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AN INVARIANCE PRINCIPLE FOR INTEGRAL TYPE FUNCTIONALS OF WEAKLY DEPENDENT RANDOM PROCESSES

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1. Introduction.

Let $\{X_i: -\infty<i<\infty\}$ be be a strictly stationary process with values in R^1, defined on a probability space $(\Omega, \mathcal{F}, P)$. Suppose that $E(X_0)=0$ and $E(X_0^2)=\sigma_0^2$ is finite. Put $S_k=X_1+\cdots+X_k$ for each positive integer k $(S_0=0)$, and $S_{n,k}=S_k/\sigma\sqrt{n}$ for nonnegative integers k where σ is a positive number defined in §2. Let $\{u_n(t, x): n=1, 2, \cdots\}$ be a sequence of functions defined on $[0, 1]\times R^1$. For each n, define a functional $Z_n(t)$ $(0\leqq t\leqq 1)$ as follows:

$$(1.1)\qquad Z_n(t)=\begin{cases}\sum_{k=0}^{[nt]} u_n(k/n, S_{n,k})\dfrac{X_{k+1}}{\sigma\sqrt{n}} & \text{if } nt\geqq 1\\ 0 & \text{otherwise}\end{cases}$$

which is a random element in $D[0, 1]$ endowed with the Skorohod J_1-topology (see [1]).

This type of the functionals have been studied by some authors, for example, [4] [5] [7] [8] and [10]. These authors have proved some weak convergence theorems of the functional Z_n to the stochastic integral $Z(t)=\int_0^t u(s, W(s))dW(s)$ ($W(t)$ denotes the standard Brownian motion in $[0, 1]$) under the condition that $\{X_n\}$ is a sequence of martingale differences. Of course, here, the authors assume that u_n converges to u as $n\to\infty$ in some sense.

It is natural to ask whether the same convergence theorem is valid when we replace the martingale property by some weak dependence condition. This problem is related with a conjecture that a process $X_n(t)$ with the stochastic differential $dX_n(t)=u_n(t, W_n(t-))dW_n(t)$ converges in distribution on $D[0, 1]$ to the process $X(t)$ with the stochastic differential $dX(t)=u(t, W(t))dW(t)$ when W_n converges in distribution on $D[0, 1]$ to the standard Brownian motion W on $[0, 1]$.

This conjecture is correct if $\{X_n\}$ is a sequence of martingale differences.

In this paper we consider the case where $\{X_n\}$ satisfies the strong mixing condition or m-dependence condition. Our results show that the conjecture above is not correct, i.e., if $\{X_n\}$ is not independent and satisfies some weak dependence condition instead of martingale property, then a term of drift may appear in limit (see Theorem 1 and 2).

2. Preliminaries and Theorems.

Let $\{X_n\}$ be the process given in the previous section. We say that $\{X_n\}$ satisfies the strong mixing condition if, as $n\to\infty$,

(2.1) $$\alpha(n)=\sup_{A\in F_{-\infty}^{0}}\sup_{B\in F_{n}^{\infty}}|P(A\cap B)-P(A)P(B)|\downarrow 0$$

where F_a^b denotes the σ-algebra generated by $X_a, X_{a+1}, \cdots, X_b$. When $\alpha(m)>0$ and $\alpha(m+1)=0$, we say that $\{X_n\}$ satisfies the m-dependence condition.

Denote by $\mathcal{A}$ the set of functions $u(t, x)$ defined on $[0, 1]\times R^1$, such that the partial derivatives $\partial u/\partial t$ and $\partial^2 u/\partial x^2$ are continuous on $[0, 1]\times R^1$.

In what follows we always assume that

(2.2) (C.0) the limit $\lim_{n\to\infty} n^{-1}E(S_n^2)=\sigma^2$ exists and is positive.

In Theorem 1 the following conditions are assumed:

(2.3)
(C.1) $E(X_0)=0$, $E(X_0^2)=\sigma_0^2$ and $E|X_0|^{4+2\delta}<\infty$ for some $0<\delta\leqq 2$,
(C.2) $\{X_n\}$ is strongly mixing with $\alpha(n)=O(n^{-p})$ for some $p>(4+2\delta)/\delta$.

In Theorem 2 the following conditions are assumed:

(2.4)
(C.3) $E(X_0)=0$, $E(X_0^2)=\sigma_0^2$ and $E|X_0|^4<\infty$,
(C.4) $\{X_n\}$ is m-dependent $(m\geqq 1)$.

Remark. Since the condition (C.2) implies $\sum_{n=1}^{\infty}\alpha^{\delta/(2+\delta)}(n)<\infty$, the existence of the limit σ^2 is assured (cf. [3]).

For a sequence $\{u_n\}$ in $\mathcal{A}$, define a sequence of processes $Z_n(t)$ $(0\leqq t\leqq 1)$ as follows:

(2.5) $$Z_n(t)=\sum_{k=0}^{[nt]}u_n(k/n, S_{n,k})\frac{X_{k+1}}{\sigma\sqrt{n}} \qquad 0\leqq t\leqq 1$$

where $S_{n,k}=S_k/\sigma\sqrt{n}$.

Now we state the main results.

Theorem 1. *Let $u(t, x)$ be in $\mathcal{A}$. Suppose that u_n and its partial derivatives up to second order converge to u and the ones of u, respectively, uniformly on each rectangle $[0, 1]\times[-T, T]$ $(T>0)$ as $n\to\infty$. Then under the conditions* (C.1) *and*

(C.2) *each finite dimensional joint distribution of the process* $Z_n(t)$ *converges weakly to that of the process*

$$Z(t)=\int_0^t u(s, W(s))dW(s)+\frac{\sigma^2-\sigma_0^2}{2\sigma^2}\int_0^t \frac{\partial u}{\partial x}(s, W(s))ds, \qquad 0\leqq t\leqq 1 \tag{2.6}$$

where $W(t)$ *is the standard Brownian motion in* $D[0, 1]$ *and the stochasic integral is the usual Ito integral.*

Theorem 2. *Let* u *and* u_n*'s be the same ones as in Theorem* 1. *Then under the conditions* (C.3) *and* (C.4) *the process* $Z_n(t)$ *converges in distribution on* $D[0, 1]$ *to the process* $Z(t)$ *defined above.*

Remark. We are not able to show the tightness of the processes Z_n under the conditions of Theorem 1. We think that perhaps it is difficult to prove the tightness for Z_n's even if the strong mixing condition is replaced by the ϕ-mixing condition or more stringent condition except for m-dependence.

3. Proof of Theorem 1.

In this section we prove the convergence of finite dimensional joint distribution of the process $Z_n(t)$, and in the next section we will prove the tightness for Z_n's under the condtions of Theorem 2. So throughout this section we assume the conditions of Theorem 1.

Lemma 3.1 ([2]). *Let* X *and* Y *be* $F_{-\infty}^0$*- and* F_n^∞*-measurable random variables, respectively, with* $\|X\|_p=\{E|X|^p\}^{1/p}<\infty$ *and* $\|Y\|_q=\{E|Y|^q\}^{1/q}<\infty$. *Then*

$$|E(XY)-E(X)E(Y)|\leqq 10\alpha^{1/s}(n)\|X\|_p\|Y\|_q \tag{3.1}$$

where p, q *and* s *are positive numbers such that* $1/p+1/q+1/s=1$.

Lemma 3.2. *There exists an absolute constant* K *not depending upon* n *such that*

$$E|S_n|^4\leqq Kn^2. \tag{3.2}$$

Proof. The condition (C.2) implies $\sum_{n=1}^{\infty} n\alpha^{2\delta/(4+2\delta)}(n)<\infty$. Hence by Theorem in [9] we have the inequality (3.2). Q.E.D.

Now remark that by the functional central limit theorem forst rongly mixing sequences [3], the family of distributions of $\{S_{n,k}: n, 0\leqq k\leqq n\}$ is tight [1]. Hence for any $\varepsilon>0$ there exist a constant $C>0$ and functions $u_n^C(t, x)$ and $u^C(t, x)$ in $\mathcal{A}$ satisfying the following conditions:

(I) for all $(t, x)\in[0, 1]\times[-C, C]$, $u_n^C(t, x)=u_n(t, x)$ for all n and $u^C(t, x)=u(t, x)$,

(II) u^C and u_n^C and their partial derivatives up to second order uniformly bounded

and

(III) $P(Z_n\neq Z_n^C)<\varepsilon$ for all n and $P(Z\neq Z^C)<\varepsilon$

where Z_n^C (resp. Z^C) denotes the process defined in (2.5) (resp. (2.6)) replacing u_n (resp. u) by u_n^C (resp. u^C).

Therefore it suffices to prove the theorem when u and u_n and their partial derivatives up to second order are uniformly bounded.

For each positive integer m, define

(3.3) $$T_k=T_{k,m}=X_k+X_{k-1}+\cdots+X_{k-(m-1)}$$

and $$U_k=U_{k,m}=X_{k-m}+X_{k-m-1}+\cdots+X_{k-(2m-1)} \qquad (k\geqq 2m).$$

Then $u_n(k/n, S_{n,k})$ can be expanded as follows $(k\geqq 2m)$.

(3.4) $$\begin{aligned} u_n\Big(\frac{k}{n}, S_{n,k}\Big) &= u_n\Big(\frac{k}{n}, S_{n,k-m}\Big)+\frac{\partial u_n}{\partial x}\Big(\frac{k}{n}, S_{n,k-m}\Big)\frac{T_k}{\sigma\sqrt{n}} \\ &\quad +\frac{1}{2}\frac{\partial^2 u_n}{\partial x^2}\Big(\frac{k}{n}, S_{n,k-m}+\theta_1^{(k)}\frac{T_k}{\sigma\sqrt{n}}\Big)\frac{T_k^2}{\sigma^2 n} \quad (0<\theta_1^{(k)}<1 \text{ a.s.}) \\ &= u_n\Big(\frac{k}{n}, S_{n,k-m}\Big)+\frac{\partial u_n}{\partial x}\Big(\frac{k}{n}, S_{n,k-2m}\Big)\frac{T_k}{\sigma\sqrt{n}} \\ &\quad +\frac{\partial^2 u_n}{\partial x^2}\Big(\frac{k}{n}, S_{n,k-2m}+\theta_2^{(k)}\frac{U_k}{\sigma\sqrt{n}}\Big)\frac{U_k}{\sigma\sqrt{n}}\frac{T_k}{\sigma\sqrt{n}} \\ &\quad +\frac{1}{2}\frac{\partial^2 u_n}{\partial x^2}\Big(\frac{k}{n}, S_{n,k-m}+\theta_1^{(k)}\frac{T_k}{\sigma\sqrt{n}}\Big)\frac{T_k^2}{\sigma^2 n} \quad (0<\theta_2^{(k)}<1 \text{ a.s.}). \end{aligned}$$

Now we specify the integer m as $m=[n^\alpha]$ where α is an arbitrarily fixed number such that $\delta/(2+2\delta)\leqq\alpha<1/3$. Here $[a]$ denotes the largest integer contained in the real number a. For convenience, in what follows, if $a>b$, the summation $\sum_{i=a}^{b}$ should be read as zero. And for an n fixed let $m'(t)$ denote $\min\{(2m-1), [nt]\}$.

By (3.4) we can rewrite $Z_n(t)$ as follows: for $0\leqq t\leqq 1$

(3.5) $$\begin{aligned} Z_n(t) &= \frac{1}{\sigma\sqrt{n}}\sum_{k=1}^{m'(t)} u_n\Big(\frac{k}{n}, S_{n,k}\Big)X_{k+1}+\frac{1}{\sigma\sqrt{n}}\sum_{k=2m}^{[nt]} u_n\Big(\frac{k}{n}, S_{n,k}\Big)X_{k+1} \\ &= \frac{1}{\sigma\sqrt{n}}\sum_{k=0}^{m'(t)} n_n\Big(\frac{k}{n}, S_{n,k}\Big)X_{k+1}+\frac{1}{\sigma\sqrt{n}}\sum_{k=2m}^{[nt]} u_n\Big(\frac{k}{n}, S_{n,k-m}\Big)X_{k+1} \\ &\quad +\frac{1}{\sigma^2 n}\sum_{k=2m}^{[nt]}\frac{\partial u_n}{\partial x}\Big(\frac{k}{n}, S_{n,k-2m}\Big)(T_k X_{k+1}-E(T_k X_{k+1})) \end{aligned}$$

$$+\frac{1}{\sigma^3 n^{3/2}}\sum_{k=2m}^{[nt]}\frac{\partial^2 u_n}{\partial x^2}\Big(\frac{k}{n},\ S_{n,k-2m}+\theta_2^{(k)}\frac{U_k}{\sigma\sqrt{n}}\Big)U_k T_k X_{k+1}$$

$$+\frac{1}{\sigma^3 n^{3/2}}\sum_{k=2m}^{[nt]}\frac{\partial^2 u_n}{\partial x^2}\Big(\frac{k}{n},\ S_{n,k-m}+\theta_1^{(k)}\frac{T_k}{\sigma\sqrt{n}}\Big)T_k^2 X_{k+1}$$

$$+\frac{1}{\sigma^2 n}\sum_{k=2m}^{[nt]}\frac{\partial n_n}{\partial x}\Big(\frac{k}{n},\ S_{n,k-2m}\Big)E(T_k X_{k+1})$$

$$=I_1(t)+I_2(t)+\cdots+I_6(t),\qquad \text{(say)}.$$

When n tends to infinity, the terms $I_2(t)$ and $I_6(t)$ play the principal role. Below we will show that the other terms converge to zero in probability as $n\to\infty$. Hereafter we shall denote by the letter K, various absolute positive constants. We, without mentions, use the assumption that u_n and their partial derivatives up to second order are uniformly bounded (see the beginning of this section).

(I_1). Since $\alpha<1/3$, we have

(3.6) $$E|I_1(t)|\leqq Kn^{-1/2}\sum_{k=0}^{2m-1}E|X_{k+1}|=O(n^{\alpha-1/2})=o(1).$$

(I_4). By the Schwarz inequality and Lemma 3.2, we have

$$E|U_k T_k X_{k+1}|\leqq(E|U_k T_k|^2)^{1/2}(E|X_{k+1}|^2)^{1/2}\leqq K(E|U_k|^4)^{1/2}=O(m).$$

Hence we have

(3.7) $$E|I_4(t)|\leqq K(n^{\alpha-1/2})=o(1).$$

(I_5). By the same calculation as was used for $I_4(t)$ we have

(3.8) $$E|I_5(t)|\leqq K(n^{\alpha-1/2})=o(1).$$

(I_3). For the brevity we denote by v_n the partial derivative $\partial u_n/\partial x$.

(3.9) $$\sigma^4 E|I_3(t)|^2=n^{-2}E\Big\{\sum_{k=2m}^{[nt]}v_n\Big(\frac{k}{n},\ S_{n,k-2m}\Big)^2(T_k X_{k+1}-\sigma(m))^2$$

$$+2\sum_{k<j}v_n\Big(\frac{k}{n},\ S_{n,k-2m}\Big)v_n\Big(\frac{j}{n},\ S_{n,j-2m}\Big)(T_k X_{k+1}-\sigma(m))(T_j X_{j+1}-\sigma(m))\Big\}$$

$$=J_1+2J_2,\qquad \text{(say)}$$

where $\sigma(m)=\sum_{r=1}^{m}E(X_0X_r)$. The first term J_1 can be estimated easily as follows.

(3.10) $$J_1\leqq n^{-2}K\sum_{k=2m}^{[nt]}E\{(T_k X_{k+1}-\sigma(m))^2\}\leqq Kn^{-1}m^2=o(1).$$

Next we estimate the second term J_2. Separate J_2 into two parts:

$$J_2=n^{-2}\sum_{k<j-2m}+n^{-2}\sum_{j-2m\leqq k<j}=J_{2,1}+J_{2,2},\qquad \text{say}.$$

By Lemma 3.1, we have

(3.11) $$|J_{2,1}| \leqq Kn^{-2} \sum_{k<j-2m} \alpha^{\delta/(2+\delta)}(m) \|T_k X_{k+1} - \sigma(m)\|_{2+\delta} \|T_j X_{j+1} - \sigma(m)\|_{2+\delta}$$
$$\leqq Kn^{-2} n^2 \alpha^{\delta/(2+\delta)}(m) m^2 = o(m^{-2}) m^2 = o(1).$$

Here we used the following facts: $0<\delta\leqq 2$ (see (C.1)) and

(3.12) $$\|T_k X_{k+1} - \sigma(m)\|_{2+\delta} \leqq K \sum_{p=0}^{m} \|X_{k-p} X_{k+1}\|_{2+\delta}$$
$$\leqq K \sum_{p=0}^{m} \|X_{k-p}\|_{4+2\delta} \|X_{k+1}\|_{4+2\delta} = O(m).$$

Next we treat the term $J_{2,2}$. By the Schwarz inequality we have

(3.13) $$|J_{2,2}| \leqq Kn^{-2} \sum_{j-2m \leqq k<j} \|T_k X_{k+1} - \sigma(m)\|_2 \|T_j X_{j+1} - \sigma(m)\|_2$$
$$= O(n^{-1} m^3) = o(1).$$

Lemma 3.3. *The quantities*

(3.14) $$\sup_{0\leqq t\leqq 1} \left| \int_0^t u\left(\frac{[qs]}{q}, W\left(\frac{[qs]}{q}\right)\right) dW(s) - \int_0^t u(s, W(s)) dW(s) \right|$$

and $$\sup_{0\leqq t\leqq 1} \left| \int_0^t \frac{\partial u}{\partial x}\left(\frac{[qs]}{q}, W\left(\frac{[qs]}{q}\right)\right) ds - \int_0^t \frac{\partial u}{\partial x}(s, W(s)) ds \right|$$

converge to zero in probability as $q\to\infty$.

The proof of this lemma is given by the standard argument in the theory of stochastic integrals.

Lemma 3.4. *As $n\to\infty$,*

(3.15) $$\sum_{k=0}^{[nt]-m-1} u_n\left(\frac{k}{n}, S_{n,k}\right) \frac{X_{k+m+1}}{\sqrt{n}} - \sum_{k=0}^{[nt]-m-1} u_n\left(\frac{k+m}{n}, S_{n,k}\right) \frac{X_{k+m+1}}{\sqrt{n}}$$
$$\xrightarrow{P} 0 \qquad (0\leqq t\leqq 1).$$

The proof of this lemma is, using the mean value theorem, easy and so is omitted.

Consider a partition $\Delta: 0=t_0<t< \cdots <t_p=1$ of $[0, 1]$ and $\delta_\Delta = \max_{1\leqq i\leqq p}(t_i - t_{i-1})$ and $p(t)$=the integer q such that $t_q \leqq t < t_{q+1}$. For each partition Δ and each positive integer n, define intervals of integers, depending on t $(0\leqq t\leqq 1)$, as

$$J_j(t) = \{k\,;\, 1\leqq k\leqq [nt_{j+1}] - [nt_j] \text{ and } k\leqq [nt]-[nt_j]\}$$

and

$$I_j(t) = \{k\,;\, [nt_j] \leqq k < [nt_{j+1}] \text{ and } k\leqq [nt]\} \qquad (j=0, 1, 2, \cdots, p-1).$$

Lemma 3.5. *For given $\varepsilon>0$ there exists a positive number γ such that, for any partition Δ with $\delta_\Delta<\gamma$, we can choose a positive integer n_Δ such that for any*

$n \geqq n_{\Delta}$ *and* $0<t\leqq 1$

(3.16)
$$E\left|\sum_{k=0}^{[nt]-m-1} u_n\left(\frac{k}{n}, S_{n,k}\right)\frac{X_{k+m+1}}{\sqrt{n}} - \sum_{j=0}^{p(t)} u_n\left(\frac{[nt_j]}{n}, S_{n,[nt_j]}\right) \times \sum_{t\in J_j(t)} \frac{X_{[nt_j]+m+t}}{\sqrt{n}}\right|^2 < \varepsilon$$

Proof. For the simplicity we denote by s_j the integer $[nt_j]$ ($j=1, 2, \cdots, p$). Fix t and write I_j and J_j instead of $I_j(t)$ and $J_j(t)$.

$$\begin{aligned}
&\text{L.H.S. of (3.16)}\\
&=E\left|\sum_{j=0}^{p(t)}\sum_{k\in I_j}\left\{u_n\left(\frac{k}{n}, S_{n,k}\right)-u_n\left(\frac{s_j}{n}, S_{n,s_j}\right)\right\}\frac{X_{k+m+1}}{\sqrt{n}}\right|^2\\
&=\sum_{j=0}^{p(t)} E\left|\sum_{k\in I_j}\left\{u_n\left(\frac{k}{n}, S_{n,k}\right)-u_n\left(\frac{s_j}{n}, S_{n,s_j}\right)\right\}\frac{X_{k+m+1}}{\sqrt{n}}\right|^2\\
&\quad+2\sum_{r<j}^{p(t)} E\left[\sum_{k\in I_j}\left\{u_n\left(\frac{k}{n}, S_{n,k}\right)-u_n\left(\frac{s_j}{n}, S_{n,s_j}\right)\right\}\frac{X_{k+m+1}}{\sqrt{n}}\right.\\
&\quad\left.\times\sum_{q\in I_r}\left\{u_n\left(\frac{q}{n}, S_{n,q}\right)-u_n\left(\frac{s_r}{n}, S_{n,s_r}\right)\right\}\frac{X_{q+m+1}}{\sqrt{n}}\right]\\
&=T_1+2T_2, \qquad \text{(say)}.
\end{aligned}$$

We will estimate T_1 and T_2 separately.

(T_1).
$$\begin{aligned}
T_1&=\sum_{j=0}^{p(t)}\sum_{k\in I_j} E\left\{\left(u_n\left(\frac{k}{n}, S_{n,k}\right)-u_n\left(\frac{s_j}{n}, S_{n,s_j}\right)\right)^2\frac{X_{k+m+1}^2}{n}\right\}\\
&\quad+2\sum_{j=0}^{p(t)}\sum_{\substack{k_1<k_2\\ k_1,k_2\in I_j}} E\left\{\left(u_n\left(\frac{k_1}{n}, S_{n,k_1}\right)-u_n\left(\frac{s_j}{n}, S_{n,s_j}\right)\right)\left(u_n\left(\frac{k_2}{n}, S_{n,k_2}\right)\right.\right.\\
&\quad\left.\left.-u_n\left(\frac{s_j}{n}, S_{n,s_j}\right)\right)\frac{X_{k_1+m+1}X_{k_2+m+1}}{n}\right\}\\
&=T_{1,1}+2T_{1,2}, \qquad \text{(say)}.
\end{aligned}$$

By the mean value theorem we have

$$\left|u_n\left(\frac{k}{n}, S_{n,k}\right)-u_n\left(\frac{s_j}{n}, S_{n,s_j}\right)\right| \leqq K\{(k-s_j)/n+|S_{n,k}-S_{n,s_j}|\}.$$

So, by Lemma 3.2.,

(3.17)
$$\begin{aligned}
T_{1,1}&\leqq K\sum_{j=0}^{p(t)}\sum_{k\in I_j}\left[\delta_{\Delta}^2\frac{E\{X_{k+m+1}^2\}}{n}+\|S_{n,k}-S_{n,s_j}\|_4^2\frac{\{E(X_{k+m+1}^4)\}^{1/2}}{n}\right]\\
&\leqq K\delta_{\Delta}.
\end{aligned}$$

Next we consider $T_{1,2}$. When $k_1<k_2\leqq k_1+m+1$, by Lemma 3.1 we have

$$\left|E\left\{\left(u_n\left(\frac{k_1}{n}, S_{n,k_1}\right)-u_n\left(\frac{s_j}{n}, S_{n,s_j}\right)\right)\left(u_n\left(\frac{k_2}{n}, S_{n,k_2}\right)-u_n\left(\frac{s_j}{n}, S_{n,s_j}\right)\right)\right.\right.$$

$$\left.\left.\times X_{k_1+m+1}X_{k_2+m+1}\right\}\right|$$

$$\leqq K\left\|\left(u_n\left(\frac{k_1}{n}, S_{n,k_1}\right)-u_n\left(\frac{s_j}{n}, S_{n,s_j}\right)\right)\left(u_n\left(\frac{k_2}{n}, S_{n,k_2}\right)-u_n\left(\frac{s_j}{n}, S_{n,s_j}\right)\right)\right.$$

$$\left.\times X_{k_1+m+1}\right\|_{(4+2\delta)/(3+\delta)}\|X_{k_2+m+1}\|_{4+2\delta}\alpha^{\delta/(4+2\delta)}(k_2-k_1)$$

$$\leqq K\left\|u_n\left(\frac{k_1}{n}, S_{n,k_1}\right)-u_n\left(\frac{s_j}{n}, S_{n,s_j}\right)\right\|_4\left\|u_n\left(\frac{k_2}{n}, S_{n,k_2}\right)-u_n\left(\frac{s_j}{n}, S_{n,s_j}\right)\right\|_4$$

$$\times\|X_0\|_{4+2\delta}^2\alpha^{\delta/(4+2\delta)}(k_2-k_1)$$

by the Hölder inequality

(3.18) $\leqq K\delta_\Delta\alpha^{\delta/(4+2\delta)}(k_2-k_1)$.

by the mean value theorem and Lemma 3.2.

When $k_1+m+1<k_2$, by Lemma 3.1 we have

$$\left|E\left\{\left(u_n\left(\frac{k_1}{n}, S_{n,k_1}\right)-u_n\left(\frac{k_2}{n}, S_{n,s_j}\right)\right)\left(u_n\left(\frac{k_2}{n}, S_{n,k_2}\right)-u_n\left(\frac{s_j}{n}, S_{n,s_j}\right)\right)\right.\right.$$

$$\left.\left.\times X_{k_1+m+1}X_{k_2+m+1}\right\}\right|$$

$$\leqq K\|X_{k_1+m+1}\|_{4+2\delta}\|X_{k_2+m+1}\|_{4+2\delta}\alpha^{(1+\delta)/(2+\delta)}(m)$$

$$=O(m^{-(2+2\delta)/\delta-\eta})\qquad(\eta=p-(4+2\delta)/\delta \text{ see (C.2)})$$

$$\leqq n^{-1}\delta_\Delta\quad\text{for all}\quad n\leqq n_\Delta$$

when n_Δ is a sufficiently large integer. Here we used that we had specified $m=m(n)$ as $m=[n^\alpha]$ $(\delta/(2+2\delta)\leqq\alpha<1/3)$

From these inequalities we have

(3.19) $$|T_{1,2}|\leqq K\delta_\Delta\qquad\text{for some } K.$$

As to T_2, as well as the case $T_{1,2}$, we know that there exists an integer $\bar{n}_\Delta\geqq 0$ such that, for any $n\geqq\bar{n}_\Delta$,

$$|T_2|\leqq K\delta_\Delta\qquad\text{for some } K.$$

Thus for given $\varepsilon>0$, we can choose a positive number γ such that for any division Δ $(\delta_\Delta<\gamma)$ there exists an integer n_Δ satisfying that for any $n\geqq n_\Delta$,

$$E\left|\sum_{j=0}^{p(t)}\sum_{k\in I_j}\left\{u_n\left(\frac{k}{n}, S_{n,k}\right)-u_n\left(\frac{[t_jn]}{n}, S_{n,[t_jn]}\right)\right\}\frac{X_{k+m+1}}{\sqrt{n}}\right|^2<\varepsilon.$$

This completes the proof of Lemma 3.5.

Lemma 3.6. *For a fixed division $\Delta: 0=t_0<t_1<\cdots<t_p=1$ and for all t $(0\leqq t\leqq 1)$,*

$$\sum_{j=0}^{p(t)} u_n\Big(\frac{[t_jn]}{n}, S_{n,[t_jn]}\Big)\sum_{i\in J_j}\frac{X_{[t_jn]+m+i}}{\sqrt{n}} - \sum_{j=0}^{p(t)} u_n\Big(\frac{[t_jn]}{n}, S_{n,[t_jn]}\Big)\sum_{i\in J_j}\frac{X_{[t_jn]+i}}{\sqrt{n}} \longrightarrow 0 \text{ in probability as } n\to\infty. \tag{3.20}$$

This lemma can be proved by a standard arguments and so the proof is omitted.

Lemma 3.7. *For a fixed t $(0\leqq t\leqq 1)$, as $n\to\infty$,*

$$\frac{1}{n}\sum_{k=2m}^{[nt]}\frac{\partial u_n}{\partial x}\Big(\frac{k}{n}, S_{n,k-2m}\Big)-\frac{1}{n}\sum_{k=0}^{[nt]}\frac{\partial u_n}{\partial x}\Big(\frac{k}{n}, S_{n,k}\Big) \longrightarrow 0 \text{ in probability.} \tag{3.21}$$

This lemma is proved using the mean value theorem.

Lemma 3.8. *For given $\varepsilon>0$, there exists a positive number γ such that for any division Δ with $\delta_\Delta<\gamma$ we can choose a positive integer n_Δ satisfying that for all $n\geqq n_\Delta$,*

$$E\Big|\frac{1}{n}\sum_{k=0}^{[nt]}\frac{\partial u_n}{\partial x}\Big(\frac{k}{n}, S_{n,k}\Big)-\sum_{j=0}^{p(t)-1}\frac{\partial u_n}{\partial x}\Big(\frac{[t_jn]}{n}, S_{n,[t_jn]}\Big)\frac{[t_{j+1}n]-[t_jn]}{n} - \frac{\partial u_n}{\partial x}\Big(\frac{[t_{p(t)}n]}{n}, S_{n,[t_{p(t)}n]}\Big)\frac{[tn]-[t_{p(t)}n]}{n}\Big|<\varepsilon \tag{3.22}$$

for all $0\leqq t\leqq 1$.

Proof. Denote by v_n the partial derivative $\partial u_n/\partial x$. By applying the mean value theorem we have the following inequality.

L.H.S. of (3.22)

$$\leqq \frac{1}{n}\sum_{j=0}^{p(t)}\sum_{k\in I_j}E\Big|v_n\Big(\frac{k}{n}, S_{n,k}\Big)-v_n\Big(\frac{[t_jn]}{n}, S_{n,[t_jn]}\Big)\Big|+K\sqrt{\delta_\Delta}$$

$$\leqq \frac{K}{n}\sum_{j=0}^{p(t)}\sum_{i\in I_j}\Big(\frac{k-[t_jn]}{n}+\|S_{n,k}-S_{n,[t_jn]}\|_2\Big)+K\sqrt{\delta_\Delta}\leqq K\sqrt{\delta_\Delta}.$$

Q.E.D.

Proof of Theorem 1. Fix a division $\Delta: 0=t_0<t_1<\cdots<t_p=1$ and define two processes as follows.

$$Z_n(\Delta, t)=\sum_{j=0}^{p(t)} u_n\Big(\frac{[t_jn]}{n}, S_{n,[t_jn]}\Big)\sum_{i\in J_j}\frac{X_{[t_jn]+i}}{\sigma\sqrt{n}}$$

$$+\frac{\sigma(m)}{\sigma^2}\sum_{j=0}^{p(t)-1}\frac{\partial u_n}{\partial x}\Big(\frac{[t_jn]}{n}, S_{n,[t_jn]}\Big)\frac{[t_{j+1}n]-[t_jn]}{n}$$

$$+\frac{\sigma(m)}{\sigma^2}\frac{\partial u_n}{\partial x}\Big(\frac{[t_{p(t)}n]}{n}, S_{n,[t_{p(t)}n]}\Big)\frac{[nt]-[t_{p(t)}n]}{n}$$

and

$$Z(\Delta, t)=\sum_{j=0}^{p(t)-1} u(t_j, W(t_j))(W(t_{j+1})-W(t_j))+u(t_{p(t)}, W(t_{p(t)}))(W(t)-W(t_{p(t)}))$$

$$+\frac{\sigma(\infty)}{\sigma^2}\sum_{j=0}^{p(t)-1}\frac{\partial u}{\partial x}(t_j, W(t_j))(t_{j+1}-t_j)$$

$$+\frac{\sigma(\infty)}{\sigma^2}\frac{\partial u}{\partial x}(t_{p(t)}, W(t_{p(t)})(t-t_{p(t)})$$

where $\sigma(m)=\sum_{i=1}^{m}E(X_0X_i)$ and $\sigma(\infty)=\sum_{i=1}^{\infty}E(X_0X_i)$.

Now by the expression (3.4) of Z_n and Lemma 3.3-3.8, in order to complete the proof of Theorem 1, it is sufficient to show that, for each finite set $\{s_i;\ s_i\in[0, 1]\ i=1, 2, \cdots, k\}$, the joint distribution of $(Z_n(\Delta, s_1), \cdots, Z_n(\Delta, s_k))$ converges weakly to the corresponding one of $(Z(\Delta, s_1), \cdots, Z(\Delta, s_k))$. But this is an immediate consequence of the functional central limit theorem for $\{S_{n,k}, k=0, 1, 2, \cdots, n\}$ which is ensured by the conditions (C.0), (C.1) and (C.2) (cf. [3]). Thus Theorem 1 has been proved.

4. Proof of Theorem 2.

Let $\{\xi_i\}$ be a sequence of identically distributed random variables defined on a probability sprce $(\Omega_0, \mathcal{A}_0, P)$ and $\{\mathcal{A}_n, n=1, 2, \cdots\}$ an increasing sequence of the sub-σ-fields of $\mathcal{A}_0$ such that $\sigma\{\xi_1, \cdots, \xi_n\}\subset\mathcal{A}_n$ and $E\{\xi_{n+1}|\mathcal{A}_n\}=0$ a.e.. Assume $E\{\xi_0^2\}=\bar{\sigma}<\infty$.

Proposition 4.1. *Let* $\{u_n(t, \omega): 0\leqq t\leqq 1, \omega\in\Omega_0\ n=1, 2, \cdots\}$ *be a family of random variables satisfying that for each n and t,* $u_n(t, \ )$ *is measurable with respect to* $\mathcal{A}_{[nt]-a}$ *(a is a fixed positive integer). Suppose that* $u_n(t, \omega)$ *are uniformly bounded. Then the family of distributions on* $D[0, 1]$ *defined by the process*

$$W_n(t, \omega)=\frac{1}{\sqrt{n}}\sum_{j=0}^{[nt]} u_n\Big(\frac{j+a}{n}, \omega\Big)\xi_{j+1} \qquad 0\leqq t\leqq 1$$

is tight.

Proof. This proposition had been proved essentially in [8], but for completeness we will give the proof.

Define

$$Y_{n,i}=\sqrt{n}\,W_n(t) \quad \text{if} \quad i\leqq nt<i+1.$$

In order to prove the proposition, it suffices to show that, for each $\varepsilon>0$, there exists a λ with $\lambda>1$, and an integer m_0 such that, if $m\geqq m_0$, then

$$P\Big\{\max_{0\leq i\leq m}|Y_{n,k+i}-Y_{n,k}|\geqq\lambda\sqrt{m}\Big\}\leqq\frac{\varepsilon}{\lambda^2} \tag{4.1}$$

for all k and n.

Since

$$P\Big\{\max_{0\leq i\leq m}|Y_{n,k+i}-Y_{n,k}|\geqq\lambda\sqrt{m}\Big\}\leqq\frac{1}{\lambda^2}E_{\lambda^2}\Big\{\frac{1}{m}\max_{0\leq i\leq m}|Y_{n,k+i}-Y_{n,k}|^2\Big\}$$

where $E_\lambda\{|Z|\}$ denotes $E\{|Z|\,;\,|Z|\geqq\lambda\}$ for a random variable Z, it is sufficient to show the uniform integrability of

$$\frac{1}{m}\max_{0\leq i\leq m}|Y_{n,k+i}-Y_{n,k}|^2.$$

Let K be a constant such that $\|u_n\|\leqq K$ for all n. Put

$$\xi_{i,\alpha}=\begin{cases}\xi_i & \text{if} \quad |\xi_i|\leqq\alpha\\ 0 & \text{otherwise}\end{cases}$$

and

$$\eta_{i,\alpha}=\xi_{i,\alpha}-E\{\xi_{i,\alpha}\,|\,\mathcal{A}_{i-1}\},$$

then $|\eta_{i,\alpha}|$ is bounded by 2α. Define

$$Y_{n,k}^{\alpha}=\sum_{i=0}^{k}u_n\Big(\frac{i+a}{n},\,\omega\Big)\eta_{i+1,\alpha}$$

and

$${}_nD_{k,\alpha}^{l}=\sum_{j=0}^{l}u_n\Big(\frac{k+j+a}{n},\,\omega\Big)\delta_{k+j+1,\alpha}$$

where $\delta_{i,\alpha}$ denotes the difference $\xi_i-\eta_{i,\alpha}$. Then, by the argument used in § 23 in [1], we have

$$E\{\max_{0\leq i\leq m}|Y_{n,k+i}^{\alpha}-Y_{n,k}^{\alpha}|^4\}\leqq\Big(\frac{4}{3}\Big)^4 12m^2K^2\alpha^4$$

and

$$E\{\max_{0\leq l\leq m}|{}_nD_{k,\alpha}^{l}|^2\}\leqq 4K^2mE_{\alpha^2}\{|\xi_0|^2\}.$$

Thus we have that, for all $\beta>0$, n and m

$$E_\beta\Big\{\frac{1}{m}\max_{0\leq i\leq m}|Y_{n,k+i}-Y_{n,k}|^2\Big\}\leqq M\Big\{\frac{\alpha^4}{\beta}+E_{\alpha^2}\{|\xi_0|^2\}\Big\}$$

for a suitable constant M. This completes the proof of the proposition.

Now we proceed to prove Theorem 2. In what follows, we always assume that $\{X_n\}$ is m-dependent ($m\geqq 1$) and $E(X_1)=0$ and $E|X_1|^4<\infty$.

Recall the expansion (3.5) of $Z_n(t)$. Firstly we show that the random elements $I_1(t)$, $I_3(t)$, $I_4(t)$ and $I_5(t)$ in $D[0, 1]$ converge in probability to zero element $\tilde{O}$ as $n\to\infty$. Remark that we can fix the integer m in the expansion (3.5).

(I_1), $E\{\sup_{0\le t\le 1}|I_1(t)|\}\le Kn^{-1/2}\sum_{k=0}^{2m-1}E|X_{k+1}|\longrightarrow 0$ as $n\to\infty$.

Hence we have that $I_1(t)\xrightarrow{P}\tilde{O}$ in $D[0, 1]$ as $n\to\infty$.

(I_3). $I_3(t)$ can be rewritten as

$$\frac{1}{\sqrt{n}}\sum_{j=0}^{2m}\frac{1}{\sqrt{n}}\sum_{k;j+k(2m+1)\le[nt]}u_n\left(\frac{k_j}{n}, S_{n,k_j-2m}\right)(T_{k_j}X_{k_j+1}-E(T_{k_j}X_{k_j+1}))$$

where k_j denotes $j+k(2m+1)$

$$=\sum_{j=0}^{2m}I_{2,j}(t) \quad \text{(say)}.$$

Fix j $(0\leqq j\leqq 2m)$. And define $\mathcal{A}_k=\sigma\{X_0, X_1, \cdots, X_{j+k(2m+1)}\}$ and $\xi_k=T_{k_j}X_{k_j+1}-E(T_{k_j}X_{k_j+1})$. Then the triplet $(\{u_n\}, \{\xi_k\}, \mathcal{A}_k)$ satisfies the assumptions of Proposition 4.1, hence we know that the family of the processes $\{\sqrt{n}I_{2,j}(t), 0\leqq t\leqq 1\}$ is tight (for a fixed j). Therefore we have $\sup|I_3(t)|$ converges to zero in probability as $n\to\infty$.

(I_4, I_5). Trivial.

Next we consider about the tightness for $I_2(t)$ and $I_6(t)$. As to $I_6(t)$ the tightness is trivial. On the other hand $I_2(t)$ can be handled in the same way as the case $I_3(t)$. Combining the tightness with the convergence result of finite joint distribution, we have completed the proof of Theorem 2.

5. Examples.

Let $\{\zeta_i : i=1, 2, \cdots\}$ be a sequence of i.i.d. random variables with $E(\zeta_i)=0$, $E(\zeta_i^2)=\sigma^2$ and finite fourth moment. Suppose that a sequence $\{u_n\}$ and u in $\mathcal{A}$ satisfy the conditions of Theorem 2.

Example A. Let $a_0, a_1\cdots, a_{m-1}$ be a set of real numbers such that $A=a_0+a_1+\cdots+a_{m-1}\neq 0$. Define $X_i=a_0\zeta_i+a_1\zeta_{i+1}+\cdots+a_{m-1}\zeta_{i+m-1}$. Then X_i ia an $(m-1)$-dependent sequence with $E(X_i)=0$ and $E(X_i^2)=\sigma^2(a_0^2+a_1^2+\cdots+a_{m-1}^2)$ $(=\sigma^2B$ say). we have

$$\lim_{n\to\infty}\frac{1}{n}E\left\{\left|\sum_{i=1}^{n}X_i\right|^2\right\}=\sigma^2A^2. \tag{5.1}$$

So, by our Theorem 2, we have that an n,

$$\sum_{k=0}^{[nt]}u_n\left(\frac{k}{n}, S_{n,k}\right)\frac{X_{k+1}}{\sigma A\sqrt{n}}\xrightarrow{D}\int_0^t u(s, W(s))dW(s)+\frac{A^2-B}{2A^2}\int_0^t\frac{\partial u}{\partial x}(s, W(s))ds$$

on $D[0, 1]$.

Example B. In example A, put $a_0=2$, $a_1=2$, $a_2=-1$ ($m=3$). Then it is easily checked that $\{X_i\}$ is 2-dependent and not independent, and that $A^2=B=9$. Hence we have that as $n\to\infty$,

$$\sum_{k=0}^{[nt]} u_n\left(\frac{k}{n}, S_{n,k}\right)\frac{X_{k+1}}{3\sigma\sqrt{n}} \xrightarrow{D} \int_0^t u(s, W(s))dW(s)$$

on $D[0, 1]$.

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