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A SUBCLASS OF ANALYTIC FUNTIONS

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Abstract: In this place, we consider about the subclass $S(u)$ analytic functions in the unit disc E. It is the purpose of this paper to obtain coefficients estimates for functions in the class $S(u)$.

Introduction.

Let $f(z)$, $g(z)$ and $\Phi(z)$ be functions analytic in the unit disc $E=\{z:|z|<1\}$. We say that $f(z)$ is subordinate to $g(z)$ in E if $f(z)=g(\Phi(z))$ where $|\Phi(z)|\leq 1$ in E. (see [5, p. 50]).

Let $S(u)$ be the class of functions

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n$$

which are analytic in E and satisfy the condition

$$\left|\frac{f(z)}{g(z)}-1\right|<\left|u\frac{f(z)}{g(z)}+1\right| \tag{1}$$

for some u $(0\leq u\leq 1)$ and for all $z\in E$, where

$$g(z)=z+\sum_{n=2}^{\infty}b_n z^n$$

is univalent and starlike in E.

Goel and Sohi [2] have obtained coefficients estimates for functions $f(z)$ belonging to the class $S(0)$ and sharp bounds for the coefficients $|a_2|$, $|a_3|$ of the class $S(u)$.

Theorem. *If the function $f(z)=z+a_2z^2+\cdots$ is in $S(u)$ then*

$$|a_n|\leq n+(n-1)(1+u)+(u+u^2)\frac{(n-1)(n-2)}{2} \qquad \textit{for} \quad n\geq 2.$$

Proof. Let

$$\phi(z)=\frac{\dfrac{f(z)}{g(z)}-1}{u\dfrac{f(z)}{g(z)}+1}.$$

From (1) we have

$$|\phi(z)|\leq|z| \quad\text{and}\quad \phi(z)=z\cdot\Phi(z) \qquad\text{where}\quad \Phi(z)=\sum_{n=0}^{\infty}C_nz^n$$

analytic in E and $|\Phi(z)|\leq 1$ in E. The equality

$$\frac{f(z)-g(z)}{z\left(u\frac{f(z)}{g(z)}+1\right)}=g(z)\cdot\Phi(z)$$

implies that

$$\frac{f-g}{z\left(u\frac{f}{g}+1\right)} \quad\text{is majorized by } g \text{ in } E.$$

If we let

$$\frac{f(z)-g(z)}{z\left(u\frac{f(z)}{g(z)}+1\right)}=h(z)=h_1z+h_2z^2+\cdots$$

then by MacGregor [3, p. 99 Theorem 2(B)]. We have $|h_n|\leq n$. Let

$$P(z)=\frac{f(z)}{g(z)}-1=\frac{(1+u)\phi(z)}{1-u\phi(z)}=p_1z+p_2z^2+\cdots$$

and

$$Q(z)=\frac{(1+u)z}{1-uz}=q_1z+q_2z^2+\cdots.$$

In this case $P(z)$ is subordinate to $Q(z)$. For $n\geq 1$ and $0\leq u\leq 1$ we obtain

$$q_{n+1}-q_n=u^{n-1}(u^2-1)\leq 0$$

and

$$q_n-2q_{n+1}+q_{n+2}=u^{n-1}(1-u)(1-u^2)\geq 0.$$

So the sequence $\{q_n\}$ consists of non negative, non increasing real numbers and $\{q_n\}$ is convex. Hence by Rogosinski [5, p. 50 and 53] we have

$$|p_n|\leq q_1=1+u.$$

Let $K(z)=u\frac{f(z)}{g(z)}+1=1+u+up_1z+up_2z^2+\cdots$. From the equality

$$f(z)-g(z)=z\cdot K(z)\cdot h(z)$$

equating of the coefficients of z^n on both sides, we have

$$a_n-b_n=(1+u)h_{n-1}+up_1h_{n-2}+up_2h_{n-3}+\cdots+up_{n-2}h_1.$$

Since

$$|p_n|\leq 1+u, \quad |h_n|\leq n \quad\text{and}\quad |b_n|\leq n,$$

we obtain $|a_n-b_n|\leq(1+u)(n-1)+u(1+u)(n-2+n-3+\cdots+1)$

and from $|a_n|\leq|a_n-b_n|+|b_n|$ we obtain

$$|a_n| \leq n+(1+u)(n-1)+(u+u^2)\frac{(n-1)(n-2)}{2}.$$

Remark. In this proof we have not been able to obtain an extremal function.

Corollary 1. *If* $u=0$ *then* $|a_n| \leq 2n-1$. *Goel* [1].

Corollary 2. *If* $u=1$ *then* $\mathrm{Re}\,\dfrac{f(z)}{g(z)}>0$, *so f is a close-to-star function. In this case we have* $|a_n| \leq n^2$. *(see* [4, *p.* 61, *Theorem* 4]).

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