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CENTRAL LIMIT THEOREMS FOR INTEGRATED SQUARE ERROR OF NONPARAMETRIC DENSITY ESTIMATORS BASED ON ABSOLUTELY REGULAR RANDOM SEQUENCES

By

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1. Introduction.

Let $\{X_n, -\infty<n<\infty\}$ be a strictly stationary sequence of random variables which take values in R^p. Assume that the process satisfies the absolute regularity condition

$$\beta(n)=E\Big[\sup_{B\in\mathcal{M}_n^\infty}|P(B|\mathcal{M}_{-\infty}^0)-P(B)|\Big]\longrightarrow 0 \quad (\text{as } n\to\infty) \tag{1.1}$$

where $\mathcal{M}_a^b$ denotes the σ-algebra generated $X_a, \cdots, X_b$ $(a\leqq b)$. Suppose that the distribution function $F(x)$ of X_0 has a density function $f(x)$ and let $f_n(x)$ be a nonparametric estimator of $f(x)$ based on $X_1, \cdots, X_n$. In this paper, we focus our attentions on a global measure of performance of $f_n(x)$ which is called an integrated square error (ISE)

$$I_n=\int\{f_n(x)-f(x)\}^2dx\,. \tag{1.2}$$

Several authors had proved central limit theorems (C. L. T.) for I_n when $\{X_n\}$ is a sequence of independent and identically distributed (i. i. d.) random variables. (cf. Bickel and Rosenblatt (1973), Csörgo and Révész (1981) and Rosenblatt (1975)).

Most recently, Hall (1984) devised an elegant approach to this problem. He set up a central part of the problem in the context of degenerate U-statistics, applied the martingale theory to derive a central limit theorem for degenerate U-statistics with variable kernels and then proved the C. L. T. for I_n by combining this result and the Lindeberg-Feller theorem.

Turning back to dependent cases, techniques used in Bickel and Rosenblatt (1973), Csörgo and Révész (1981) and Rosenblatt (1975) seem to have some drawbacks in proving the C. L. T. for I_n when $\{X_n\}$ is an absolutely regular strictly stationary sequence and $f_n(x)$ is the Rosenblatt-Parzen type estimator.

In this paper, using Yoshihara's technique (1976) and modifying Hall's method we prove the C. L. T. for I_n in that case. Our result generalizes Hall's one (1984).

2. The main results.

In this and following sections, we assume that $\{X_n\}$ is a strictly stationary sequence of random vectors which are defined on a probability space $(\Omega, \mathcal{A}, P)$ and take values in R^p $(p \geqq 1)$ and which have the marginal probability density function $f(x)$ with respect to the Lebesgue measure.

Let $K(x)$ be a bounded, non-negative function on R^p such that

$$\int K(z)dz=1, \quad \int z_i K(z)dz=0 \quad \text{and} \quad \int z_i z_j K(z)dz=2\tau\delta_{ij} \tag{2.1}$$

for each i and j $(1 \leqq i,\ j \leqq p)$. Here, $z=(z_1, \cdots, z_p)$, $dz=dz_1 \cdots dz_p$, τ is a constant which does not depend on i and j and $\delta_{ii}=1$ and $\delta_{ij}=0$ $(i \neq j)$.

As usual, for a sample $(X_1, \cdots, X_n)$ we define the Rosenblatt-Parzen estimator $f_n(x)$ for $f(x)$ by

$$f_n(x)=\frac{1}{nh^p}\sum_{i=1}^{n} K\left(\frac{x-X_i}{h}\right) \tag{2.2}$$

where $h=h(n)$ is a bandwidth parameter such that as $n\to\infty$, $h\to 0$ and $nh^p\to\infty$ (p being the number of dimension).

For brevity, we use the following notations:

$$K(x, y)=K_n(x, y)=K\left(\frac{x-y}{h}\right), \tag{2.3}$$

$$\begin{aligned} H(x, y)&=H_n(x, y) \\ &=\int \{K(u, x)-EK(u, X_1)\}\{K(u, y)-EK(u, X_1)\}du, \end{aligned} \tag{2.4}$$

$$\tilde{H}(X_i, X_j)=\tilde{H}_n(X_i, X_j)=H(X_i, X_j)-EH(X_i, X_j), \tag{2.5}$$

$$S_n=\sum_{1\leq j<i\leq n} H(X_i, X_j). \tag{2.6}$$

We consider the following set of conditions:

Condition A. A(i). $\{X_n\}$ is an absolutely regular strictly stationary sequence with coefficient $\beta(n)=O(e^{-\gamma n})$, γ being some positive constant.

A(ii). $nh^p(\log n)^{-3}\to\infty$ and $h^p(\log n)^2\to 0$.

Next, we put

$$\tau^2\sigma_1^2=\lim_{n\to\infty}\frac{1}{nh^{2p+4}}E\left[\sum_{j=1}^{n}\int\{K(x,\ X_j)-EK(x,\ X_j)\}\{Ef_n(x)-f(x)\}dx\right]^2 \tag{2.7}$$

if exist and

$$\sigma_2^2=\lim_{n\to\infty}\frac{2}{n^2h^{3p}}ES_n^2. \tag{2.8}$$

(The existence of σ_2 is assured by Lemma 5, below.) Further, let

$$d(n)=\begin{cases} n^{1/2}h^{-2} & \text{if } nh^{p+4}\to\infty \\ nh^{p/2} & \text{if } nh^{p+4}\to 0 \\ nh^{(p+8)/2(p+4)} & \text{if } nh^{p+4}\to\lambda(0<\lambda<\infty). \end{cases} \tag{2.9}$$

Now, we state the main result.

Theorem. *Suppose Condition A holds. Then, $\sigma_2>0$ exists. Further, if σ_1 exists, then*

$$d(n)\{I_n-EI_n\}\xrightarrow{D}\begin{cases} 2\tau\sigma_1 Z & \text{if } nh^{p+4}\to\infty, \\ 2^{1/2}\sigma_2 Z & \text{if } nh^{p+4}\to 0, \\ (4\tau^2\sigma_1^2\lambda^{4/(p+4)}+2\sigma_2^2\lambda^{-p/(2+4)})^{1/2}Z & \text{if } nh^{p+4}\to\lambda(0<\lambda<\infty), \end{cases} \tag{2.10}$$

as $n\to\infty$, where Z has the standard normal distribution.

Remark 1. We consider the following set of conditions:

Condition B. B(i). $\{X_n\}$ is a strictly stationary m-dependent sequence.

B(ii). The joint density functions $f_{ij}(x,\ y)$ of $(X_i,\ X_j)$ exist for all i and j ($|i-j|\leqq m$) and are uniformly continuous on $R^p\times R^p$ for each i and j. $f(x)$ and $f_{ij}(x,\ y)$ have uniformly bounded second partial derivatives. Further,

$$\int f_{ij}(u,\ u)du<\infty$$

holds.

By the same method as the proof of Theorem, we can prove that the same conclusion as Theorem holds under Condition B instead of Condition A. In this case, σ_1 and σ_2 are given explicitely as follows:

$$\sigma_1^2=\bar{\sigma}_0^2+2\sum_{i=1}^{m}\bar{\sigma}_{0i}, \tag{2.11}$$

$$\sigma_2^2=\left\{\int f^2(x)dx\right\}\int\left\{\int K(u)K(u+v)du\right\}^2dv. \tag{2.12}$$

Here,

$$\bar{\sigma}_0^2=\int\{\Delta f(x)\}^2 f(x)dx-\left[\int\{\Delta f(x)\}f(x)dx\right]^2$$

$$\bar{\sigma}_{0i}=\iint\{\Delta f(x)\Delta f(y)\}f_{0i}(x, y)dxdy-\left[\int\{\Delta f(x)\}f(x)dx\right]^2$$

and $\Delta=\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}$ is the Laplacian. The conclusion is a generalization of Hall's result (1984).

Remark 2. For an absolutely regular strictly stationary sequence, in order to show that the conclusion of Theorem holds with $\sigma_1^2=\bar{\sigma}_0^2+2\sum_{i=1}^{\infty}\bar{\sigma}_{0i}$ and σ_2^2 defined by (2.12), it is needed to assume some additional conditions to Condition A. For example, additional conditions are as follows:

C(i) $\int|z_i z_j z_k|K(z)dz\leqq M<\infty$ for all i, j and k

C(ii) Second partial derivatives of $f(x)$ and of $f_{0j}(x)$ exist and uniformly bounded and moreover satisfy the Lipschitz condition of order one. Further, they belong to some ball in the space $L^1(R^d)$ or $L^1(R^d\times R^p)$.

But, we do not treat this problem in details.

The proof of Theorem is given in Section 3, but the proofs of all lemmas, needed there, are given in Appendix.

3. Proof of Theorem.

In this and following sections, we shall agree that c, with or without subscript, denotes some absolute constant and $\|\zeta\|_t=\{E|\zeta|^t\}^{1/t}$ for any $t\geqq 1$.

Firstly, we consider I_n-EI_n. Since

$$\begin{aligned}I_n&=\int\{f_n(x)-f(x)\}^2dx\\&=\int\{f_n(x)-Ef_n(x)\}^2dx+2\int\{f_n(x)-Ef_n(x)\}\{Ef_n(x)-f(x)\}dx\\&\quad+\int\{Ef_n(x)-f(x)\}^2dx,\end{aligned}\tag{3.1}$$

we decompose I_n-EI_n as follows:

$$\begin{aligned}I_n-EI_n&=\frac{1}{n^2h^{2p}}\sum_{1\leq j<i\leq n}\tilde{H}(X_i, X_j)\\&\quad+\frac{2}{nh^p}\sum_{j=1}^{n}\int\{K(x, X_j)-EK(x, X_j)\}\{Ef_n(x)-f(x)\}dx\end{aligned}$$

$$+\frac{1}{n^2h^{2p}}\sum_{j=1}^{n}\Big[\int\{K(x,\,X_j)-EK(x,\,X_j)\}^2dx$$
$$-\int E\{K(x,\,X_j)-EK(x,\,X_j)\}^2dx\Big]$$
$$=L_1+L_2+L_3,\quad \text{(say)}. \tag{3.2}$$

(cf. Hall (1984)).

Now, the main part of the proof of Theorem is broken up into proofs of the following four facts:

(i) the C.L.T. for L_1 (Proposition 3.1),
(ii) the C.L.T. for L_2 (Proposition 3.2),
(iii) the asymptotic uncorrelatedness of L_1 and L_2 (Proposition 3.3),
(iv) the asymptotic negligibility of L_3 (Proposition 3.4).

To prove these facts, let c_0 be a sufficiently large number such that

$$\beta(m)=o(n^{-8}) \tag{3.3}$$

where $m=[c_0\log n]$ and $[s]$ denotes the integral part of s. Further, let $r=r_n=[n^{1/4}]$ and $k=k_n=[n/(r+m)]$. Define a sequence $\{(a_i,\,b_i)\ (i=1,\,\cdots,\,k)\}$ of pairs of integers inductively as follows;

$$b_0=0,\quad a_i=b_{i-1}+m,\quad b_i=a_i+r-1\quad (i=1,\,2,\,\cdots,\,k). \tag{3.4}$$

Let $F_\alpha=\mathcal{M}_1^{a_\alpha-m}$ $(\alpha=1,\,\cdots,\,k)$. Put

$$T_\alpha=T_{n\alpha}=\sum_{i=a_\alpha}^{b_\alpha}\sum_{j=1}^{b_\alpha-m}H(X_i,\,X_j)\quad(\alpha=1,\,\cdots,\,k) \tag{3.5}$$

and

$$U_n=\sum_{\alpha=1}^{k}(T_{n\alpha}-ET_{n\alpha}). \tag{3.6}$$

Proposition 3.1. *If the conditions of Theorem are satisfied, then, as $n\to\infty$,*

$$\frac{1}{s_n}S_n\xrightarrow{D}N(0,\,1) \tag{3.7}$$

where $s_n^2=ES_n^2$.

Proof. Let $\bar{s}_n^2=\operatorname{Var}U_n$. Then, as $n\to\infty$,

$$\frac{1}{\bar{s}_n}\sum_{\alpha=1}^{k}E\{T_\alpha\,|\,F_\alpha\}\xrightarrow{P}0, \qquad \text{(Lemma 6)}$$

$$\frac{1}{\bar{s}_n^2}\sum_{\alpha=1}^{k}[E\{T_\alpha^2\,|\,F_\alpha\}-(E\{T_\alpha\,|\,F_\alpha\})^2]\xrightarrow{P}1 \qquad \text{(Lemma 6)}$$

and

$$\frac{1}{\bar{s}_n^4}\sum_{\alpha=1}^{k} ET_\alpha^4 \longrightarrow 0. \qquad \text{(Lemma 8)}$$

Hence, by Dvoretzky's theorem (1972)

$$\frac{1}{\bar{s}_n}U_n \xrightarrow{D} N(0, 1) \qquad (n\to\infty)$$

and so (3.7) is obtained, since

$$s_n^2=\bar{s}_n^2(1+o(1)), \qquad \text{(Lemma 7)} \quad (3.8)$$

$$\bar{s}_n^2\sim\frac{n^2}{2}EH^2(\tilde{X}_1, \tilde{X}_2)\sim\frac{n^2}{2}h^{3p}\sigma_2^2 \qquad \text{(Lemma 5)} \quad (3.9)$$

and

$$\frac{1}{\bar{s}_n^2}E(S_n-U_n)^2 \longrightarrow 0. \qquad \text{(Lemma 7)} \quad (3.10)$$

where $\{\tilde{X}_i\}$ is a sequence of independent and identically distributed random variables such that each $\tilde{X}_i$ has the same distribution as that of X_i.

Proposition 3.2. *Let* $\bar{L}_2=\frac{nh^p}{2}L_2$. *If the conditions of Theorem are satisfied aud if* $nh^{p+4}\to\infty$ *as* $n\to\infty$, *then, as* $n\to\infty$,

$$\frac{1}{n^{1/2}h^{p+2}}\bar{L}_2 \xrightarrow{D} N(0, \sigma_1^2).$$

Proof. Let

$$K_j=\int\{K(x, X_j)-EK(x, X_j)\}\{Ef_n(x)-f(x)\}dx \quad (j=1, \cdots, n).$$

Then, $\{K_j\}$ is an absolutely regular strictly stationary sequence of random variable with coefficient $\beta(n)$ and $EK_j=0$. Since $EK_j^6\leqq ch^{6(p+2)}$ (cf. Hall (1984)), so by Lemma 4

$$E\Big(\sum_{j=1}^{n}K_j\Big)^4\leqq cn^2(EK_j^6)^{2/3}\leqq cn^2h^{4(p+2)}.$$

Hence, using Lemma 1 we have

$$E\Big(\exp it\frac{1}{n^{1/2}h^{p+2}}\sum_{j=1}^{n}K_j\Big)\sim\prod_{\alpha=1}^{k}E\Big(\exp it\frac{1}{n^{1/2}h^{p+2}}\sum_{j=a_\alpha}^{b_\alpha}K_j\Big)+ck\beta(m)$$

$$=\Big\{1-\frac{t^2}{2nh^{2(p+2)}}E\Big(\sum_{j=1}^{r}K_j\Big)^2+O\Big(\frac{|t|^3}{n^{3/2}h^{3(p+2)/2}}E\Big|\sum_{j=1}^{\alpha}K_j\Big|^3\Big)\Big\}^k+o(n^{-1})$$

$$=\exp\Big\{-\frac{t^2}{2}\frac{k}{nh^{2(p+2)}}E\Big(\sum_{j=1}^{r}K_j\Big)^2+O\Big(k\Big(\frac{r}{n}\Big)^{3/2}|t|^3\Big)\Big\}+o(n^{-1}).$$

Thus, (3.11) follows, since

$$\lim_{n\to\infty} \frac{k}{nh^{2(p+2)}} E\Big(\sum_{j=1}^{r} K_j\Big)^2 = \tau^2\sigma_1^2 .$$

Proposition 3.3. *If $nh^{p+4}\to\lambda$ $(0<\lambda<\infty)$ as $n\to\infty$, then $n^{-1/2}h^{-(p+2)}L_2$ and $s_n^{-1}S_n$ are asymptotically uncorrelated as $n\to\infty$.*

Proof. By Lemmas 1 and 4, Schwarz's inequality and (A.7)

$$\begin{aligned}
|E\{L_2S_n\}| &\leqq \sum_{i=1}^{n} \sum_{1\leq i<j\leq n} |EK_i\tilde{H}(X_j, X_k)| \\
&\leqq \Big\{ \sum_{\max(|i-j|, |j-k|, |k-i|)\leq m} + \sum_{\max(|i-j|, |j-k|, |k-i|)>m} \Big\} |EK_i\tilde{H}(X_j, X_k)| \\
&\leqq c\Big[nm\|K_i\|_2 \max_{2\leq j\leq n} \|\tilde{H}(X_1, X_j)\|_2 + n^3\beta(m)\Big] \\
&\leqq c[nm^2h^{p+2}h^p + o(n^{-5})]
\end{aligned}$$

since $\|K_i\|_2 \leqq c\,h^{p+2}$.

Hence, from Condition A(ii) the desired conclusion follows.

Proposition 3.4. *If the conditions of Theorem are satisfied, then*

$$\mathrm{Var}(L_3) = O(n^{-3}h^{-2p}). \tag{3.12}$$

Proof. Let

$$M_i = \int \{K(x, X_i) - EK(x, X_i)\}^2 dx \quad (i=0, \pm 1, \pm 2, \cdots),$$

Then, $\{M_i\}$ is an absolutely regular, strictly stationary sequence of random variables with mixing coefficient $\beta(n)$. We note that by Lemma 2 in Hall (1984)

$$EM_i^j = O(h^{jp}) \quad (j=1, 2, 3).$$

So, by Lemma 3 we have

$$n^4h^{4p}\,\mathrm{Var}(L_3) = E\Big[\sum_{j=1}^{n} \{M_j - EM_j\}\Big]^2 \leqq cn\|M_1 - EM_1\|_3^2 \leqq cn\|M_1\|_3^2 \leqq cnh^{2p},$$

which implies (3.12) and the proof is completed.

The rest of the proof of Theorem is obvious and so is omitted.

Appendix

In this appendix, we prove lemmas needed in the proof of Theorem.

Lemma 1. (cf. Lemma 2.1 in Yoshihara (1976)). *Let $\xi_1, \cdots, \xi_n$ be random vectors satisfying an absolute regularity condition with mixing condition $\beta(n)$. Let $h(x_1, \cdots, x_k)$ be a bounded Borel measurable function, i.e., $|h(x_1, \cdots, x_k)| \leqq c_1$. Then*

$$\Big|Eh(\xi_{i_1}, \cdots, \xi_{i_k})-\int\cdots\int h(x_1, \cdots, x_j, x_{j+1}, \cdots, x_k)$$
$$\times dF^{(1)}(x_1, \cdots, x_j)dF^{(2)}(x_{j+1}, \cdots, x_k)\Big| \leqq 2c_1\beta(i_{j+1}-i_j) \qquad \text{(A.1)}$$

where $F^{(1)}$ and $F^{(2)}$ are distribution functions of random vectors $(X_{i_1}, \cdots, X_{i_j})$ and $(X_{i_{j+1}}, \cdots, x_{i_k})$, respectively, and $i_1<i_2<\cdots<i_k$.

Lemma 2. *Let $\xi_1, \cdots, \xi_n$ be random vectors as in Lemma 1. Let $h(y, z)$ be a Borel measurable function such that $|h(y, z)| \leqq c_1$ for all y and z. Let η be an M_1^k-measurable random variable. Further, let $H(y)=Eh(y, \zeta)$. Then*

$$E|E\{h(\eta, \zeta)|M_1^h\}-H(\eta)| \leqq 2c_1\beta(m). \qquad \text{(A.2)}$$

Proof. Let Q and R be probability distribution of η and ζ, respectively. Let $\bar{P}$ be the joint distribution of (η, ζ) and let $P(z|y)$ be a regular conditional probability distribution of ζ given $\eta=y$. Then

$$\begin{aligned}\text{L.H.S. of (A.2)} &= \int\Big|\int h(y, z)P(dz|y)-\int h(y, z)R(dz)\Big|Q(dy)\\ &\leqq c_1\int\int|P(dz|y)-R(dz)|Q(dy)\\ &= c_1\text{Var}[\bar{P}-Q\times R]=2c_1\beta(m)\end{aligned}$$

where $\text{Var}[\bar{P}-Q\times R]$ denotes the total variation of $\bar{P}-Q\times R$. (cf. Rozanov and Volkonski (1961)). Thus, the proof is completed.

Lemma 3. *Let η and ζ be as in Lemma 2. If $\|\eta\|_s$ and $\|\zeta\|_t$ exist for $s>2$ and $t>2$ and $E\eta=E\zeta=0$, then*

$$|E\eta\zeta| \leqq c\beta^{1-(1/s)-(1/t)}(m)\|\eta\|_s\|\zeta\|_t. \qquad \text{(A.3)}$$

Lemma 4. (cf. Theorem 2.2 in Utev). *Let $\xi_1, \cdots, \xi_n$ be as in Lemma 1. Assume $E\xi_i=0$ and $E|\xi_i|^{t+\delta} \leqq c_2$ $(i=1, \cdots, n)$ for some $t \geqq 2$ and $\delta>0$. If $\beta(n)=O(e^{-\gamma n})$ for some $\gamma>0$, then*

$$E\Big|\sum_{i=1}^n \xi_i\Big|^t \leqq c(t)\max\{L_t(n, \delta), (L_2(n, \delta))^{t/2}\} \qquad \text{(A.4)}$$

where $c(t)$ is some absolute constant depending only on t and

$$L_t(n, \delta)=\sum_{i=1}^{n} \|\xi_i\|_{t+\delta}^{t}. \tag{A.5}$$

In what follows, we always assume that all conditions of Theorem are satisfied. Let $\{\tilde{X}_i\}$ be i.i.d. random vectors each of which has the same distribution as that of X_0. Put

$$M_0=\sup_{x,y}|H(x, y)|=O(h^p).$$

$$G(x, y)=G_n(x, y)=\int H(u, x)H(u, y)dF(u),$$

and

$$Y_\alpha(x)=\sum_{j=1}^{a_\alpha-m} H(x, X_j) \quad (\alpha=1, \cdots, k).$$

Let Q_α be the distribution function of $(X_{a_\alpha}, \cdots, X_{b_\alpha})$. By Hall's results (1984) and Lemma 1, the following inequalities are easily obtained:

$$EH(\tilde{X}_1, \tilde{X}_2)=0;\ E\{H(\tilde{X}_1, \tilde{X}_2)|\tilde{X}_1\}=0 \quad \text{a.s.} \tag{A.6}$$

$$M(2j)=M_n(2j)=EH_n^{2j}(\tilde{X}_1, \tilde{X}_2)=O(h^{2jp+1}) \quad (j=1, 2, 3) \tag{A.7}$$

$$\sup_{x,y} H(x, y)=O(h^p) \tag{A.8}$$

$$EH^2(X_i, X_j)=E\left[\int\{K(x, X_1)-EK(x, X_1)\}^2dx\right]^2 \leqq ch^{2p} \quad \text{for all } i \text{ and } j; \tag{A.9}$$

$$|EH(X_i, X_j)|\leqq ch^p\beta(|i-j|) \quad \text{for all } i \text{ and } j; \tag{A.10}$$

$$EG(\tilde{X}_1, \tilde{X}_2)=0:\ EG^2(\tilde{X}_1, \tilde{X}_2)=O(h^{7p}); \tag{A.11}$$

$$|EG(X_i, X_j)|\leqq c\beta^{1/2}(|i-j|)h^{(7/2)p}, \quad \text{for all } i \text{ and } j. \tag{A.12}$$

Lemma 5. *As $n\to\infty$*

$$\bar{s}_n^2=\operatorname{Var} U_n=E\Big(\sum_{\alpha=1}^{k} T_\alpha\Big)^2-\Big\{E\Big(\sum_{\alpha=1}^{k} T_\alpha\Big)\Big\}^2\sim\frac{n^2}{2}EH^2(\tilde{X}_1, \tilde{X}_2)=O(n^2h^{3p}).$$

Proof. Firstly, we note that by (A.9)

$$\Big|E\Big(\sum_{\alpha=1}^{k} T_\alpha\Big)\Big|\leqq\sum_{\alpha=1}^{k}\sum_{i=a_\alpha}^{b_\alpha}\sum_{j=1}^{a_\alpha-m}|EH(X_i, X_j)|\leqq cn^2\beta(m)=o(n^{-6}). \tag{A.14}$$

Now, we consider

$$E\Big(\sum_{\alpha=1}^{k} T_\alpha\Big)^2=\sum_{\alpha=1}^{k} ET_\alpha^2+2\sum_{1\leqq\alpha<\alpha'\leqq k} ET_\alpha T_{\alpha'}=I_{11}+I_{12}, \quad \text{(say)}. \tag{A.15}$$

Since by Lemma 1, (A.5) and the fact $|T_\alpha H(x, y)|\leqq rnM_0^2$

$$ET_\alpha T_{\alpha'} \leqq \sum_{i'=a_{\alpha'}}^{b_{\alpha'}} \sum_{j'=1}^{a_{\alpha'}-m} |ET_\alpha H(X_{i'}, X_{j'})|$$
$$\leqq \sum_{i'=a_{\alpha'}}^{b_{\alpha'}} \sum_{j'=1}^{a_{\alpha'}-m} 2rnM_0^2\beta(m) \leqq cr^2n^2\beta(m)$$

if $\alpha<\alpha'$, so

$$I_{12}=o(n^{-1}). \tag{A.16}$$

Next, by Lemma 1 and the fact $|T_\alpha|^2 \leqq cr^2n^2$

$$ET_\alpha^2=E\Big(\sum_{i=a_\alpha}^{b_\alpha} Y_\alpha(X_i)\Big)^2 \sim \int E\Big(\sum_{i=a_\alpha}^{b_\alpha} Y_\alpha(x_i)\Big)^2 dQ_\alpha + cr^2n^2\beta(m)$$
$$=\sum_{i=a_\alpha}^{b_\alpha} \int EY_\alpha^2(x_i)dF(x_i)+2\sum_{a_\alpha \leq i<i' \leq b_\alpha} \int EY_\alpha(x_i)Y_{\alpha'}(x_{i'})dQ_\alpha + o(n^{-5})$$
$$=J_{1\alpha}+2J_{2\alpha}+o(n^{-5}), \quad \text{(say)}. \tag{A.17}$$

By (A.12)

$$J_{1\alpha}=\sum_{i=a_\alpha}^{b_\alpha}\Big[\sum_{j=1}^{a_\alpha-m}\int EH^2(x_i, X_j)dF(x_i)$$
$$+2\sum_{1\leq j<j'\leq a_\alpha-m}\int E\{H(x_i, X_j)H(x_i, X_{j'})\}dF(x_i)\Big]$$
$$\leqq \sum_{i=a_\alpha}^{b_\alpha}\Big[a_\alpha EH^2(\tilde{X}_1, \tilde{X}_2)+2\sum_{1\leq j<j'\leq a_\alpha-m} EG(X_j, X_{j'})\Big]$$
$$=a_\alpha r[EH^2(\tilde{X}_1, \tilde{X}_2)+O(h^{(7/2)p})].$$

On the other hand, if $i'-i>m$ or $|j'-j|>m$, then by the Lemma 1 and (A.6)

$$\Big|\int E\{H(x_i, X_j)H(x_{i'}, X_{j'})\}dQ_\alpha\Big| \leqq c\beta(m)$$

and if $i'-i\leqq m$ and $|j'-j|\leqq m$, then by Schwarz's inequality and (A.9)

$$\Big|\int E\{H(x_i, X_j)H(x_{i'}, X_{j'})\}dQ_\alpha\Big| \leqq ch^{4p}.$$

Hence by (A.11) and Condition A(ii)

$$|J_{2\alpha}| \leqq \sum_{a_\alpha \leq i<i' \leq b_{\alpha'}, 1\leq j, j' \leq a_\alpha - m}\Big|\int E\{H(x_i, X_j)H(x_{i'}, X_{j'})\}dQ_\alpha\Big|$$
$$\leqq \Sigma_{(1)}c\beta(m)+\Sigma_{(2)}ch^{4p}+\Sigma_{(3)}|EG(X_i, X_{i'})|$$
$$\leqq cr^2n^2\beta(m)+cnrm^2h^{4p}+cnrh^{(7/2)p}$$
$$=o(nrh^{3p}), \tag{A.18}$$

where $\Sigma_{(1)}$ denotes the summation over all i, i', j and j' such that $a_\alpha \leqq i<i' \leqq b_\alpha$, $1\leqq j$, $j'\leqq a_\alpha-m$ and $i'-i>m$ (or $|j'-j|>m$), $\Sigma_{(2)}$ denotes the summation over all i, i', j and j' such that $i'-i\leqq m$ and $1\leqq|j'-j|\leqq m$ and $\Sigma_{(3)}$ denotes the summation over all i, i' and j such that $i'-i\leqq m$ and $1\leqq j\leqq a_\alpha-m$. Thus, we have

$$I_{11}=\sum_{\alpha=1}^{k}(J_{1\alpha}+J_{2\alpha})=\sum_{\alpha=1}^{k}[a_\alpha r\{EH^2(\tilde{X}_1,\ \tilde{X}_2)+O(h^{(7/2)p})\}+o(nrh^{3p})]$$
$$=\frac{n^2}{2}EH^2(\tilde{X}_1,\ \tilde{X}_2)\{1+o(1)\}. \tag{A.19}$$

Now, (A.13) follows from (A.16) and (A.19) and the proof is completed.

Lemma 6. *As* $n\to\infty$

$$\bar{s}_n^{-1}\sum_{\alpha=1}^{k}E\{T_\alpha|F_\alpha\}\xrightarrow{P}0 \tag{A.20}$$

and

$$\bar{s}_n^{-2}\sum_{\alpha=1}^{k}[E\{T_\alpha^2|F_\alpha\}-(E\{T_\alpha|F_\alpha\})^2]\xrightarrow{P}1. \tag{A.21}$$

Proof. Since by Lemma 2 and (A.6)

$$E\left|\sum_{\alpha=1}^{k}E\{T_\alpha|F_\alpha\}\right|\leqq\sum_{\alpha=1}^{k}\sum_{i=a_\alpha}^{b_\alpha}\sum_{j=1}^{a_\alpha-m}E|E\{H(X_i,\ X_j)|F_\alpha\}|$$
$$\leqq\sum_{\alpha=1}^{k}\sum_{i=a_\alpha}^{b_\alpha}\sum_{j=1}^{a_\alpha-m}\{|EH(\tilde{X}_1,\ \tilde{X}_2)|+c\beta(m)\}$$
$$\leqq cn^2\beta(m)=o(n^{-1}\bar{s}_n),$$

so (A.20) follows.

To prove (A.21) it suffices to show that

$$I_{21}=\bar{s}_n^{-2}\sum_{\alpha=1}^{k}E|E\{T_\alpha^2|F_\alpha\}-ET_\alpha^2|\longrightarrow 0 \tag{A.22}$$

and

$$I_{22}=\bar{s}_n^{-2}[E|E\{T_\alpha|F_\alpha\}|^2+(ET_\alpha)^2]\longrightarrow 0. \tag{A.23}$$

(A.22) follows since by Lemmas 1 and 2

$$\bar{s}_n^2 I_{21}\leqq\sum_{\alpha=1}^{k}\sum_{a_\alpha\leq i,i'\leq b_\alpha}\sum_{1\leq j,j'\leq a_\alpha-m}E|E\{H(X_i,\ X_j)H(X_{i'},\ X_{j'})|F_\alpha\}$$
$$-E\{H(X_i,\ X_j)H(X_{i'},\ X_{j'})\}|$$
$$\leqq\sum_{\alpha=1}^{k}\sum_{a_\alpha\leq i,i'\leq b_\alpha}\sum_{1\leq j,j'\leq a_\alpha-m}\{E|E\{H(X_i,\ X_j)H(X_{i'},\ X_{j'})|F_\alpha\}$$
$$-D_\alpha(i,\ i';\ j,\ j')|+|E\{H(X_i,\ X_j)H(X_{i'},\ X_{j'})\}-D_\alpha(i,\ i';\ j,\ j')|$$
$$\leqq\sum_{\alpha=1}^{k}\sum_{a_\alpha\leq i,i'\leq b_\alpha}\sum_{1\leq j,j'\leq a_\alpha-m}[c\beta(m)+c\beta(m)]$$
$$\leqq cn^3r\beta(m)=o(n^{-1}\bar{s}_n^{-2}),$$

where

$$D_\alpha(i, i' : j, j') = \int E\{H(x_i, X_j)H(x_{i'}, X_{j'})\}dQ_\alpha .$$

On the other hand, by Lemma 1

$$\begin{aligned}
&|E(E\{H(X_i, X_j)|F_\alpha\}E\{H(X_{i'}, X_{j'})|F_\alpha\})| \\
&\quad = |E\{H(X_i, X_j)E\{H(X_{i'}, X_{j'})|F_\alpha\}| \\
&\quad \leqq \left|\int E\{H(x_i, X_j)E\{H(X_{i'}, X_{j'})|F_\alpha\}dF(x_i)\right| + c\beta(m) \\
&\quad = c\beta(m)
\end{aligned}$$

for any i, i', j and j' such that $a_\alpha \leqq i$, $i' \leqq b_\alpha$ and $1 \leqq j$, $j' \leqq a_\alpha - m$. Hence, we have

$$\begin{aligned}
&\sum_{\alpha=1}^{k} E(E\{T_\alpha|F_\alpha\})^2 \\
&\quad \leqq \sum_{\alpha=1}^{k} \sum_{a_\alpha \leq i, i' \leq b_\alpha} \sum_{1 \leq j, j' \leq a_\alpha - m} |E[E\{H(X_i, X_j)|F_\alpha\}E\{H(X_{i'}, X_{j'})|F_\alpha\}| \\
&\quad \leqq cn^8 r\beta(m) = o(n^{-1}\bar{s}_n^2).
\end{aligned} \tag{A.24}$$

Thus, (A.23) is obtained from (A.14) and (A.24), and the proof is completed.

Lemma 7. *As $n \to \infty$*

$$E(S_n - U_n)^2 \longrightarrow 0. \tag{A.25}$$

Proof. Since

$$S_n - U_n = \sum_{\alpha=1}^{k} \sum_{i=a_\alpha}^{b_\alpha} \sum_{j=a_\alpha-m+1}^{a_\alpha-1} \{H(X_i, X_j) - EH(X_i, X_j)\}$$

and by Lemma 1 and (A.6)

$$\begin{aligned}
&\sum_{\alpha=1}^{k} \sum_{i=a_\alpha}^{b_\alpha} \sum_{j=a_\alpha-m+1}^{a_\alpha-1} |EH(X_i, X_j)| \\
&\quad \leqq \sum_{\alpha=1}^{k} \left\{\sum_{i=a_\alpha+m+1}^{b_\alpha} + \sum_{i=a_\alpha+1}^{a_\alpha+m}\right\} \sum_{j=a_\alpha-m+1}^{a_\alpha} |EH(X_i, X_j)| \\
&\quad \leqq cn\beta(m) + ckm^2 M^{1/2}(2) \leqq cn^{3/4}(\log n)^2 h^{(3/2)p},
\end{aligned}$$

so to prove (A.25) it is enough to show

$$E\left\{\sum_{\alpha=1}^{k} \sum_{i=a_\alpha}^{b_\alpha} \sum_{j=a_\alpha-m+1}^{a_\alpha-1} H(X_i, X_j)\right\} = o(\bar{s}_n^2) \quad \text{as } n \to \infty . \tag{A.26}$$

We note that by Lemma 1

$$\text{L. H. S. of (A.26)} \leqq k \sum_{\alpha=1}^{k} E\Big\{\sum_{i=a_\alpha}^{b_\alpha} \sum_{j=a_\alpha-m+1}^{a_\alpha-1} H(X_i, X_j)\Big\}^2$$

$$\leqq 2k \sum_{\alpha=1}^{k} \Big[E\Big\{\sum_{i=a_\alpha-m+1}^{b_\alpha} \sum_{j=a_\alpha-m+1}^{a_\alpha-1} H(X_i, X_j)\Big\}+E\Big\{\sum_{i=a_\alpha}^{a_\alpha-m} \sum_{j=a_\alpha-m+1}^{a_\alpha-1} H(X_i, X_j)\Big\}^2\Big]$$

$$\leqq 2k \sum_{\alpha=1}^{k} \Big[\Big\{\int E\Big(\sum_{i=a_\alpha-m+1}^{b_\alpha} \sum_{j=a_\alpha-m+1}^{a_\alpha-1} H(x_i, X_j)\Big)^2 dQ_\alpha + cr^2m^2\beta(m)\Big\}+cm^4\Big]. \tag{A.27}$$

Further, by Lemma 4

$$E\Big\{\int\Big(\sum_{i=a_\alpha-m+1}^{b_\alpha} \sum_{j=a_\alpha-m+1}^{a_\alpha-1} H(x_i, X_j)\Big)^2 dQ_\alpha\Big\}$$

$$\leqq c \sum_{i=a_\alpha-m+1}^{b_\alpha} E\Big\{\int\Big(\sum_{j=a_\alpha-m+1}^{a_\alpha-1} H(x_i, X_j)\Big)^4 dQ_\alpha\Big\}^{1/2}$$

$$\leqq cm^2(b_\alpha-a_\alpha)=cm^2r. \tag{A.28}$$

Hence, (A.26) follows from (A.28) and Lemma 5.

Lemma 8. *As* $n\to\infty$

$$\sum_{\alpha=1}^{k} ET_\alpha^4=o(\bar{s}_n^4). \tag{A.29}$$

Proof. Since $|T_\alpha^4|\leqq cn^4r^4$, so by Lemma 1

$$ET_\alpha^4\leqq E\Big\{\int\Big(\sum_{i=a_\alpha}^{b_\alpha} Y_\alpha(x_i)\Big)^4 dQ_\alpha\Big\}+cn^4r^4\beta(m)$$

$$=\sum_{i=a_\alpha}^{b_\alpha} EY_\alpha^4(x_i)dQ_\alpha+\int \sum_{\substack{a_\alpha\leq i,i'\leq b_\alpha \\ i\neq i'}} \int EY_\alpha^3(x_i)Y_\alpha(x_{i'})dQ_\alpha$$

$$+2\sum_{a_\alpha\leq i<i'\leq b_\alpha} \int E\Big\{Y_\alpha^2(x_i)Y_\alpha^2(x_{i'})\Big\}dQ_\alpha$$

$$+\sum_{\substack{a_\alpha\leq i_1,i_2,i_3\leq b_\alpha \\ i_s\neq i_t\ (s\neq t)}} \int E\{Y_\alpha^2(x_{i_1})Y_\alpha(x_{i_2})Y_\alpha(x_{i_3})\}dQ_\alpha$$

$$+\sum_{\substack{a_\alpha\leq i_1,i_2,i_3,i_4\leq b_\alpha \\ i_s\neq i_t\ (s\neq t)}} \int E\Big\{\prod_{d=1}^{4} Y_\alpha(x_{i_d})\Big\}dQ_\alpha+o(n^{-1})$$

$$=I_{\alpha1}+I_{\alpha2}+I_{\alpha3}+I_{\alpha4}+I_{\alpha5}+o(n^{-1}) \quad \text{(say)}.$$

(i) Firstly, we prove $\sum_{\alpha=1}^{k} I_{\alpha1}=o(\bar{s}_n^4)$. Let $\sum_{(k)}$ be the summation over all $j_1, \cdots, j_k$ such that $1\leqq j_1, \cdots, j_k\leqq a_\alpha-m$ and $j_s\neq j_t$ $(s\neq t)$. By Lemma 1, Hölder's inequality and (A.8) we have the following inequalities:

$$\sum_{(1)}\int EH^4(x,\ X_j)dF(x)\leqq a_\alpha M(4)$$

$$\sum_{(2)}\left|\int E\{H^3(x,\ X_{j_1})H(x,\ X_{j_2})\}dF(x)\right|$$

$$\leqq\sum_{(2)}cM^{2/3}(6)\beta^{1/3}(|j_2-j_1|)\leqq ca_\alpha M^{1/2}(4)$$

$$\sum_{(2)}\int E\{H^2(x,\ X_{j_1})H^2(x,\ X_{j_2})\}dF(x)$$

$$\leqq\sum_{(2)}\{M^2(2)+cM^{2/3}(6)\beta^{1/3}(|j_2-j_1|)\}\leqq a_\alpha^2M^2(2)+ca_\alpha M^{2/3}(6).$$

$$\sum_{(3)}\left|\int E\{H^2(x,\ X_{j_1})H(x,\ X_{j_2})H(x,\ X_{j_3})\}dF(x)\right|$$

$$\leqq\Big[\sum_{(3)\atop \max(|j_2-j_1|,\ |j_3-j_1|)>m}+\sum_{(3)\atop \max(|j_2-j_1|,\ |j_3-j_1|)\leqq m}\Big]$$

$$\left|\int E\{H^2(x,\ X_{j_1})H(x,\ X_{j_2})H(x,\ X_{j_3})\}dF(x)\right|$$

$$\leqq\sum_{(3)\atop \max(|j_2-j_1|,\ |j_3-j_1|)>m}c\beta(m)+\sum_{(3)\atop \max(|j_2-j_1|,\ |j_3-j_1|)\leqq m}cM(4)$$

$$\leqq ca_\alpha^3\beta(m)+a_\alpha m^2M(4)=a_\alpha m^2M(4)+o(n^{-4}).$$

If $j_1<j_2<j_3<j_4$ and $\max(j_2-j_1,\ j_4-j_3)>m$, then by Lemma 1

$$\left|E\Big\{\prod_{d=1}^4 H(x,\ X_{j_d})\Big\}dF(x)\right|\leqq c\beta(m),$$

and if $j_1<j_2<j_3<j_4$, $\max(j_2-j_1,\ j_4-j_3)\leqq m$ and $j_3-j_2>m$, then by Hölder's inequality

$$\left|\int E\Big\{\prod_{d=1}^4 H(x,\ X_{j_d})\Big\}dF(x)\right|$$

$$\leqq\left|\int E\{H(x,\ X_{j_1})H(x,\ X_{j_2})\}E\{H(x,\ X_{j_3})H(x,\ X_{j_4})\}dF(x)\right|+c\beta(m)$$

$$\leqq M^2(2)+c\beta(m).$$

Further, if $j_1<j_2<j_3<j_4$ and $\max(j_2-j_1,\ j_3-j_2,\ j_4-j_3)\leqq m$, then

$$\left|\int E\Big\{\prod_{d=1}^4 H(x,\ X_{j_d})\Big\}dF(x)\right|\leqq M(4).$$

Accordingly, we have

$$\sum_{(4)}\left|\int E\Big\{\prod_{d=1}^4 H(x,\ X_{j_d})\Big\}dF(x)\right|$$

$$\leqq ca_\alpha^4\beta(m)+a_\alpha^2\{M^2(2)+c\beta(m)\}+a_\alpha m^3M(4)$$

$$\leqq a_\alpha^2M^2(2)+a_\alpha m^3M(4)+o(n^{-4}).$$

Hence, we have

$$J_{\alpha i}^{(4)}=\int Y_{\alpha}^{4}(x_i)dF(x_i)\leqq c\{a_{\alpha}^{2}M^{2}(2)+a_{\alpha}m^{3}M(4)+a_{\alpha}M^{2/3}(6)+o(n^{-4})\}. \qquad \text{(A.30)}$$

Thus, by (A.13) and Lemma 5

$$\sum_{\alpha=1}^{k} I_{\alpha 1}=\sum_{\alpha=1}^{k}\sum_{i=a_\alpha}^{b_\alpha}\int EY_{\alpha}^{4}(x_i)dF(x_i)$$
$$\leqq c\{n^{3}M^{2}(2)+n^{2}m^{3}M(4)+n^{2}M^{2/3}(6)+o(n^{-3})\}=o(\bar{s}_n^4).$$

(ii) Next, by Schwarz's inequality, (A.30) and Lemma 5 we have

$$\sum_{\alpha=1}^{k} I_{\alpha 3}=2\sum_{\alpha=1}^{k}\sum_{a_\alpha\leq i<i'\leq b_\alpha} E\{Y_{\alpha}^{2}(x_i)Y_{\alpha}^{2}(x_{i'})\}dQ_{\alpha}$$
$$\leqq 2\sum_{\alpha=1}^{k}\sum_{a_\alpha\leq i<i'\leq b_\alpha}\{J_{\alpha i}J_{\alpha i'}\}^{1/2}$$
$$\leqq c\sum_{\alpha=1}^{k}(b_\alpha-a_\alpha)^{2}\{a_{\alpha}^{2}M^{2}(2)+a_{\alpha}m^{3}M(4)+a_{\alpha}M^{2/3}(6)+o(n^{-4})\}=o(s_n^4).$$

Similarly, we have

$$\sum_{\alpha=1}^{k} I_{\alpha_2}=o(\bar{s}_n^4).$$

(iii) Thirdly, we note that if $|i_3-i_2|>2m$, then $\max(|i_2-i_1|, |i_3-i_1|)>m$ and so by Lemma 1 and the fact that $|Y_{\alpha}(x)|\leqq ca_{\alpha}$, we have

$$I(i_1, i_2, i_3)=\left|E\int Y_{\alpha}^{2}(x_{i_1})Y_{\alpha}(x_{i_2})Y_{\alpha}(x_{i_3})dQ_{\alpha}\right|\leqq ca_{\alpha}^{4}\beta(m).$$

Hence, by Hölder's inequality and (A.30) we have

$$\sum_{\alpha=1}^{k}|I_{\alpha 4}|\leqq\sum_{\alpha=1}^{k}\Bigg\{\sum_{\substack{a_\alpha\leq i_1, i_2, i_3\leq b_\alpha\\ |i_3-i_2|>2m}}+\sum_{\substack{a_\alpha\leq i_1, i_2, i_3\leq b_\alpha\\ |i_3-i_2|\leq 2m}}\Bigg\}I(i_1, i_2, i_3)$$
$$\leqq\sum_{\alpha=1}^{k}\Bigg[\sum_{\substack{a_\alpha\leq i_1, i_2, i_3\leq b_\alpha\\ |i_3-i_2|>2m}} ca_{\alpha}^{4}\beta(m)+\sum_{\substack{a_\alpha\leq i_1, i_2, i_3\leq b_\alpha\\ |i_3-i_2|\leq 2m}}\{J_{\alpha i_1}\}^{1/2}\{J_{\alpha i_2}\}^{1/4}\{J_{\alpha i_3}\}^{1/4}\Bigg]$$
$$\leqq o(n^{-1})+c\sum_{\alpha=1}^{k} m(b_\alpha-a_\alpha)^{2}\{a_{\alpha}^{2}M^{2}(2)+a_{\alpha}m^{3}M(4)+a_{\alpha}M^{2/3}(6)\}=o(\bar{s}_n^4).$$

(iv) To prove $\sum_{\alpha=1}^{k}|I_{\alpha 5}|=o(\bar{s}_n^4)$, it suffices to show that

$$\sum_{\alpha=1}^{k}\sum_{a_\alpha\leq i_1<i_2<i_3<i_4\leq b_\alpha}\int E\Big\{\prod_{d=1}^{4}Y(x_{i_d})\Big\}dQ_{\alpha}=o(\bar{s}_n^4). \qquad \text{(A.31)}$$

This relation is proved, since by Lemma 1, Hölder's inequality and the method used in (i)

$$\text{L.H.S. of (A.31)} \leqq \sum_{\alpha=1}^{k} \{\Sigma_{(1)}^{(*)} + \Sigma_{(2)}^{(*)}\} \left| \int E\left\{ \prod_{d=1}^{4} Y_\alpha(x_{i_d}) \right\} dQ_\alpha \right|$$

$$\leqq \sum_{\alpha=1}^{k} [\Sigma_{(1)}^{(*)} c\beta(m) + \Sigma_{(2)}^{(*)}] \prod_{d=1}^{4} \{J_{\alpha i_d}\}^{1/4}$$

$$\leqq ckn^7 o(n^{-8}) + c \sum_{\alpha=1}^{k} m^4 (b_\alpha - a_\alpha)^2 \{a_\alpha M^2(2) + a_\alpha m^3 M(4) + a_\alpha M^{2/3}(6)\}$$

$$= o(\bar{s}_n^4),$$

where $\Sigma_{(1)}^{(*)}(\Sigma_{(2)}^{(*)})$ denotes the summation over all i_d, j_d $(d=1, \cdots, 4)$ such that $a_\alpha \leqq i_1 < i_2 < i_3 < i_4 \leqq b_\alpha$, $1 \leqq j_1 < j_2 < j_3 < j_4 \leqq a_\alpha - m$ and $\max(i_2 - i_1, i_4 - i_3, j_2 - j_1, j_4 - j_3) > m (\max(i_2 - i_1, i_4 - i_3, j_2 - j_1, j_4 - j_3) < m)$.

Now, (A.29) follows from (i)-(iv) and the proof of Lemma 8 is completed.

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