YOKOHAMA MATHEMATICAL
JOURNAL VOL. 35, 1987

A WEAK CONVERGENCE THEOREM FOR FUNCTIONALS OF SUMS OF ϕ-MIXING SEQUENCES

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(Received October 15, 1986)

1. Introduction.

Let $\{X_n, n\geq 1\}$ be a sequence of random variables on a probability space $(\Omega, \mathcal{F}, P)$. Let $F_n^m=\sigma(X_i : n\leq i\leq m)$, $1\leq n\leq m\leq\infty$. Define the following measure of dependence between F_1^n and F_{n+m}^{∞} by

$$\varphi_n(m)=\sup_n\{|P(B|A)-P(B)| : A\in F_1^n,\ P(A)>0,\ B\in F_{n+m}^{\infty}\},$$

and put $\varphi(m)=\sup\varphi_n(m)$. The sequence $\{X_n, n\geq 1\}$ is said to be φ-mixing if $\varphi(m)\to 0$ as $m\to\infty$.

Let $D[0, 1]$ be the space of functions on $[0, 1]$ that are right continuous and have left-hand limits. We give the Skorokhod J_1-topology in $D[0, 1]$ (cf. [1]).

Set $S_n=\sum_{i=1}^{n}X_i$ if $n>0$ and $S_k=0$ if $k\leq 0$. Skorokhod and Slobodeneuk [5] proved that

$$n^{-1/2}\sum_{i=0}^{n-1}f_n(n^{-1/2}S_i)X_{i+1}\xrightarrow{\mathcal{D}}\int_0^1 f(W(t))dW(t)\qquad(n\to\infty)$$

when $\{X_i, i\geq 1\}$ is a sequence of i.i.d. random variables and $W=\{W(t): 0\leq t\leq 1\}$ is a standard Wiener process.

In [9], [7], [8], [3], [4] and [6] the authors proved functional weak convergence theorems of the same type concerning various classes of stochastic processes. In this paper we will give a similar weak convergence theorem when $\{X_i, i\geq 1\}$ is some φ-mixing sequence.

2. Conditions and the main result.

Let F be a space of functions defined on $[0, 1]\times(-\infty, \infty)$ satisfying the following condition: there exist some positive constants M and α such that for $f\in F$

$$|Df(s, x)| \leq M(1+|x|^{\alpha}), \tag{2.1}$$

where D denotes either the identity operator or a first derivative.

Let $\{X_i, i\geq 1\}$ be a centered sequence of random variables. Throughout the paper we assume that for some $\delta>0$

$$\sup_i E|X_i|^{2+\delta}<\infty, \tag{2.2}$$

$$\lim_{n\to\infty} n^{-1}ES_n^2=\sigma^2>0 \tag{2.3}$$

and there exists a sequence $\{\alpha(n), n\geq 1\}$ of positive integers such that

$$\lim_{n\to\infty} n^{-1}\alpha(n)=0 \quad \text{and} \quad \sup_n \sum_{i=1}^{n}\varphi^{1/2}(i\wedge\alpha(n))<\infty. \tag{2.4}$$

Remark 2.1. It is known (cf. Peligrad [2, Corollary (2.4)]) that if condinions (2.2)-(2.4) are satisfied, then

$$(n^{1/2}\sigma)^{-1}S_{[nt]} \xrightarrow{\mathcal{D}} W \qquad (\text{in } D[0, 1]) \tag{2.5}$$

when $\sigma>0$.

Theorem 2.1. *Let $\{X_i, i\geq 1\}$ be a centered sequence satisfying conditions* (2.2)-(2.4). *If $f, f_n\in F$, $n\geq 1$, are functions such that for each $s\in[0, 1]$*

$$Df_n(s, x) \longrightarrow Df(s, x) \qquad \text{as} \quad n\to\infty \tag{2.6}$$

uniformly in x on every finite interval, then

$$(\sigma^2 n)^{-1/2}\sum_{i=0}^{n-\alpha(n)} f_n(i/n, S_i/\sigma n^{1/2})X_{i+\alpha(n)} \xrightarrow{\mathcal{D}} \int_0^1 f(t, W(t))dW(t) \qquad \text{as} \quad n\to\infty \tag{2.7}$$

Here, the stochastic integral in (2.7) *is taken in the L^2-sense.*

From Theorem 2.1 and Lemma 3 [1, § 20] we immediately obtain the following theorem:

Theorem 2.2. *Let $\{X_i, i\geq 1\}$ be a centered wide stationary sequence of random variables. Assume that for some $\delta\in(0, 1)$*

$$\sup_i E|X_i|^{2+\delta}<\infty \quad \textit{and} \quad \sum_{i=1}^{\infty} i(\varphi(i))^{1-\delta}<\infty.$$

If a sequence $\{f, f_n, n\geq 1\}\subset F$ satisfies (2.6) *and $\sigma^2=EX_1^2+2\sum_{j=2}^{\infty}EX_1X_j>0$, then* (2.7) *holds with $\alpha(n)=[n^{1-\delta}]$.*

Remark 2.2. Although a φ-mixing sequence is strong mixing (i.e.

$\sup\{|P(A\cap B)-P(A)P(B)|: A\in F_1^m, B\in F_{m+n}^\infty, m\geq 1\}\to 0$ as $n\to\infty$), the result presented is not implied by that one given in [8], because our moment and mixing rate assumptions are much weaker than those considered by Yoshihara [8].

3. Proof of Theorem 2.1.

Firstly, we shall prove some lemmas. For each $C>0$, define

$$f^C(s, x)=\begin{cases} f(s, x) & \text{if } |x|\leq C \\ 0 & \text{if } |x|>C+1 \\ f(s, x)(C+1-x\,\mathrm{sgn}(x)), & \text{otherwise} \end{cases}$$

In what follows, we shall assume that $\sigma^2=1$.

Lemma 3.1. *Let $\{f_n, n\geq 1\}$ be a sequence of functions such that $f_n\in F$, $n\geq 1$, and let $0=t_0<t_1<\cdots<t_b=1$ be a partition of the interval $[0, 1]$. Then, under the assumptions of Theorem 2.1, for any $\varepsilon>0$ and every $C>0$*

(2.8)
$$\lim_{\gamma\to 0}\lim_{n\to\infty} P_1(\varepsilon, \gamma, n, C)=0,$$

where

$$P_1(\varepsilon, \gamma, n, C)=P\Big(\Big|\sum_{i=0}^{n-\alpha(n)} f_n^C(i/n, S_i/n^{1/2})X_{i+\alpha(n)}/n^{1/2}$$
$$-\sum_{j=0}^{b-1} f_n^C(t_j, S_{[nt_j]-\alpha(n)}/n^{1/2})(S_{[nt_{j+1}]}-S_{[nt_j]})/n^{1/2}\Big|>\varepsilon\Big)$$

and

$$\gamma=\max_{1\leq i\leq b}(t_i-t_{i-1}).$$

Proof. For every i ($[nt_j]<i+\alpha(n)\leq[nt_{j+1}]$) define

$$W_{ij}=W_{ij(i)}=f_n^C(i/n, S_i/n^{1/2})-f_n^C(t_j, S_{[nt_j]-\alpha(n)}/n^{1/2}).$$

Denote

$$W_n(t)=S_{[nt]}/n^{1/2} \quad\text{and}\quad W_n'(t)=S_{[nt]-\alpha(n)}/n^{1/2}, \qquad t\in[0, 1],\ n\geq 1.$$

We have

$$P_1(\varepsilon, \gamma, n, C)=P\Big(\Big|\sum_{j=0}^{b-1}\sum_{i=[nt_j]-\alpha(n)+1}^{[nt_{j+1}]-\alpha(n)} W_{ij}X_{i+\alpha(n)}/n^{1/2}\Big|>\varepsilon\Big).$$

On the other hand, by (2.1), for every $f\in F$

(2.9)
$$|f^C(s, x)-f^C(s', x')|\leq M(1+C^\alpha)(|s-s'|+|x-x'|).$$

Thus, taking into account (2.9), (2.3) and (2.4), for every $0\leq j\leq b-1$ and $[nt_j]<i+\alpha(n)\leq[nt_{j+1}]$ we get

$$\text{(2.10)}\qquad EW_{ij}^2 \le K(C)(|i/n-t_j|^2+E(S_i/n^{1/2}-W_n'(t_j))^2)$$
$$\le K(C)\{(t_{j+1}-t_j+2\alpha(n)/n)^2+\sup_{\substack{0\le t\le 1-\delta\\ 0\le\delta\le\gamma}} E(W_n(t+\delta)-W_n(t))^2\}.$$

But from well known inequality (cf. Peligrad [2, Lemma (1.1)]) we obtain

$$\text{(2.11)}\qquad E(S_{i+j}-S_i)^2=\sum_{k=i+1}^{i+j} EX_k^2+2\sum_{k=i+1}^{i+j}\sum_{l=k+1}^{i+j} EX_kX_l$$
$$\le \sum_{k=i+1}^{i+j} EX_k^2+4\sum_{k=i+1}^{i+j}\sum_{l=k+1}^{i+j}\varphi^{1/2}(l-k)(EX_k^2EX_l^2)^{1/2}$$
$$\le j\sup_k EX_k^2+4j\sum_{i=1}^{\infty}\varphi^{1/2}(i)\sup_k EX_k^2$$
$$=j\Big[\sup_k EX_k^2\Big(1+4\sum_{i=1}^{\infty}\varphi^{1/2}(i)\Big)\Big].$$

Hence, from (2.2), (2.4) and (2.11) it follows that

$$\lim_{\gamma\to 0}\limsup_{n\to\infty}\sup_{0\le\delta\le\gamma}\sup_t E(W_n(t+\delta)-W_n(t))^2=0,$$

so we have

$$\text{(2.12)}\qquad \lim_{\gamma\to 0}\lim_{n\to\infty} M(n)=0,$$

where $M(n)=\sup\{EW_{ij}^2:\ [nt_j]<i+\alpha(n)\le[nt_{j+1}],\ 0\le j\le b-1\}$.

Furthermore, by Holder's and Czebyshev's inequalities

$$\text{(2.13)}\quad P_1(\varepsilon,\gamma,n,C)\le n^{-1}\varepsilon^{-2}\Big\{\sum_{i=0}^{n-\alpha(n)} EW_{ij(i)}^2X_{i+\alpha(n)}^2$$
$$+\sum_{i=0}^{n-\alpha(n)}\sum_{k=i+1}^{n-\alpha(n)} 2EW_{ij(i)}X_{i+\alpha(n)}W_{kj(k)}X_{k+\alpha(n)}\Big\}$$
$$=n^{-1}\varepsilon^{-2}\Big\{\sum_{i=0}^{n-\alpha(n)} EW_{ij(i)}^2X_{i+\alpha(n)}^2+4\sum_{i=0}^{n-\alpha(n)}\sum_{k=i+1}^{n-\alpha(n)}$$
$$\varphi^{1/2}((k-i)\wedge\alpha(n))(E(W_{ij(i)}^2X_{i+\alpha(n)}^2W_{kj(k)}^2)EX_{k+\alpha(n)}^2)^{1/2}\Big\}$$
$$\le n^{-1}\varepsilon^{-2}\Big\{n\sup_{0\le i\le n-\alpha(n)}(E|W_{ij(i)}|^{2+4/\delta})^{\delta/(2+\delta)}(\sup_i E|X_i|^{2+\delta})^{2/(2+\delta)}$$
$$+4n\Big(\sum_{i=1}^{n}\varphi^{1/2}(i\wedge\alpha(n))\Big)\sup_{0\le i,\,k\le n-\alpha(n)}(E|W_{ij(i)}W_{kj(k)}|^{2+4/\delta})^{\delta/2(2+\delta)}$$
$$\times(\sup_i E|X_i|^{2+\delta})^{1/(2+\delta)}(\sup_i EX_i^2)^{1/2}\Big\}.$$

Finally (2.8) is a consequence of (2.13), (2.12), (2.2), (2.4) and the following inequality (cf. (2.9))

(2.14)
$$\sup_i |W_{ij(i)}| \le K(C),$$

where $K(C)$ is a constant depending only on C.

Lemma 3.2. *Under the assumptions of Lemma* 3.1, *for every* $C>0$ *and* $\varepsilon>0$

(2.15)
$$\lim_{\gamma\to 0}\lim_{n\to\infty} P_2(\varepsilon, \gamma, n, C)=0,$$

where

$$P_2(\varepsilon, \gamma, n, C)=P\Big(\Big|\sum_{j=0}^{b-1}\Big\{f_n^C(t_j, W_n'(t_j))-f^C(t_j, W_n'(t_j))\Big\}(W_n(t_{j+1})-W_n(t_j))\Big|>\varepsilon\Big).$$

Proof. Let $V_{nj}(x)=f_n^C(t_j, x)-f^C(t_j, x)$, $1\le j\le b$. Then

(2.16)
$$\begin{aligned} P_2(\varepsilon, \gamma, n, C) &\le \varepsilon^{-2}E\Big\{\sum_{j=0}^{b-1} V_{nj}(W_n'(t_j))(W_n(t_{j+1})-W_n(t_j))\Big\}^2 \\ &=\varepsilon^{-2}\sum_{1\le i\le j\le b-1} E\{V_{ni}(W_n'(t_i))V_{nj}(W_n'(t_j)) \\ &\qquad \times(W_n(t_{i+1})-W_n(t_i))(W_n(t_{j+1})-W_n(t_j))\} \\ &=n^{-1}\varepsilon^{-2}\Big(\sum_{i=0}^{n-\alpha(n)} EV_{nj(i)}^2(W_n(t_{j(i)}))X_{i+\alpha(n)}^2 \\ &\qquad +2\sum_{i=0}^{n-\alpha(n)}\sum_{k=i+1}^{n-\alpha(n)} E(V_{nj(i)}(W_n(t_{j(i)})) \\ &\qquad \times X_{i+\alpha(n)}V_{nj(k)}(W_n(t_{j(k)}))X_{k+\alpha(n)}\Big), \end{aligned}$$

where $j(i)$ denotes a positive integer such that

$$[nt_{j(i)}]<i+\alpha(n)\le[nt_{j(i)+1}].$$

Furthermore, by (2.6),

$$\max_{0\le j\le b-1}\sup_x |V_{nj}(x)| \longrightarrow 0, \qquad \text{as} \quad n\to\infty.$$

Now, using the methods of Lemma 3.1, it is easy to see that (2.15) holds.

Lemma 3.3. *Under the assumptions of Lemma* 3.1, *for every* $f\in F$ *and any given* $C>0$

(2.17)
$$\sum_{j=0}^{b-1} f^C(t_j, W_n'(t_j))(W_n(t_{j+1})-W_n(t_j)) \xrightarrow{\mathscr{D}} \sum_{j=0}^{b-1} f^C(t_j, W(t_j))(W(t_{j+1})-W(t_j)).$$

Proof. Let us observe (cf. (2.11)) that for each $t\in[0, 1]$

$$E(W_n(t)-W_n'(t))^2 \longrightarrow 0 \qquad \text{as} \quad n\to\infty,$$

and f^C is a continuous function. Thus Lemma 3.3 is a simple consequence of Remark 2.1.

For the sake of completeness we give the following lemma:

Lemma 3.4 (cf. Yoshihara [7]) *If $f\in F$, then for every $\varepsilon>0$ and any given* $C>0$

$$P\left(\left|\sum_{j=0}^{b-1} f^C(t_j, W(t_j))(W(t_{j+1})-W(t_j))-\int_0^1 f^C(t, W(t))dW(t)\right|>\varepsilon\right)\longrightarrow 0$$

as $\gamma=\max_{1\le i\le b}(t_i-t_{i-1})\to 0$, *where* $0=t_0<t_1<\cdots<t_b=1$ *is a partition of the interval* $[0, 1]$.

Proof of Theorem 2.1. It is well known that

$$P(\sup_{0\le t\le 1}|W(t)|>C)\longrightarrow 0 \quad \text{as} \quad C\to\infty. \tag{2.18}$$

From this fact and Remark 2.1 we obtain (cf. Billingsley [1, § 10])

$$\lim_{C\to\infty}\lim_{n\to\infty}P(\max_{1\le i\le n}|S_i|>Cn^{1/2})=0. \tag{2.19}$$

Now, it is easy to see that (2.7) follows from (2.8), (2.15), (2.17), (2.18), (2.19) and Lemma 3.4.

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