1=\lambda^Z$ and T_1 is the shift on X^Z, i.e. $(T_1\omega)_i=\omega_{i+1}$. Let $T_2: X\to X$ be defined by $T_2(1)=-1$, $T_2(-1)=1$. $(X, \mathcal{F}, T_2, \lambda)$ is a dynamical system. Let us put $\Omega=\Omega_1\times X$, $\mathcal{A}=\mathcal{A}_1\otimes\mathcal{F}$, $\mu=\mu_1\otimes\lambda$ and for $(\omega', \omega'')\in\Omega$ let $T(\omega', \omega'')=(T_1\omega', T_2\omega'')$. By q_i, $i=1, 2$, we denote the projections of Ω to Ω_1, X and p_i is the projection of Ω_1 to the i-th coordinate, $i\in Z$. Let $f=(p_0\circ q_1)\cdot q_2$, $\mathcal{C}_1=q_1^{-1}(\mathcal{A}_1)$, $\mathcal{C}_2=q_2^{-1}(\mathcal{F})$, $\mathcal{M}_1=\sigma\{p_i: i\geqq 0\}$, $\mathcal{M}=q_1^{-1}(\mathcal{M}_1)$, P_i be the difference projection operators generated by $\mathcal{M}$ and $\bar{P}_i$ by $\mathcal{M}\vee\mathcal{C}_2$. We shall show that

$$f=\bar{P}_0 f\,,$$

but for each $i\in Z$

$$P_i f=0\,.$$

The function f is $\mathcal{M}\vee\mathcal{C}_2$-measurable; we shall prove that $E(f\,|\,T^{-1}(\mathcal{M}\vee\mathcal{C}_2))=0$. The function q_2 is $\mathcal{C}_2$-measurable so $E(f\,|\,\mathcal{C}_2\vee T^{-1}\mathcal{M})=q_2\cdot E(p_0\circ q_1\,|\,\mathcal{C}_2\vee T^{-1}\mathcal{M})$. The σ-algebras $\mathcal{C}_1$ and $\mathcal{C}_2$ are independent so $E(p_0\circ q_1\,|\,\mathcal{C}_2\vee T^{-1}\mathcal{M})=E(p_0\circ q_1\,|\,T^{-1}\mathcal{M})$. For $\mathcal{M}_1$ it holds $E_\mu(p_0\circ q_1\,|\,T^{-1}\mathcal{M})=E_{\mu_1}(p_0\,|\,T^{-1}\mathcal{M}_1)\circ q_1$. The σ-algebras $\sigma\{p_0\}$ and $T^{-1}\mathcal{M}_1$ are independent (w.r. to μ_1) hence $E_{\mu_1}(p_0\,|\,T^{-1}\mathcal{M}_1)=E_{\mu_1}p_0$. This proves the first equality.

The function $p_0\circ q_1$ is $\mathcal{M}_\infty$-measurable therefore $E(f\,|\,\mathcal{M}_\infty)=(p_0\circ q_1)\cdot E(q_2\,|\,\mathcal{M}_\infty)$. The σ-algebras $\mathcal{C}_2$ and $\mathcal{M}_\infty$ are independent so $E(q_2\,|\,\mathcal{M}_\infty)=Eq_2=0$ which proves the second equality.

If f is an integrable function and if there exists an invariant σ-algebra $\mathcal{M}\subset\mathcal{R}$ generating operators P_i such that

$$E(f\,|\,\mathcal{R})=\sum_{i\in Z}P_i f$$

where $\mathcal{R}=\bigcap_{n=1}^{\infty}T^{-n}\mathcal{A}$ is the Rohlin σ-algebra (compare [9]) we say that f is *difference decomposable*.

Theorem 4. *For each integrable function* f *there exist uniquely determined integrable functions* f' *and* f'' *such that* f' *is difference decomposable,* f'' *is measurable with respect to the Pinsker* σ*-algebra and*

$$f=f'+f''.$$

The process $(f\circ T^i)$ can be thus expressed as a sum of a process measurable with respect to the Pinsker σ-algebra and at most countably many reversed martingale difference sequences.

Corollary. *The set of all difference decomposable functions from $L^2(\mathcal{A})$ is equal to $L^2(\mathcal{A})\ominus L^2(\mathcal{P})$.*

3. Decomposition of invariant measure.

Let the σ-algebra $\mathcal{A}$ be separable (in the sense of [7]) i.e. there exists a countable collection $\mathcal{Q}$ such that $\mathcal{A}=\sigma\mathcal{Q}$, the smallest σ-algebra containing $\mathcal{Q}$. The set of all $A\in\mathcal{A}$ such that $A=TA$ forms a σ-algebra $\mathcal{I}$. If the measure μ of each set from $\mathcal{I}$ is 0 or 1 we say that μ is ergodic.

Let us suppose that there exists a family of regular conditional probabilities $(m_\omega; \omega\in\Omega)$ induced by $\mathcal{I}$ with respect to μ. Following [7] this family exists if $\Omega=R^Z$, $\mathcal{A}$ is the σ-algebra of Borel sets and T is the shift $(T\omega)_i=\omega_{i+1}$. Choosing a finite set instead of R gives the same result. Using the technique of conditional probabilities we can prove the following proposition known from Kryloff-Bogoliouboff theory (compare [8]).

Theorem 5. *For almost all (μ) $\omega\in\Omega$ the measure m_ω is T-invariant and ergodic.*

Let ξ be a finite partition of Ω and ξ^- be the σ-algebra generated by $T^{-k}\xi$, $k>0$. From Theorem 2 we can easily derive (compare [2]):

Theorem 6. $$H_\mu(\xi\,|\,\xi^-)=\int H_{m_\omega}(\xi\,|\,\xi^-)\,d\mu(\omega).$$

Remark. If f is an integrable function and if for each invariant σ-algebra $\mathcal{M}$ it is $E(f\,|\,\mathcal{M})=E(f\,|\,T^{-1}\mathcal{M})$ we say that f is absolutely undecomposable in $L^1(\mathcal{A}, \mu)$ (according to Theorem 4 these functions form $L^1(\mathcal{P}, \mu)$).

From Theorem 6 it follows that if f is absolutely undecomposable in $L^1(\mathcal{A}, \mu)$ then it is absolutely undecomposable in $L^1(\mathcal{A}, m_\omega)$ for almost all (μ) ω.

Let $\mathcal{M}$ be a separable and invariant σ-algebra. In the spaces $L^2(\mathcal{A}, \mu)$ and $L^2(\mathcal{A}, m_\omega)$ where m_ω is an ergodic probability measure, $\mathcal{M}$ generates difference projection operators P_i, P_i^ω.

Theorem 7. *Let $f\in L^1(\mathcal{A}, \mu)$ and P_i, P_i^ω be the difference projection operators defined above. Then for almost all (μ) $\omega\in\Omega$*

$$P_i f=P_i^\omega f \quad a.s.\ (m_\omega)$$

whenever P_i *exists and*

$$E_\mu(f \mid \mathcal{M}_{-\infty}) = E_{m_\omega}(f \mid \mathcal{M}_{-\infty}) \quad a.s.\ (m_\omega).$$

II. Applications to the martingale limit theory.

Let $f \in L^2(\mathcal{A})$. Let us denote $\rho(f) = \limsup_{n\to\infty} \|s_n(f)\|$ where $s_n(f) = (n)^{-1/2} \cdot \sum_{j=1}^n f \circ T^j$. We shall say that f is *finitely approximable* if there exists a sequence of functions $f_k \in L^2(\mathcal{A})$ such that $(f_k \circ T^j)$ is a reversed martingale difference sequence and $\rho(f - f_k) \to 0$. If f is finitely approximable then the measures $\mu s_n^{-1}(f)$ weakly converge to a probability measure. This-CLT was given by M. I. Gordin in 1969 for μ ergodic ([4]) and in [3], [11] and [12] for a general invariant measure μ. The hard problem is to show which function f is finitely approximable. From Theorem 2 we can easily deduce the following statement (see [12]).

Theorem 8. *If f is finitely approximable then there exist a difference decomposable function f' and f'' measurable with respect to the Pinsker σ-algebra such that $f = f' + f''$ and $\rho(f'') = 0$.*

A more detailed investigation of the relation of $L^2(\mathcal{P})$ and the space of difference decomposable functions was done in [12].

Theorem 7 gives a lot of opportunities to derive limit theorems for $(f \circ T^i)$ from their ergodic versions.

Let $f \in L^2(\mathcal{A}, \mu)$ and $(f \circ T^i)$ be a reversed martingale difference sequence. Let us define a mapping $\psi : \Omega \to R^N$ by $\psi(\omega)_i = f(T^i \omega)$. $(R^N, \mathcal{F}, S, \nu)$ where $\mathcal{F}$ is the Borel σ-algebra on R^N, $(Sx)_i = x_{i+1}$ and $\nu = \mu\psi^{-1}$, is a dynamical system. If p_i are the coordinate projections then $p_i = p_0 \circ S^i$ and the process (p_i) has the same distribution as $(f \circ T^i)$. Hence we can assume that $\mathcal{A}$ is separable and the family of regular conditional probabilities $(m_\omega ; \omega \in \Omega)$ induced by $\mathcal{J}$ exists. From Theorem 7 we get:

Corollary. *For almost all (μ) $\omega \in \Omega$, $(f \circ T^i)$ is a reversed martingale difference sequence in $L^2(\mathcal{A}, m_\omega)$.*

From the Corollary we obtain nonergodic versions of limit theorems which are known in the ergodic case: the central limit theorem, law of iterated logarithm, invariance principle (see [11], [15]).

In a similar way, nonergodic versions of limit theorems for other processes can be derived as well (ese [13] for the nonergodic version of Gordin's CLT for integrable random variables). It is not difficult to derive a limit theorem in $L^2(\mathcal{A}, \mu)$ if the convergence takes place in almost all (μ) $L^2(\mathcal{A}, m_\omega)$. In the case of Gordin's central limit theorem [5] the assumptions which are given in $L^1(\mathcal{A}, \mu)$ are in $L^1(\mathcal{A}, m_\omega)$ preserved (the $\mathcal{J}$-measurable functions become constants). On the other hand, however, there exists a function $f \in L^2(\mathcal{A}, \mu)$ such that $\mu s_n^{-1}(f)$ weakly converge to a probability measure but $m_\omega s_n^{-1}(f)$ does not converge for any ergodic component m_ω of μ ([14]).

III. Proofs.

First let us give some comments on the existence of the Pinsker σ-algebra. The Pinsker σ-algebra was originally defined as the σ-algebra generated by all finite measurable partitions ξ such that $H(\xi|\xi^-)=0$. In [9] it is proved that a finite partition ξ is measurable w.r. to so defined σ-algebra $\mathcal{P}$ if and only if $H(\xi|\xi^-)=0$. It holds that $H(\xi|\xi^-)=0$ if and only if $\xi \subset \xi^- \bmod \mu$ (this means that for each $A \in \xi$ there is a set $A' \in \xi^-$ such that $\mu(A \Delta A')=0$ where Δ means the symmetric difference). From this it follows that the Pinsker σ-algebra as defined in [9] is characterized by the properties (i) and (ii) given in section I.

Let us recall some other properties of entropy H.

Lemma 1. *Let $\mathcal{C}$, $\mathcal{D}$ be σ-algebras and ξ, ζ, η be at most countable partitions of Ω with finite entropy. Then*

a) *if $\mathcal{C} \subset \mathcal{D} \bmod \mu$ then $H(\xi|\mathcal{C}) \geqq H(\xi|\mathcal{D}) \geqq 0$,*

b) *if $\mathcal{C}_n$ are σ-algebras and $\mathcal{C}_n \downarrow \mathcal{C}$ or $\mathcal{C}_n \uparrow \mathcal{C}$ then*

$$\lim_{n\to\infty} H(\xi|\mathcal{C}_n)=H(\xi|\mathcal{C}),$$

c) *if $\zeta \geqq \xi$ (ζ is finer then ξ) then $\lim_{n\to\infty} H(\xi|\zeta^- \vee T^{-n}\eta^-)=H(\xi|\zeta^-)$ where $\mathcal{C} \vee \mathcal{D}$ denotes $\sigma(\mathcal{C} \cup \mathcal{D})$; if η is $\mathcal{P}$-measurable we thus have $H(\xi|\zeta^- \vee T^{-n}\eta^-)=H(\xi|\zeta^-)$ for each $n=1, 2, \cdots$*

d) *if η is $\mathcal{P}$-measurable, k is a positive integer and $\eta^{(k)}$ denotes the σ-algebra generated by $T^{-n\cdot k}\eta$, $n=1, 2, \cdots$, then $H(\eta|\eta^{(k)})=0$,*

e) *$H(\xi \vee \zeta|\mathcal{C})=H(\zeta|\mathcal{C})+H(\zeta|\hat{\xi} \vee \mathcal{C})$ where $\xi \vee \zeta$ means the common refinement of ξ, ζ and $\hat{\xi}$ means the σ-algebra generated by ξ (i.e. $H(\xi|\mathcal{C}) \leqq H(\eta|\mathcal{C})$ for $\xi \leqq \eta$),*

f) *$H(\xi \vee \zeta|\mathcal{C})=H(\xi|\mathcal{C})+H(\zeta|\mathcal{C})$ if and only if ξ and ζ are conditionally independent* with respect *to $\mathcal{C}$.*

The proofs of (a)-(e) can be found in standard textbooks dealing with ergodic theory (for example [1], [9]). We shall give the proof of (f) here.

For $\mathcal{C}$ trivial the proposition is given e.g. in [10]. We can suppose that for each $\omega \in \Omega$ there exists a probability measure m_ω on the σ-algebra generated by $\xi \vee \zeta$ sucht hat $m_\omega(\cdot)=\mu(\cdot|\mathcal{C})(\omega)$. We have $H_\mu(\xi|\mathcal{C})=\int H_{m_\omega}(\xi)d\mu(\omega)$ and similarly for ζ, $\xi \vee \zeta$. Let $H(\xi \vee \xi|\mathcal{C})=H(\xi|\mathcal{C})+H(\xi|\mathcal{C})$. Then $\int H_{m_\omega}(\xi \vee \zeta)d\mu(\omega)$ $=\int [H_{m_\omega}(\xi)+H_{m_\omega}(\zeta)]d\mu(\omega)$. As $0 \leqq H_{m_\omega}(\xi \vee \zeta) \leqq H_{m_\omega}(\xi)+H_{m_\omega}(\zeta)$ (see (a), (e)) the partitions are independent w.r. to m_ω. So, for $A\in\xi$ and $B\in\zeta$ we have $\mu(A\cap B|\mathcal{C})=\mu(A|\mathcal{C})\cdot\mu(B|\mathcal{C})$. Let ξ and ξ be conditionally independent with respect to $\mathcal{C}$. Then $H_{m_\omega}(\xi\vee\xi)=H_{m_\omega}(\xi)+H_{m_\omega}(\xi)$ hence $H(\xi<\xi|\mathcal{C})=H(\xi|\mathcal{C})+H(\xi|\mathcal{C})$.

Proof of Theorem 1. First, we shall suppose that ξ is $\mathcal{M}$-measurable and $\mathcal{M}$, Q are separable. Let $\xi=\xi_1\leqq\xi_2\leqq\cdots$, $\pi_1\leqq\pi_2\leqq\cdots$ (where $\eta\leqq\zeta$ means that ζ refines η) be at most countable partitions with finite entropies generating $\mathcal{M}$, resp. Q. By Lemma 1 (c), (b) we get

$$H(\xi|T^{-1}\mathcal{M})=H(\xi|Q\vee T^{-1}\mathcal{M}).$$

From (d) we obtain the statement for $T^{-k}\mathcal{M}$.

In the case of general σ-algebras $\mathcal{M}$, Q it is not difficult to find invariant separable σ-algebras $\mathcal{M}^*$ and Q^* such that $\mu(A|T^{-k}\mathcal{M})=\mu(A|T^{-k}\mathcal{M}^*)$, $\mu(A|Q\vee T^{-k}\mathcal{M})=\mu(A|Q^*\vee T^{-k}\mathcal{M}^*)$, $A\in\xi$. So, we can restrict ourselves to the case of separable $\mathcal{M}$ and Q.

We shall prove $H(\xi|\mathcal{M}_{-\infty})=H(\xi|Q\vee\mathcal{M}_{-\infty})$. Let us denote $\mathcal{D}=\bigcap_{k=0}^{\infty}T^{-k}(\mathcal{M}\vee Q)$. From Lemma 1 (a) it follows that $H(\xi|\mathcal{M}_{-\infty})\geqq H(\xi|Q\vee\mathcal{M}_{-\infty})\geqq H(\xi|\mathcal{D})$ and from (b) we have $\lim_{n\to\infty}H(\xi|T^{-n}\mathcal{M})=H(\xi|\mathcal{D})$ from which we obtain the desired equality.

Let all σ-algebras $T^i\mathcal{M}$, $i\in Z$, exist and ξ be $\mathcal{M}_\infty$-measurable. Let $\xi=\{A_1, A_2, \cdots\}$, partition ξ' be generated by $A_1, \cdots, A_n$ and η' be generated by A_j, $j\geqq n+1$. It is $\xi=\xi'\vee\eta'$ and for each $\varepsilon>0$ there exists n large enough so that $H(\eta')<\varepsilon$. For each $\delta>0$ there exists k, a $T^k\mathcal{M}$-measurable partition ξ'' and a partition η'' such that $\xi'\leqq\xi''\vee\eta''$, $\xi''\leqq\xi'\vee\eta''$, $H(\eta'')\leqq\delta$,

Following Lemma 1 it holds $0\leqq H(\xi|\mathcal{C})-H(\xi'|\mathcal{C})\leqq H(\eta')<\varepsilon$, $H(\xi''|\mathcal{C})\leqq H(\xi'|\mathcal{C})+\delta$, $H(\xi'|\mathcal{C})\leqq H(\xi''|\mathcal{C})+\delta$. So we have

$$|H(\xi|\mathcal{C})-H(\xi''|\mathcal{C})|<\varepsilon+\delta$$

for each σ-algebra $\mathcal{C}$.

The partition ξ'' is $T^k\mathcal{M}$-measurable hence $H(\xi''|T^i\mathcal{M})=H(\xi''|Q\vee T^i\mathcal{M})$, $i<k$, $H(\xi''|\mathcal{M}_{-\infty})=H(\xi''|Q\vee\mathcal{M}_{-\infty})$. The numbers ε, δ can be chosen arbitrarilly small so the equations hold for ξ as well.

Proof of Theorem 2. Let A be a set from $\mathcal{M}$ (resp. $\mathcal{M}_\infty$) and let B be a set from Q, ξ be the partition generated by A and η be the partition generated by B. Let $\mathcal{C}$ be one from the σ-algebras $T^{-k}\mathcal{M}$, $\mathcal{M}_{-\infty}$, $k=1, 2, \cdots$. By Lemma 1 (a) we have $H(\xi|\mathcal{C}\vee Q)\leqq H(\xi|\mathcal{C}\vee\hat{\eta})\leqq H(\xi|\mathcal{C})$ and by Theorem 1 $H(\xi|\mathcal{C}\vee Q)=H(\xi|\mathcal{C})$ so $H(\xi|\mathcal{C}\vee\hat{\eta})=H(\xi|\mathcal{C})$. From Lemma 1 (e), (f) it follows that ξ, η are conditionally independent w.r. to $\mathcal{C}$ so Q and $\mathcal{M}$ (resp. $\mathcal{M}_\infty$) are conditionally independent w.r. to $\mathcal{C}$. The second statement of the theorem now follows from [7].

Proof of Theorem 3. From Theorem 2 it follows that $\bar{P}_iP_if=P_if$,

$\bar{P}_i E(f\,|\,\mathcal{M}_{-\infty})=0$. Theorem 3 is an immediate consequence of this and of the mutual orthogonality of operators $\bar{P}_i$.

Proof of Theorem 4. We can restrict ourselves to $g=E(f\,|\,\mathcal{R})$. For $\mathcal{C}=\sigma\{g\circ T^k:\ k\geqq 0\}$, $\mathcal{C}_\infty$ is a sub-σ-algebra of $\mathcal{R}$ and is separable. Following the Rohlin-Sinai theorem (see [9]) there exists an invariant σ-algebra $\mathcal{M}$ such that $\mathcal{M}_{-\infty}$ is an invariant sub-σ-algebra of the Pinsker σ-algebra $\mathcal{P}$ and $\mathcal{M}_\infty=\mathcal{C}_\infty$, Hence the decomposition $f=f'+f''$ exists. From Theorem 2 we have $E(f'\,|\,\mathcal{P})=0$ so the decomposition is unique.

Proof of Theorem 5. For each $\mathcal{I}$-measurable function f we have $f\circ T=f$. Let $\mathcal{G}$ be a countable set algebra generating $\mathcal{A}$. For each $A\in\mathcal{G}$ it holds $\mu(A\,|\,\mathcal{I})=\mu(A\,|\,\mathcal{I})\circ T=\mu(T^{-1}A\,|\,\mathcal{I})$. As $m_\omega(A)=\mu(A\,|\,\mathcal{I})(\omega)$, the measures m_ω are T-invariant.

By Birkhoff's pointwise ergodic theorem we have $n^{-1}\cdot\sum_{j=1}^n 1_A\circ T^j\to\mu(A\,|\,\mathcal{I})$ a.s. (μ) and following [7] $\mu(A\,|\,\mathcal{I})=m_\omega(A)$ a.s. (m_ω). The averages $n^{-1}\sum_{j=1}^n 1_A\circ T^j$ thus converge to a constant a.s. (m_ω) for each A from $\mathcal{G}$, so m_ω is ergodic (see [1], p. 17).

Lemma 2. *Let $\mathcal{C}$ be a separable σ-algebra, $\mathcal{I}\subset\mathcal{C}$ mod μ and f be an integrable function. Then for almost all (μ) $\omega\in\Omega$ it holds*

$$E_\mu(f\,|\,\mathcal{C})=E_{m_\omega}(f\,|\,\mathcal{C})\quad \text{a.s. } (m_\omega)$$

Proof. Let $\mathcal{G}$ be a countable set algebra generating $\mathcal{C}$. For each $A\in\mathcal{G}$ it is $E(1_A\cdot E(f\,|\,\mathcal{C})\,|\,\mathcal{I})=E(1_A\cdot f\,|\,\mathcal{I})$, so $\int_A E(f\,|\,\mathcal{C})dm_\omega=\int_A f dm_\omega$.

Proof of Theorem 6. From the separability of $\mathcal{A}$ it follows that there exists a separable σ-algebra $\mathcal{I}'\subset\mathcal{I}$ such that $\mathcal{I}'=\mathcal{I}$ mod μ. From Theorem 2 and Lemma 2 it followst hat for each $A\in\xi$ $\int\phi(\mu(A\,|\,\xi^-))d\mu=\int\phi(\mu(A\,|\,\mathcal{I}'\vee\xi^-))d\mu=\iint\phi(m_\omega(A\,|\,\mathcal{I}'\vee\xi^-))dm_\omega d\mu(\omega)=\iint\phi(m_\omega(A\,|\,\xi^-))dm_\omega d\mu(\omega)$ from which we obtain the statement of Theorem 6.

Proof of Theorem 7. According to Theorem 3 we can suppose that $\mathcal{I}\subset\mathcal{M}$ mod μ. Now, the theorem follows from Lemma 2.

Remark. R. Yokoyama ([15]) has shown that for the proof of Theorem 7 the equation $E(f\,|\,\mathcal{M}_{-\infty})=E(f\,|\,\mathcal{I}\vee\mathcal{M}_{-\infty})$ where $f\in L^1(\mathcal{M}_\infty)$ is sufficient: Using properties of conditional expectations we can deduce that $E(E(f\,|\,T^i\mathcal{M})\,|\,\mathcal{I})=E(f\,|\,\mathcal{I})$ hence $E(1_A\cdot E(f\,|\,T^i\mathcal{M})\,|\,\mathcal{I})=E(1_A\cdot f\,|\,\mathcal{I})$ for each $A\in T^i\mathcal{M}$. Therefore $E_\mu(f\,|\,T^i\mathcal{M})=E_{m_\omega}(f\,|\,T^i\mathcal{M})$ a.s. (m_ω).

The proof of Theorem 8. Follows from the fact that for $f' \in L^2(\mathcal{A})$ difference decomposable and $f'' \in L^2(\mathcal{P})$, $f=f'+f''$, it holds $Es_n^2(f)=Es_n^2(f')+Es_n^2(f'')$ (see [12] for more details).

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